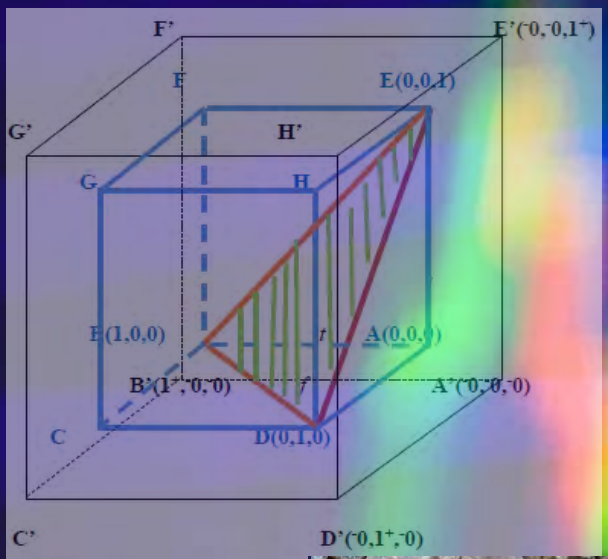
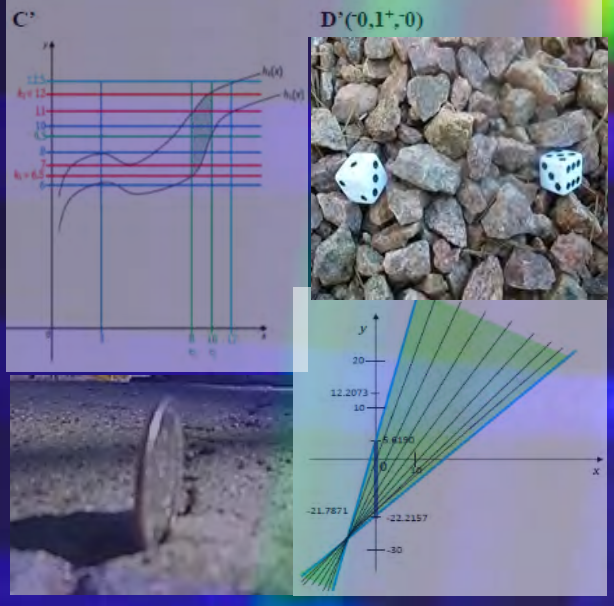
Volume 70,2024

Neutrosophic Sets and Systems

An International Journal in Information Science and Engineering





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Editors-in-Chief

ISSN 2331-6055 (Print)
ISSN 2331-608X (Online)

Neutrosophic Science
International Association (NSIA)

ISSN 2331-6055 (print) *ISSN 2331-608X (online)*

Neutrosophic Sets and Systems

An International Journal in Information Science and Engineering

University of New Mexico

ISSN 2331-6055 (print) ISSN 2331-608X (online)

University of New Mexico

Neutrosophic Sets and Systems

An International Journal in Information Science and Engineering

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The submitted papers should be professional, in good English, containing a brief review of a problem and obtained results.

Neutrosophy is a new branch of philosophy that studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra.

This theory considers every notion or idea <A> together with its opposite or negation <antiA> and with their spectrum of neutralities <neutA> in between them (i.e. notions or ideas supporting neither <A> nor <antiA>). The <neutA> and <antiA> ideas together are referred to as <nonA>.

Neutrosophy is a generalization of Hegel's dialectics (the last one is based on <A> and <antiA> only).

According to this theory every idea <A> tends to be neutralized and balanced by <antiA> and <nonA> ideas - as a state of equilibrium.

In a classical way <A>, <neutA>, <antiA> are disjoint two by two. But, since in many cases the borders between notions are vague, imprecise, Sorites, it is possible that <A>, <neutA>, <antiA> (and <nonA> of course) have common parts two by two, or even all three of them as well.

Neutrosophic Set and *Neutrosophic Logic* are generalizations of the fuzzy set and respectively fuzzy logic (especially of intuitionistic fuzzy set and respectively intuitionistic fuzzy logic). In neutrosophic logic a proposition has a degree of truth (*T*), a degree of indeterminacy (*I*), and a degree of falsity (*F*), where *T, I, F* are standard or non-standard subsets of $]^{-}0, 1^{+}[$.

Neutrosophic Probability is a generalization of the classical probability and imprecise probability.

Neutrosophic Statistics is a generalization of the classical statistics.

What distinguishes the neutrosophics from other fields is the <neutA>, which means neither <A> nor <antiA>.

<neutA>, which of course depends on <A>, can be indeterminacy, neutrality, tie game, unknown, contradiction, ignorance, imprecision, etc.

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ISSN 2331-6055 (print) ISSN 2331-608X (online)

University of New Mexico

NSS

Neutrosophic Sets and Systems

An International Journal in Information Science and Engineering

**** NSS has been accepted by SCOPUS. Starting with Vol. 19, 2018, the NSS articles are indexed in Scopus.**

NSS ABSTRACTED/INDEXED IN
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ISSN 2331-6055 (print) NSS *ISSN 2331-608X (online)*

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Contents

Raed Hatamleh, Ayman Hazaymeh, Finding Minimal Units In Several Two-Fold Fuzzy Finite Neutrosophic Rings……..1-16
Jun Jiang, Fangwei Zhang, Yiying Pan, Rui Wang, and Zhu Han, An adaptive neutrosophic large neighborhood search and its application……..……..……17-36
Yaser Ahmad Alhasan, Florentin Smarandache, Mohamed Elghazali Ali Mohieldin Mohamed and Raja Abdullah Abdulfatah, The indefinite integrals of refined neutrosophic trigonometric functions…….……...37-44
Le Mai Trang, Nguyen Tho Thong and Nguyen Thi Lan Nhi, Utilizing New Temporal Complex Neutrosophic Aczel-Alsina Aggregation Operators for Multi-Criteria Decision Making: An Application to Tourist Destination Selection……..45-63
Pandiselvi. M, Jeyaraman. M, Fixed-Point Results in Neutrosophic b- Metric Like Spaces……...64-72
Nouran Ajabnoor, A new approach of tree soft set with MCDM technique for leadership in international climate policy: measuring renewable energy sources and policies……..……73-84
M.Anandhkumar, A. Bobin, S. M. Chithra, V. Kamalakannan, Generalized Symmetric Fermatean Neutrosophic Fuzzy Matrices…….……....85-106
Abdullah Kargın, Memet Şahin and Kübra Arıtı Şiğva, Operators Based On Multiple Generalized Set-Valued Neutrosophic Quadruple Sets…….……107-136
Lu Niu and Jun Ye, MAGDM Technique Based on Linguistic Neutrosophic Tangent Dombi Aggregation Operators and Its Application in Selecting New Energy Design Schemes.....137-154
Memet Şahin, Abdullah Kargın and Damla Yalvaç, Some Operators for Interval Generalized Set Valued Neutrosophic Quintuple Numbers and Sets.....155-187
Murhaf Riad Alabdullah, Neutrosophic Regular Rings and Properties of Their Ideals…….……188-196
Monika, Rakesh Kumar Bajaj, Aman Sharma, Developing Neutrosophic Cubic Spherical Fuzzy Sets along with their Exponential Aggregation Operators for Decision-Making Problems……..……...197-216
Pragnaleena Debroy, Priyanka Majumder, Surapati Pramanik, and Lalu Seban, TrF-BWM-Neutrosophic-TOPSIS Strategy under SVNS Environment Approach and Its Application to Select the Most Effective Water Quality Parameter of Aquaponic System……..……..217-251
Bimal Shil, Suman Das, Rakhal Das, and Binod Chandra Tripathy, Harmonic Mean Operator Based MADM-Strategy under SVPN-Set Environment……252-263
Lilia Esther Valencia Cruzaty, Adriana Mercedes Ruiz Reynoso, Blanca Estela Hernández Boni-lla, New Neutrosophic Sentiment Analysis Method Based on NeutroAlgebra for the Evaluation of Ethics in Companies……..……264-270
Luong Thi Hong Lan, Nguyen Tho Thong, Nguyen Long Giang and Florentin Smarandache, A New Development of Entropy and Similarity Measures in Temporal Complex Neutrosophic Environments for Tourist Destination Selection ……..…...271-294
J. Johnsy and M. Jeyaraman, Statistical Convergence Sequences in Neutrosophic Metric Spaces…….……...295-304
T.P. Sreelakshmi, K. Uma Samundesvari, Total Domination in Uniform Single Valued Neutrosophic Graph.305-313
Mohanapriya G,Muthukumar S, Santhosh Kumar S, Shanmugapriya M.M, Kalman Bucy Filtered Neuro Fuzzy Image Denoising for Medical Image Processing…….……..314-330
P. Chellamani, R. Sundareswaran, M.Shamugapriya, Said Broumi, Identification of most impact factors of CIBIL score using Fermatean Neutrosophic Dombi Fuzzy Graphs……...331-349
Florentin Smarandache, Mohammad Abobala, Operations with n-Refined Literal Neutrosophic Numbers using the Identification Method and the n-Refined AH-Isometry……350-358
Nawar Hazim Mohammed, On The Solutions of Some Different Types of Two-Fold Fuzzy and Neutrosophic Differential Equations……..359-368
Dolores Hernandez De la Cantera, Rosangela Caicedo Quiroz, Mairim Lago Queija, Jeanntte Rodriguez Gonzalez, Maikel Y. Leyva Vazquez, Analyzing Interdisciplinary Education in General Medicine Using Smarandache's Multivalued Logic Hypothesis Theory and Plithogenic Probability……..……....369-377
Víctor Gustavo Gómez-Rodríguez, Raidell Avello-Martínez, Tomasz Gajderowicz, Noemí Bá-rbara Delgado Álvarez, Johanna Irene Escobar Jara, Noel Batista Hernández, Segress García Hevia , Daniel D. Iturburu Salvador, Assessment of three strategies for teaching an AI literacy program, based on a neutrosophic 2-tuple linguistic mod-el hybridized with the ARAS method…….……..……..378-389

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NSS
Neutrosophic Sets and Systems, Vol. 70, 2024
University of New Mexico

Finding Minimal Units In Several Two-Fold Fuzzy Finite Neutrosophic Rings

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Abstract:

In this paper, we have defined the concept of minimal units in finite two-fold fuzzy neutrosophic rings modulo integers as a generalization of classical elements of the group of units of the mentioned neutrosophic rings. We apply our results to characterize all minimal units in the following two-fold neutrosophic rings $\left(Z_n(I)\right)_{f_I}$ for $n \in \{2,3,4,5\}$.

Keywords: two-fold algebra, finite neutrosophic ring, modulo integer ring, minimal unit.

Introduction

The determination of the invertible elements in a ring has attracted the attention of many researchers, specifically in some modern rings such as neutrosophic rings, and n-cyclic refined neutrosophic rings [6-8]. Many interesting results have been proved about the problem of classifying the group of units in a number of special solutions, and also tables that calculate the value of these units and determine their exact number [9-10].

A unit in a ring R is the concept of being an invertible element concerning multiplication operation which make a group together.

Two-fold neutrosophic algebras are new algebraic structures presented by Smarandache [1] by combining neutrosophic values of truth, falsity, and indeterminacy with classical algebraic sets. These ideas were used by many authors to generalize other famous algebraic structures such as two-fold fuzzy number theoretical systems [2-3], two-fold modules and spaces [4], and two-fold fuzzy rings [5]. Also, they were used in the study of some special two-fold complex functions such as Gamma function [13], and in extending n-refined neutrosophic rings [11-15]. The complex fuzzy ring initiated on complex fuzzy space was given by Al-Husban [14-17]

In this work, we define the minimal two-fold neutrosophic finite ring modulo integers, and we determine all maximal units in these rings for the special values of n between 2 and 5. We have also classified all the units that have been calculated in tables showing their values as well as their number.

Main Discussion

Definition 2.1.

Let $Z_n = \{0,1,\dots,n-1\}$ be the ring of integers modulo n, and $Z_n(I) = \{a+bI \ ; a,b \in Z_n \ , I^2 = I \ \}$ be the corresponding neutrosophic ring. Assume that $f: Z_n \to [0,1]$ be a fuzzy mapping with $\begin{cases} f(0)=0 \\ f(1)=1 \end{cases}$; $f_I: Z_n(I) \to [0,1]$ such that:

$$f_I(a+bI) = \min(f(a), f(b)).$$

The minimal two-fold neutrosophic ring$(Z_n(I))_{f_I}$ is defined as:

$$(Z_n(I))_{f_I} = \{(a+bI)_{f_I(c+dI)} \quad ; \quad a+bI, c+dI \in Z_n(I)\}.$$

Example 2.1.

Consider $(Z_5 = \{0,1,2,3,4\}, +, \cdot)$ the ring of integers modulo 5, take $f : Z_5 \to [0,1]$;

$$f(x) = \begin{cases} 1 & ; x = 1 \\ \frac{1}{2} & ; x \in \{2,3\} \\ \frac{1}{4} & ; x = 4 \\ 0; x = 0 \end{cases},$$

$Z_5(I) = \{0,1,2,3,4, I, 2I, 3I, 4I, 1+I, 1+2I, 1+3I, 1+4I, 2+I, 2+2I, 2+3I, 2+4I, 3+I, 3+2I, 3+3I, 3+4I, 4+I, 4+2I, 4+3I, 4+4I\}$.

$$f_I(3+4I) = \min(f(3), f(4)) = \frac{1}{4},\ f_I(1+2I) = \min(f(1), f(2)) = \frac{1}{2}\ .$$

By a similar approach, we can see:

$f_I(1+4I)=f_I(2+4I)=\frac{1}{4}$, $f_I(0)=f_I(I)=f_I(2I)=f_I(3I)=f_I(4I)=0, f_I(1+3I)=\frac{1}{2}$,

$f_I(2+I)=f_I(3+I)=f_I(2)=f_I(3)=1\backslash 2$, and so on .

Therefor, $(Z_5(I))_{f_I}=\{(a+bI)_0,(a+bI)_1,(a+bI)_{\frac{1}{2}},(a+bI)_{\frac{1}{4}}; for\ \ all\ \ a,b\in Z_5\}$.

Definition 2.2.

An element $X=(a+bI)_{f_I(c+dI)}$ is called a minimal unit in the minimal two-fold neutrosophic ring $(Z_n(I))_{f_I}$ if and only if there exists:

$$Y=(m+nI)_{f_I(t+kI)}\in(Z_n(I))_{f_I}$$

such that:

$$X\circ Y=[(a+bI)(m+nI)]_{f_I[(c+dI)(t+kI)]}=1_0.$$

Example 2.2.

Consider $Z_3=\{0,1,2\}$, $Z_3(I)=\{0,1,2,I,2I,1+I,1+2I,2+I,2+2I\}$,

$$f:\ Z_3\to[0,1]\ \ ;\ \begin{cases}f(0)=0\\ f(1)=1\\ f(2)=\frac{1}{2}\end{cases}\ \ ,f_I: Z_n(I)\to[0,1]$$

such that:

$f_I(0)=f_I(I)=f_I(2I)=f_I(1)=0, f_I(1+I)=1, f_I(2)=0, f_I(1+2I)=f_I(2+I)=$

$f_I(2+2I)=\frac{1}{2}$.

Hence

$\left(Z_3(I)\right)_{f_I}=\left\{0_0,0_1,0_{\frac{1}{2}},1_0,1_1,1_{\frac{1}{2}},I_0,I_1,I_{\frac{1}{2}},(2I)_0,(2I)_1,(2I)_{\frac{1}{2}},(1+I)_0,(1+I)_1,(1+I)_{\frac{1}{2}},(2+\right.$

$\left.I)_0,(2+I)_1,(2+I)_{\frac{1}{2}},(1+2I)_0,(1+2I)_1,(1+2I)_{\frac{1}{2}},(2+2I)_0,(2+2I)_1,(2+2I)_{\frac{1}{2}},2_0,2_1,2_{\frac{1}{2}}\right\}$.

Theorem 2.1.

An element $X=(a+bI)_{f_I(m+nI)}\in\left(Z_n(I)\right)_{f_I}$ is a minimal unit if and only if:

1] $\gcd(a,n)=\gcd(a+b,n)=1$.

2] $\exists\ \ t+kI\in Z_n(I)$ such that: $mt\equiv 0(mod\ \ n)$ or $(m+n)(t+k)-mt\equiv 0(mod\ \ n)$.

Proof.

X is a minimal unit if and only if there exists $Y=(c+dI)_{f_I(t+kI)}$ with $X\circ Y=1_0$, so that:

$$(a+bI)(c+dI)=1 \Longrightarrow \begin{cases} ac=1 \\ ad+bc+bd=0 \end{cases} \Longrightarrow \begin{cases} ac=1 \\ (a+b)(c+d)-ac=0 \end{cases}$$

$$\Longrightarrow \begin{cases} ac=1 \\ (a+b)(c+d)=ac=1 \end{cases} \Longrightarrow \begin{cases} \gcd(a,n)=1 \\ \gcd(a+b,n)=1 \end{cases}.$$

On the other hand:

$f_I[(m+nI)(t+kI)]=0 \Longrightarrow f_I[mt+I(mk+nt+nk)]=0 \Longrightarrow f_I\big[mt+I[(m+n)(t+k)-mt]\big]=0$, thus:

$$f(mt)=0 \quad \text{or} \quad f[(m+n)(t+k)-mt]=0,$$

hence

$$\begin{cases} mt=0\equiv 0(mod\ \ n) \\ or \\ (m+n)(t+k)-mt\equiv 0(mod\ \ n) \end{cases}.$$

Minimal units of $\big(Z_2(I)\big)_{f_I}$:

$Z_2=\{0,1\}$, $Z_2(I)=\{0,1,I,1+I\}$,$\cup(Z_2)=\cup\big(Z_2(I)\big)=\{1\}$.

For any f: $Z_2\to[0,1]$,

we have:

$$f(0)=0, f(1)=1,$$

that is because (f) is a fuzzy mapping, hence: $f_I: Z_2(I)\to[0,1]$ with:

$$f_I(0)=f_I(1)=f_I(I)=0, f_I(1+I)=1.$$

The minimal units of $\big(Z_2(I)\big)_{f_I}$ are : $\{1_0,1_1\}$, that is because $0\cdot(1+I)=0$.

Remark 2.1.

For any two-fold minimal neutrosophic ring $\big(Z_n(I)\big)_{f_I}$ and for any $m+nI\in Z_n(I)$, there exists $0\in Z_n(I)$ such that: $0\cdot(m+nI)=0$.

So that the minimal units of $Z_n(I)$ are:

$$M_u\big(Z_n(I)\big)=\{(a+bI)_x \ \ ; a+bI\in\cup\big(Z_n(I)\big) \ \ , x\in f(Z_n)\}$$

Minimal units of $Z_3(I)$, $Z_4(I)$:

$$Z_3=\{0,1,2\}, Z_3(I)=\{0,1,2,I,2I,1+I,1+2I,2+I,2+2I\},$$

$$\cup(Z_3)=\{1,2\},\cup\big(Z_3(I)\big)=\{1,2,1+I,2+2I\}.$$

For any fuzzy mapping $f: Z_3\to[0,1]$,

we have

$$f(0) = 0, f(1) = 1, f(2) \in]0,1[,$$

$Z_4 = \{0,1,2,3\}, Z_4(I) = \{0,1,2,3, I, 2I, 3I, 1 + I, 1 + 2I, 1 + 3I, 2 + I, 2 + 3I, 2 + 2I, 3 + I, 3 + 2I, 3 + 3I\}$,

$$\cup (Z_4) = \{1,3\}, \cup \left(Z_4(I)\right) = \{1,3,1 + 2I, 3 + 2I\}.$$

For any fuzzy mapping $f: Z_4(I) \to [0,1]$,

we have

$$f(0) = 0, f(1) = 1, f(2), f(3) \in]0,1[.$$

Minimal units of $\left(Z_3(I)\right)_{f_I}$:

	$M_u\left(Z_3(I)\right)_{f_I}$
1	1_0
2	1_1
3	$1_{f(2)}$
4	2_1
5	2_0
6	$2_{f(2)}$
7	$(1 + I)_0$
8	$(1 + I)_1$
9	$(1 + I)_{f(2)}$
10	$(2 + 2I)_0$
11	$(2 + 2I)_1$
12	$(2 + 2I)_{f(2)}$

Minimal units of $\left(Z_4(I)\right)_{f_I}$ in the case of $f(2) = f(3)$:

	$M_u\left(Z_4(I)\right)_{f_I}$
1	1_0
2	1_1
3	$1_{f(2)}$

4	3_0
5	3_1
6	$3_{f(2)}$
7	$(1+2I)_0$
8	$(1+2I)_1$
9	$(1+2I)_{f(2)}$
10	$(3+2I)_0$
11	$(3+2I)_1$
12	$(3+2I)_{f(2)}$

Minimal units of $\left(Z_4(I)\right)_{f_I}$ in the case of $f(2) \neq f(3)$:

	$M_u\left(Z_4(I)\right)_{f_I}$
1	1_0
2	1_1
3	$1_{f(2)}$
4	$1_{f(3)}$
5	3_0
6	3_1
7	$3_{f(2)}$
8	$3_{f(3)}$
9	$(3+2I)_0$
10	$(3+2I)_1$
11	$(3+2I)_{f(2)}$
12	$(3+2I)_{f(3)}$
13	$(1+2I)_0$
14	$(1+2I)_1$
15	$(1+2I)_{f(2)}$
16	$(1+2I)_{f(3)}$

Minimal units of $\left(Z_5(I)\right)_{f_I}$:

$Z_5 = \{0,1,2,3,4\}, Z_5(I) = \{0,1,2,3,4, I, 2I, 3I, 4I, 1+I, 1+2I, 1+3I, 1+4I, 2+I, 2+2I, 2+3I, 2+4I, 3+I, 3+2I, 3+3I, 3+4I, 4+I, 4+2I, 4+3I, 4+4I\}$,

$\cup (Z_5) = \{1,2,3,4\}, \cup \left(Z_5(I)\right) = \{1,2,3,4,1+I, 1+2I, 1+3I, 2+I, 2+2I, 2+4I, 3+I, 3+3I, 3+4I, 4+2I, 4+3I, 4+4I\}$.

Minimal units of $\left(Z_5(I)\right)_{f_I}$ in the case of $f(2) = f(3) = f(4)$:

	$M_u\left(Z_5(I)\right)_{f_I}$
1	1_0
2	1_1
3	$1_{f(2)}$
4	2_0
5	2_1
6	$2_{f(2)}$
7	3_0
8	3_1
9	$3_{f(2)}$
10	4_0
11	4_1
12	$4_{f(2)}$
13	$(1+I)_0$
14	$(1+I)_1$
15	$(1+I)_{f(2)}$
16	$(1+2I)_0$
17	$(1+2I)_1$

18	$(1+2I)_{f(2)}$
19	$(1+3I)_0$
20	$(1+3I)_1$
21	$(1+3I)_{f(2)}$
22	$(2+I)_0$
23	$(2+I)_1$
24	$(2+I)_{f(2)}$
25	$(3+I)_0$
26	$(3+I)_1$
27	$(3+I)_{f(2)}$
28	$(2+2I)_0$
29	$(2+2I)_1$
30	$(2+2I)_{f(2)}$
31	$(2+4I)_0$
32	$(2+4I)_1$
33	$(2+4I)_{f(2)}$
34	$(3+4I)_0$
35	$(3+4I)_1$
36	$(3+4I)_{f(2)}$
37	$(4+2I)_0$
38	$(4+2I)_1$
39	$(4+2I)_{f(2)}$
40	$(4+3I)_0$
41	$(4+3I)_1$
42	$(4+3I)_{f(2)}$
43	$(4+4I)_0$
44	$(4+4I)_1$
45	$(4+4I)_{f(2)}$

46	$(3+3I)_0$
47	$(3+3I)_1$
48	$(3+3I)_{f(2)}$

Minimal units of $\left(Z_5(I)\right)_{f_I}$ in the case of $f(2) \neq f(3) \neq f(4)$:

	$M_u\left(Z_5(I)\right)_{f_I}$
1	1_0
2	1_1
3	$1_{f(2)}$
4	$1_{f(3)}$
5	$1_{f(4)}$
6	2_0
7	2_1
8	$2_{f(2)}$
9	$2_{f(3)}$
10	$2_{f(4)}$
11	3_0
12	3_1
13	$3_{f(2)}$
14	$3_{f(3)}$
15	$3_{f(4)}$
16	4_0
17	4_1
18	$4_{f(2)}$
19	$4_{f(3)}$
20	$4_{f(4)}$
21	$(1+I)_0$
22	$(1+I)_1$

23	$(1+I)_{f(2)}$
24	$(1+I)_{f(3)}$
25	$(1+I)_{f(4)}$
26	$(1+2I)_0$
27	$(1+2I)_1$
28	$(1+2I)_{f(2)}$
29	$(1+2I)_{f(3)}$
30	$(1+2I)_{f(4)}$
31	$(1+3I)_0$
32	$(1+3I)_1$
33	$(1+3I)_{f(2)}$
34	$(1+3I)_{f(3)}$
35	$(1+3I)_{f(4)}$
36	$(2+I)_0$
37	$(2+I)_1$
38	$(2+I)_{f(2)}$
39	$(2+I)_{f(3)}$
40	$(2+I)_{f(4)}$
41	$(2+2I)_0$
42	$(2+2I)_1$
43	$(2+2I)_{f(2)}$
44	$(2+2I)_{f(3)}$
45	$(2+2I)_{f(4)}$
46	$(2+4I)_0$
47	$(2+4I)_1$
48	$(2+4I)_{f(2)}$
49	$(2+4I)_{f(3)}$
50	$(2+4I)_{f(4)}$

51	$(3+I)_0$
52	$(3+I)_1$
53	$(3+I)_{f(2)}$
54	$(3+I)_{f(3)}$
55	$(3+I)_{f(4)}$
56	$(3+3I)_0$
57	$(3+3I)_1$
58	$(3+3I)_{f(2)}$
59	$(3+3I)_{f(3)}$
60	$(3+3I)_{f(4)}$
61	$(3+4I)_0$
62	$(3+4I)_1$
63	$(3+4I)_{f(2)}$
64	$(3+4I)_{f(3)}$
65	$(3+4I)_{f(4)}$
66	$(4+2I)_0$
67	$(4+2I)_1$
68	$(4+2I)_{f(2)}$
69	$(4+2I)_{f(3)}$
70	$(4+2I)_{f(4)}$
71	$(4+3I)_0$
72	$(4+3I)_1$
73	$(4+3I)_{f(2)}$
74	$(4+3I)_{f(3)}$
75	$(4+3I)_{f(4)}$
76	$(4+4I)_0$
77	$(4+4I)_1$
78	$(4+4I)_{f(2)}$

79	$(4+4I)_{f(3)}$
80	$(4+4I)_{f(4)}$

Minimal units of $\left(Z_5(I)\right)_{f_I}$ in the case of $f(a)=f(b)\neq f(c)$; where $a,b,c\in\{2,3,4\}$:

	$M_u\left(Z_5(I)\right)_{f_I}$
1	1_0
2	1_1
3	$1_{f(a)}$
4	$1_{f(c)}$
5	2_0
6	2_1
7	$2_{f(a)}$
8	$2_{f(c)}$
9	3_0
10	3_1
11	$3_{f(a)}$
12	$3_{f(c)}$
13	4_0
14	4_1
15	$4_{f(a)}$
16	$4_{f(c)}$
17	$(1+I)_0$
18	$(1+I)_1$
19	$(1+I)_{f(a)}$
20	$(1+I)_{f(c)}$
21	$(1+2I)_0$
22	$(1+2I)_1$

23	$(1+2I)_{f(a)}$
24	$(1+2I)_{f(c)}$
25	$(4+4I)_0$
26	$(4+4I)_1$
27	$(4+4I)_{f(a)}$
28	$(4+4I)_{f(c)}$
29	$(1+3I)_0$
30	$(1+3I)_1$
31	$(1+3I)_{f(a)}$
32	$(1+3I)_{f(c)}$
33	$(2+I)_0$
34	$(2+I)_1$
35	$(2+I)_{f(a)}$
36	$(2+I)_{f(c)}$
37	$(2+2I)_0$
38	$(2+2I)_1$
39	$(2+2I)_{f(a)}$
40	$(2+2I)_{f(c)}$
41	$(2+4I)_0$
42	$(2+4I)_1$
43	$(2+4I)_{f(a)}$
44	$(2+4I)_{f(c)}$
45	$(3+I)_0$
46	$(3+I)_1$
47	$(3+I)_{f(a)}$
48	$(3+I)_{f(c)}$
49	$(3+3I)_0$
50	$(3+3I)_1$

51	$(3+3I)_{f(a)}$
52	$(3+3I)_{f(c)}$
53	$(3+4I)_0$
54	$(3+4I)_1$
55	$(3+4I)_{f(a)}$
56	$(3+4I)_{f(c)}$
57	$(4+2I)_0$
58	$(4+2I)_1$
59	$(4+2I)_{f(a)}$
60	$(4+2I)_{f(c)}$
61	$(4+3I)_0$
62	$(4+3I)_1$
63	$(4+3I)_{f(a)}$
64	$(4+3I)_{f(c)}$

Conclusion

In this paper, we defined the concept of minimal units in finite two-fold finite neutrosophic rings modulo integers as a generalization of classical elements of the group of units of the mentioned neutrosophic rings. We applied our results to characterize all minimal units in the following two-fold neutrosophic rings $\left(Z_n(I)\right)_{f_I}$ for $n \in \{2,3,4,5\}$.

In the future, we aim to classify all minimal units for higher orders cases $n \geq 6$.

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Received: March 1, 2024. Accepted: July 25, 2024

Neutrosophic Sets and Systems, Vol. 70, 2024
University of New Mexico

An adaptive neutrosophic large neighborhood search and its application

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Abstract: An adaptive large neighborhood search (ALNS) lacks a clear measure for assessing the improvement degree in the new solution, which causes fuzziness and uncertainty in the operator score. To solve the aforementioned problem, the main innovation of this study is to propose an adaptive neutrosophic large neighborhood search (ANLNS). Specifically, the main work is as follows. Firstly, the number of times each operator scores are quantified by constructing NSs, thereby analyzing algorithm performance and preventing the idealization of scores. Secondly, a novel neutrosophic utility function and score function are proposed to assign an appropriate score for the operator within a reasonable interval. Finally, the effectiveness and robustness of the proposed ANLNS is validated by the capacitated vehicle routing problem benchmarks with three varying scales and comparative analyses. The compared results indicate that the proportion of best solutions for ANLNS are 50%, 100%, and 37.5%, which significantly highlight the robustness and reliability of the proposed algorithm when the degree of destruction is 0.3, 0.5, and 0.7, respectively. Meanwhile, the proposed ANLNS is efficient and flexible, providing a novel method for addressing other situation optimization problems.

Keywords: Neutrosophic sets; Utility function; Adaptive neutrosophic large neighborhood search; Capacitated vehicle routing problem

1. Introduction

As the complexity and particularity of contemporary work environments, the efficacy of solutions and the rationality of algorithm workflows are of concern dramatically. Therefore, it is noted that an increasing number of heuristic algorithms have been widely applied and researched, such as adaptive large neighborhood search (ALNS) [1]. Usually, the six primary sections of ALNS process are initialization, roulette wheel selection (RWS), destroy operators, repair operators, adaptive weight adjustment, and acceptance and stopping criteria [2]. Especially, in the adaptive weight adjustment stage, the ALNS only focuses on whether the

new solution is better than the best solution or current solution. If this condition is met, the operator increases the fixed score. However, this established scoring mechanism does not evaluate the specific improvement degree in the new solution, which makes it difficult to assign the appropriate score for each operator and exhibits fuzziness and uncertainty.

Based on the aforementioned problems, this study mainly focuses on how to assign an appropriate score for each operator. Thereafter, this study proposes an adaptive neutrosophic large neighborhood search (ANLNS) algorithm to better assign the operator score. Specifically, the main contributions are as follows.

Firstly, this study keeps track of the number of times that each operator scores under four different situations, and quantifies these number of times by constructing neutrosophic sets (NSs). Among them, NSs facilitate the ANLNS to comprehensively analyze operator performance and enhance the ability to evaluate actual contributions of each operator.

Secondly, this study proposes a neutrosophic utility function based on neutrosophic sets and a score function. This combination of functions not only better measures the algorithm's preference for different operators but also assign a suitable score within a reasonable interval. Moreover, the scores are rational, reducing the existence of subjective factors.

Finally, this study verifies the robustness and validity of ANLNS through the capacitated vehicle routing problem (CVRP) benchmark instances. Additionally, it provides a novel method to address the CVRP by applying ANLNS.

The rest of this study is organized as follows. Section 2 introduces the literature review of ALNS, NSs, and CVRP. Section 3 indicates some definitions of ALNS and NSs, introduces a mathematical model on CVRP, and proposes the ANLNS. Section 4 provides a study case and proves its validity and robustness of the proposed ANLNS. In Section 5, this study presents some conclusions and insights.

2. Literature review

For convenience, this section reviews previous studies on ALNS, NSs, and CVRP, providing an overview of existing deficiencies that need attention.

ALNS was proposed by Pisinger and Ropke [3] to address the vehicle routing problem (VRP) under various transportation conditions. Subsequently, scholars have carried out extensive expansion and in-depth discussion on the ALNS in recent decades. For instance, Cai et al. [4] employed a hybrid heuristic algorithm that integrates ALNS and tabu search to address the electric vehicle relocation problem. Then, aiming at a bi-objective green VRP, Amiri et al. [5] presented an approach integrating three multi-objective solution methods with the ALNS. In the same year, by enhancing the ALNS algorithm, Fathollahi-Fard et al. [6] resolved the general quadratic allocation problem efficiently. Chen et al. [7] designed a new ALNS to effectively address the routing of delivery robots. Simultaneously, Rolim et al. [8] constructed a new earliness-tardiness extension of the ALNS, which contained several insertion operators and an embedded procedure.

Meanwhile, NSs offer a powerful framework for effectively analyzing and solving uncertain, fuzzy, and contradictory information. In these regards, some scholars have delved into the application of NSs. For example, Rashno et al. [9] designed a new clustering algorithm based on NSs, effectively addressing boundary and outlier points by defining data indeterminacy and optimizing a cost function through gradient descent methods. Then,

Saeed et al. [10] extended and proposed the neutrosophic hypersoft graph to hand neutrosophic hypersoft information. Later, Ye [11] extended the application of neutrosophic (indeterminate fuzzy) multivalued sets and achieved great advancements in teaching quality assessment of teachers. Then, Smarandache [12] reviewed the definitions and properties of six new topologies introduced between 2019 and 2022. Barbosa and Smarandache [13] introduced the Pura Vida Neutrosophic Algebra based on neutrosophic numbers, which apply to the Neutrosophic matrices and vectors. Smarandache [14] designed a novel SuperHyperSoft Set and Fuzzy Extension SuperHyperSoft Set, which provides examples to present these applications. Smarandache [15] applied the n-th PowerSet of a set to describe the organizational architecture in the real world more comprehensively and discussed SuperHyperFunction, SuperHyperStructure, Neutrosophic SuperHyperFunction, and Neutrosophic SuperHyperStructure. Smarandache [16] identified nine groundbreaking topologies, providing a comprehensive overview and delving into current trends and future challenges. Al-Hijjawi and Alkhazaleh [17] proposed a concept of the possibility of neutrosophic hypersoft sets and gave the basic operations and characteristics. To better understand Neutrosophic Numbers' operation in the neutrosophic statistics framework, Smarandache [18] innovatively presented the Appurtenance Equation and Inclusion Equation and gave the solution method. Mishra et al. [19] proposed an integrated multi-attribute decision-making methodology using single-valued NSs to evaluate and prioritize energy storage technologies.

Besides, CVRP is NP-hard. CVRP has made some improvements and extensions on the basis of the traditional traveling salesman problem [20]. For example, Vidal [21] implemented a simple and open-source Hybrid Genetic Search algorithm, which specially contained a new neighborhood for the CVRP. Then, Rezaei et al. [22] proposed solving the CVRP, combining a refined imperialist competitive algorithm with a hybrid genetic search algorithm for enhancing optimization. After that, Jiang et al. [23] designed an evolutionary algorithm for solving CVRP based on the relevance matrix. At the same time, Wang et al. [24] effectively improved the genetic algorithm and successfully solved the multi-objective CVRP with workload balancing. Simultaneously, Chi and He [25] addressed the pickup CVRP with three-dimensional loading constraints by improving a traditional branch-and-price algorithm.

Based on the aforementioned, it indicates that many improved ALNS algorithms satisfy some practical work requirements. However, the issues of fuzziness and uncertainty need to be resolved in the ALNS algorithm. Specifically, the traditional scoring mechanism of the ALNS is to subjectively assign a fixed score to the operator, failing to effectively reflect the specific improvement degree in the new solution and leading to uncertainty and ambiguity in the operator score. Although NSs are widely applied, they have not yet been integrated into ALNS, which provides a novel direction for research. Aiming at the above problems, this study proposes the integration of NSs into the ALNS to better assign operator scores and reflect their performance.

3. Materials and methods

For convenience, this section briefly introduces the basic definitions of ALNS and NSs, and basic problem description on CVRP.

3.1. Preliminaries

Definition 1. [26] The ALNS algorithm flow refers to the pseudo-code of Algorithm 1.

ALGORITHM 1 ALNS.

1: **Input**: the initial solution x;
2: $x_{current} \leftarrow x$; $x_{best} \leftarrow x_{current}$;
3: $q = \mu \cdot C_N$; $t = 0$;
4: **While** $t \leq max\_item$ **do**
5: $x_s \leftarrow x_{current}$;
6: $x' \leftarrow$ select a destroy operator by using the RWS, and remove q requests from x_s;
7: $x_{new} \leftarrow$ select a repair operator by using the RWS, and insert q requests into x';
8: **If** $f(x_{new}) < f(x_{best})$ **then**
9: $x_{best} \leftarrow x_{new}$; $x_{current} \leftarrow x_{new}$; increase the score σ_1 for the utilized operators;
10: **Else if** $f(x_{new}) < f(x_{current})$ **then**
11: $x_{current} \leftarrow x_{new}$; increase the score σ_2 for the utilized operators;
12: **Else if** the simulated annealing criteria is met, **then**
13: $x_{current} \leftarrow x_{new}$; increase the score σ_3 for the utilized operators;
14: **Else**
15: increase the score σ_4 for the utilized operators;
16: **End if**
17: update scores and weights of all operators;
18: $t \leftarrow t + 1$;
19: **End while**
20: **Output**: the best solution x_{best}.

Here, Line 1 indicates that the ALNS selects the initial feasible solution x as the input parameter. Line 2 indicates that the current solution $x_{current}$ is assigned the value of x, followed by setting the best solution x_{best} equal to $x_{current}$. Line 3 indicates that q requests are removed from the solution, and the start iteration number is set to 0. Among them, the parameter C_N represents the size of the solution, and the parameter μ denotes the degree of destruction to which requests are removed from the solution. Line 5 indicates that a solution x_s is a copy of the $x_{current}$. Lines 6-7 execute the RWS to select the operators that construct a new solution x_{new}. Apply lines 8-11 to update x_{best} and $x_{current}$. If x_{new} is better than x_{best}, x_{best} and $x_{current}$ replace x_{new}. Otherwise, if x_{new} is better than $x_{current}$, $x_{current}$ replaces x_{new}. Then, if the operators produce x_{new} that is worse than $x_{current}$, accept x_{new} using the simulated annealing acceptance criterion (lines12-16). Line 17 performs the update of the operator scores and weights. Finally, the algorithm outputs x_{best} in line 20.

Definition 2. [27, 28] Let X be a space of points (objects), and each element of X is represented by x. A truth-membership function $T_S(x)$, an indeterminacy-membership function $I_S(x)$, and a falsity-membership function $F_S(x)$ are used to describe a NS S in X. Then, use $S = \{\langle x, T_S(x), I_S(x), F_S(x)\rangle | x \in X\}$ as a representation of S. Functions $T_S(x)$, $I_S(x)$, and $F_S(x)$ are real standard or non-standard subsets of $]0^-, 1^+[$, denoted by $T_S(x): X \rightarrow]0^-, 1^+[$, $I_S(x): X \rightarrow]0^-, 1^+[$, and $F_S(x): X \rightarrow]0^-, 1^+[$ respectively. There is no restriction on the sum of $T_S(x)$, $I_S(x)$, and $F_S(x)$, resulting in $0^- \leq T_S(x) + I_S(x) + F_S(x) \leq 3^+$. For simplicity, if X has only one element, S is expressed as $S = \langle T_S(x), I_S(x), F_S(x)\rangle$ and can be called a neutrosophic number.

Example: The student's performance grade is considered a neutrosophic number to

effectively represent a student's performance grade in a subject: ⟨7.2,7.5,7.8⟩. The student's grade in the truth-membership degree is 7.2. The student's grade in the indeterminacy-membership degree is 7.5. The student's grade in the falsity-membership degree is 7.8.

3.2. Problem description

The CVRP was defined on a connected graph $G = (V, E)$. Among them, $V = \{V_0, V_1, \ldots, V_{n+1}\}$ is a vertex set and $E = \{(V_i, V_j) | V_i, V_j \in V, V_i \neq V_j\}$ is an arc set. Each arc represents the distance between two vertices. The vertex V_0 represents the depot, and the other vertices represent the customer point. Each vehicle has the same loading capacity and carries the same kind of goods, that is, constitutes a group of homogeneous fleets. Each vehicle begins its route at the depot and ultimately returns to the depot after completing the distribution task. The demand for goods by all customers is known, with each customer being served by exactly one vehicle. The loading capacity and total distance of each vehicle are not allowed to exceed the maximum loading capacity and maximum distance. The CVRP aims at minimizing total cost, which, in practical terms, translates to reducing the overall transportation distance. Then, the CVRP is expressed as follows:

$$\min f = \sum_{i=0}^{N}\sum_{j=0}^{N}\sum_{v=1}^{K} d_{ij} x_{ijv} \tag{1}$$

$$s.t.$$

$$\sum_{i=0}^{N}\sum_{j=0}^{N} d_{ij} x_{ijv} \leq T_v, 1 \leq v \leq K; \tag{2}$$

$$\sum_{v=1}^{K}\sum_{j=1}^{N} x_{0jv} = \sum_{v=1}^{K}\sum_{j=1}^{N} x_{j0v} \leq 1; \tag{3}$$

$$\sum_{v=1}^{K}\sum_{j=1}^{N}\sum_{i=0}^{N} x_{ijv} = 1; \tag{4}$$

$$\sum_{i=0}^{N}\sum_{j=0}^{N} x_{ijv} M_i \leq Q_v, v \in \{1, \ldots, K\}; \tag{5}$$

$$\sum_{v=0}^{K}\sum_{i=0}^{N} x_{ijk} \leq K, \quad for \quad i = 0, \tag{6}$$

where Eq. (1) indicates an objective function of CVRP. Eq. (2) indicates that each vehicle follows the maximum traveling distance. Eq. (3) indicates that each vehicle begins and ends in the depot. Eq. (4) makes sure each customer only accesses one vehicle. Eq. (5) indicates the constraint of maximum loading capacity. Eq. (6) indicates that the number of available routes after leaving the depot, which is no more than K. x_{ijv} is a 0-1 binary variable, where $x_{ij} = 1$ indicates that vehicle v travels from customer i to customer j, and 0 otherwise. d_{ij} represents the distance from customer i to customer j, N indicates the total number of customers served, K represents the maximum number of vehicles utilized, Q_v represents the maximum loading

constraint for vehicle v, M_i represents customer i demand, T_v represents the maximum distance traveling of vehicle v. Meanwhile, Figure 1 illustrates an instance of CVRP to enhance understanding.

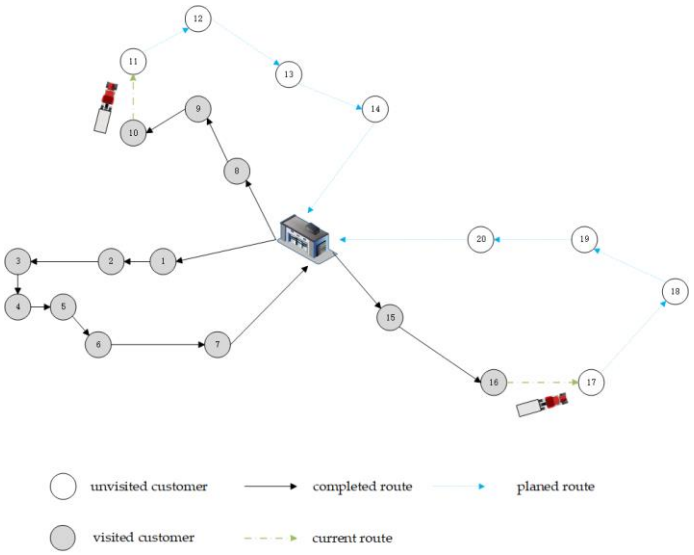


Figure 1. Illustration of the capacitated vehicle routing problem.

3.3. Proposed approach

This study introduces the concrete steps of the ANLNS algorithm, which is applied to solve the CVRP. The initial solution stage is expressed in subsection 3.3.1. Then, the general principles of the RWS are explained in subsection 3.3.2. The destroy operators are introduced in subsection 3.3.3. The repair operators are introduced in subsection 3.3.4. Furthermore, NSs and neutrosophic utility functions are used to assign an appropriate score for the actual contribution of the used operator in subsection 3.3.5. The acceptance and stopping criteria are introduced in subsection 3.3.6.

3.3.1. Initial solution

A high-quality initial feasible solution has a crucial impact on the final best solution, which aids the algorithm rapidly to converge, prevents falling into the local best solution, and reduces computation time. Due to its simplicity, efficiency, and scalability, this study chooses the greedy insertion to generate the high-quality initial feasible solution. Before initialization, customers N with different demands are known. In general, under the constraint of the maximum loading capacity Q_v of vehicle v, the greedy insertion algorithm chooses the customer nearest to the current customer as the next direction for transportation. According to reference [29], the main step for the initial solution is described in Algorithm 2.

ALGORITHM 2 Initial solution.

1: **Input**: a set of customers N, the maximum loading capacity Q_v;
2: create an empty set of routes P;
3: create an empty boole set V of uninvited customers;
4: create an empty current route R and set the start point as the depot;
5: **Repeat**
6: insert one customer into the R, all constraints are suitable;

7: set the status of the visited customer as true in the V;
8: **If** the total demand of customers in the R exceeds Q_v **then**
9: add the depot to the tail of R, put R into the P, and set R empty again;
10: return to step 4;
11: **Else**
12: return to step 6;
13: **End if**
14: **Until** all customers are served in the N as well and all access status is true in the V, respectively;
15: get the set of routes P as the initial solution x;
16: **Output**: the initial feasible solution x.

3.3.2. Selection of operators

The combination of different destroy operators and repair operators not only enhances the diversity of solutions but also balances the relationship between exploration and exploitation effectively. For each iteration, this study has k operators with weight $\omega_i, i \in \{1, \ldots, k\}$, and uses the RWS to choose the operators based on the probability φ. Thereafter, the φ of the jth operator is given as

$$\varphi = \frac{\omega_j}{\sum_{i=1}^{k} \omega_i}. \tag{8}$$

Noting the destroy operator is selected independently before the repair operator. Specially, before the algorithm starts, the weight of each operator is initialized to 1.

3.3.3. Removal operators

This study describes three types of destroy operators in detail, namely random removal, Shaw removal, and worst removal. For each iteration, these heuristics remove the number of customers $q = \mu \cdot N$ from the solution x_s. Among them, N indicates the total number of customers in $x_{current}$. Next, the removed customers are stored in set D.

1. Random removal

The main principle is to remove q customers from x_s randomly. Next, these removed customers are stored in set D.

2. Shaw removal

The Shaw removal as a new local search to address the VRP. The core idea of Shaw removal is to remove the customer that is similar in geographical location, demand, or being visited at similar times [30-31]. To do so, a relatedness function $R(i, j)$ is defined to better measure the similarity of customer i and customer j. Therefore, the function is defined as

$$R(i, j) = \alpha d_{ij} + \delta \left| q_i - q_j \right|, \tag{9}$$

where d_{ij} presents the distance of customer i and customer j as the first term of $R(i, j)$. $\left| q_i - q_j \right|$ is the second term of $R(i, j)$. Meanwhile, the parameters q_i and q_j denote the demand of customer i and customer j, respectively. The above two terms are weighted using weights α, and δ, respectively. According to reference [32], the entire Shaw removal process is expressed by Algorithm 3. In particular, the parameter P_1 is added to realize the process of selection randomly, thereby increasing the diversity of solutions.

ALGORITHM 3 Shaw removal.

Function Shaw removal (solution x_s, randomized selection parameter P_1)
  create an empty set D to store the removed customers;
  a randomly selected customer ψ from $x_{current}$;
  assume that the set D contains the customer ψ;
  while $|D| < q$ **do**
    a randomly selected customer RN from D;
    a set L containing the similarity $R(i,j)$ of RN with all customers from x_s not in D;
    sort L in ascending order according to the values of $R(i,j)$;
    generate a random number γ, where $\gamma \in [0,1]$;
    remove customer $L[\gamma^{P_1}|L|]$ from L and insert it into D;
  End while
End function

3. Worst removal

The fundamental principle of the worst removal is to select and remove the customer that brings high costs. At the beginning, let $\Delta_i = f(x_s) - f_{-i}(x_s)$ be the cost of customer i. Among them, $f_{-i}(x_s)$ is the cost for x_s after removing customer i. Note that the high Δ_i equals the longest distance from customer i to customer j in this study. Therefore, get the cost Δ_i for each customer and sort it in descending order. Subsequently, according to the parameter q, the worst removal uses the randomized selection parameter P_2 to remove customers and store them in set D. The procedure of worst removal is referred to the reference [32]. Meanwhile, the main steps are shown in Algorithm 4.

ALGORITHM 4 Worst removal.

Function Worst removal (solution x_s, randomized selection parameter P_2)
  create an empty set D to store the removed customers;
  record the total distance of solution $x_{current}$ as $f(x_s)$;
  while $q > 0$ **do**
    calculate the cost Δ_i of each customer in x_s;
    a set L containing the cost Δ_i of each customer in x_s;
    sort L in ascending order according to the values of Δ_i;
    generate a random number γ, where $\gamma \in [0,1]$;
    remove customer $L[\gamma^{P_2}|L|]$ from L and insert it into D;
    $q \leftarrow q - 1$;
  End while
End function

3.3.4. Insertion operators

In this subsection, each removed customer reinserts the broken x_s to construct the new solution. The insertion operators are regret insertion and best insertion, respectively. Then, the repair work is not completed until all removed customers are inserted in each iteration.

1. Regret insertion

The main work of this operator is to reinsert the removed customer into the lowest insertion cost position so that there is not most regretful. According to the reference [33], the regret insertion calculates the lowest insertion cost Δ_1 to determine the corresponding optimal position in route j_1. Then, the second lowest insertion cost Δ_2 follows the same approach to do so. For each user, the regret value is calculated as $RC = \Delta_2 - \Delta_1$. After that, the high regret value for customer i is selected for reinsert operation, aiming to realize minimization for the objective cost.

2. Best insertion

As indicated by reference [34], the best insertion position is selected by using the best insertion strategy, which aims to generate the lowest insertion cost for the removed customer. Hence, at each iteration, record the insertion cost Δ_i that the removed customers i reinsert each position of all routes in the broken solution. Then, find the lowest insertion cost to determine the best insertion position.

3.3.5. Adaptive neutrosophic weight adjustment

As described in Algorithm 1, the scores for each operator are assigned by σ_1, σ_2, σ_3, or σ_4, respectively in the four different situations. Among them, these scores are integers satisfying $\sigma_1 > \sigma_2 > \sigma_3 > \sigma_4 > 0$. However, according to the quality of x_{new}, the ALNS simply increases the same score for different operators in each scoring situation, rather than assigning the score corresponding to the specific improvement degree of x_{new}. To do so, this study introduces NSs, designing the neutrosophic utility function and the score function to assign the appropriate score for each operator. Besides, the adaptive neutrosophic weight adjustment follows the basic principle of reference [30].

The process of ANLNS is divided into many segments. Each segment iterates 100 times. The score of all operators equals 0 before each segment runs. During the segment, let NN_1, NN_2, NN_3, and NN_4 be the number of times the operator scores under the four different scoring levels. Specially, this way applies to each operator in the segment, as shown in Table 1.

Table 1. The number of times for operators with four different situations.

Parameter	Score level	Description
NN_1	σ_1	If x_{new} is better than x_{best}. Then, NN_1 increases 1.
NN_2	σ_2	If x_{new} is worse than x_{best} and better than $x_{current}$. Then, NN_2 increases 1.
NN_3	σ_3	If x_{new} is worse than $x_{current}$, x_{new} is accepted with a certain probability. Then, NN_3 increases 1.
NN_4	σ_4	If x_{new} is worse than $x_{current}$, x_{new} is unacceptable. Then, NN_4 increases 1.

Therefore, to analyze the uncertainty of operator performance, the neutrosophic number $NN = \langle N_T, N_I, N_F \rangle$ is constructed by using NN_1, NN_2, NN_3, and NN_4. For convenience, N_T, N_I, and N_F are defined as

$$N_T = \frac{\lambda NN_1 + NN_2}{\lambda NN_1 + \sum_{i=2}^{4} NN_i}, N_I = \frac{NN_3}{\lambda NN_1 + \sum_{i=2}^{4} NN_i}, N_F = \frac{NN_4}{\lambda NN_1 + \sum_{i=2}^{4} NN_i}, \tag{10}$$

where N_T indicates the truth-membership degree that x_{new} is better than x_{best} or $x_{current}$ in the segment of ANLNS. N_I indicates the indeterminacy-membership degree that x_{new} is worse than $x_{current}$, but x_{new} is accepted with a certain probability in the segment of ANLNS. N_F indicates the falsity-membership degree that x_{new} is unacceptable in the segment of ANLNS. And, λ is a correction factor. Subsequently, in order to express the degree to which ANLNS prefers the operator, inspired by reference [35], this study constructed a novel

neutrosophic utility function by introducing NN into the quadratic utility function. Thus, the neutrosophic utility function is obtained as follows:

$$U = \left(N_T + N_F\right)\left(N_T - N_F + 1\right)/2. \tag{12}$$

It is noteworthy that the parameters NN_1, NN_2, NN_3, and NN_4 have the upper limits of the scores level, which are σ_1, σ_2, σ_3, and σ_4 respectively. Due to the constraints of the conditions, the lower limits of the scores for each parameter are also σ_2, σ_3, σ_4, and 0 respectively. In order to assign an appropriate score, four different score intervals $(\sigma_2, \sigma_1]$, $(\sigma_3, \sigma_2]$, $(\sigma_4, \sigma_3]$, and $(0, \sigma_4]$ are redistributed to correspond to the four different scoring situations respectively. Therefore, the new score function for each operator is obtained as

$$\sigma_c^* = \begin{cases} U\left(\sigma_c - \sigma_{c+1}\right) + \sigma_{c+1}, c \in \{1,2,3\}, \\ U\left(\sigma_c - 0\right) + 0, \qquad c = 4, \end{cases} \tag{13}$$

where σ_c^* indicates the new score of the operator in the cth scoring situations. $\sigma_c - \sigma_{c+1}$ indicates the size of the score interval consisting of the score σ_c and σ_{c+1}. In particular, σ_{c+1} is equal to 0 when $c = 4$. The neutrosophic utility value U serves as a weight, which influences the magnitude of the score change that the operator has within the score interval. After 100 iterations of segment s, the weight for all operators i is updated to apply in segment $s + 1$, and the $\omega_{i,s+1}$ is defined as

$$\omega_{i,s+1} = \omega_{i,s}\left(1 - \phi\right) + \phi\frac{\eta_i}{\tau_i}, \tag{14}$$

where, $\phi \in (0,1)$ is a reaction factor. η_i is the total score obtained by operators i in segment s. τ_i indicates the utilized times for the operator i in the segment s.

3.3.6. Acceptance and stopping criteria

In ANLNS, if $f(x_{new}) > f(x_{current})$, the simulated annealing acceptance criterion is adopted. x_{new} with a certain probability $e^{-(f(x_{new}) - f(x_{current}))/T}$ is accepted. T is the current temperature, which equals the start temperature T_0 before this segment starts. With the continuous iteration in each segment, T is decreased with the cooling rate $\beta \in (0,1)$ such that $T = T \cdot \beta$. It is increasingly difficult for ANLNS to accept the probability of poor solutions. To ensure the continuous convergence of the algorithm, the setting of T_0 is extremely critical. T_0 is $-t \cdot |f(x_{current})|/lnT_p$, where t is the start temperature control parameter. lnT_p expresses that if x_{new} is worse than $x_{current}$ at T_0, x_{new} is accepted with a probability of T_p%.

4. Results and Discussion

For convenience, this section contains four subsections, which are detailed as follows. Subsection 4.1 introduces details about a set of benchmark instances. Subsection 4.2 gives the setting parameter of ANLNS. Subsection 4.3 applies the CVRP benchmark instances to test the effectiveness and practicability of ANLNS. Subsection 4.4 provides a comparative analysis to verify the robustness and validity of ANLNS.

4.1. Benchmark

In this subsection, to verify the effectiveness of the proposed ANLNS, this study selected

representative instances C1-C3, C6-C8, C12, and C14 in reference [36] as case studies. More specifically, these instances are categorized into three scales based on the number of customers served. Among them, instances C1 and C6 are classified as small-scale, each serving 50 customers. Instances C2 and C7 are classified as medium-scale, each serving 75 customers. Finally, instances C3, C8, C10, C12, and C14 are classified as large-scale, each serving 100 customers. Then, instances C1, C2, C3, and C12 are constrained by the loading capacity. In contrast, instances C6, C7, C8, and C14 are subject to additional constraints, including the maximum traveling distance and drop time. Meanwhile, the customer demands vary for each instance. Furthermore, instances C1-C3, C6-C8, C12, and C14 are designed based on the customer distribution, where instances C1-C3, and C6-C8 are randomly distributed, and instances C12 and C14 are clustered distributed. In detail, the information for CVRP benchmark instances please refers to Table 2.

Table 2. The information for CVRP benchmark instances.

Instances	Number of customers	Loading capacity	Maximum traveling distance	Drop time
1	50	160	999999	0
2	75	140	999999	0
3	100	200	999999	0
6	50	160	200	10
7	75	140	160	10
8	100	200	230	10
12	100	200	999999	0
14	100	200	1040	90

4.2. Parameters setting

The experimental results are presented for ANLNS in the CVRP benchmark. The ANLNS is coded in Python with the PyCharm. The experiments are all conducted on a Windows 11 environment, using an Intel Core i7-14650HX CPU clocked at 2.2 GHz with 16 GB RAM.

To optimize the algorithm's performance, this study defines several sets of algorithm parameters anticipated to bring high quality solutions, and these parameters are tested on several instances to keep a balance between solution quality and runtime. This study identifies that the randomness parameters of Shaw removal and worst removal, the maximum number of iterations, and the start temperature, have a great impact on the performance of the algorithm. Therefore, this study tunes these parameters, and the remaining parameters are adopted as reference [30]. Then, the operating parameters of ANLNS are explained in Table 3.

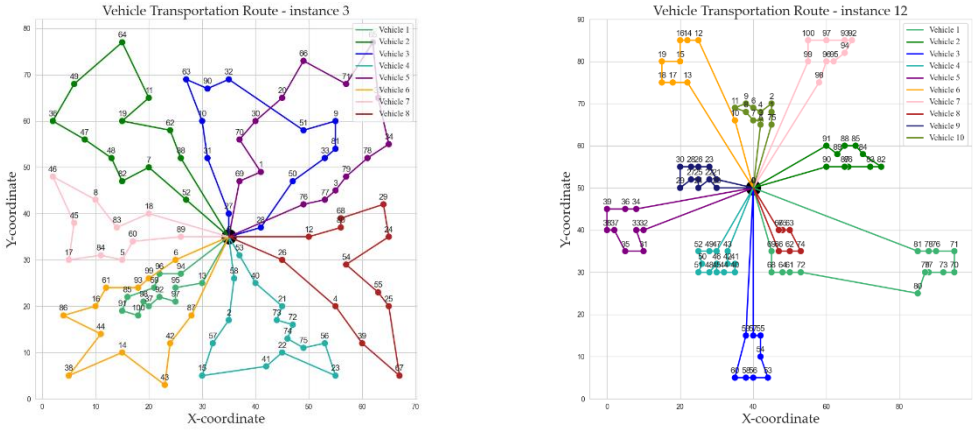
Table 3. The operating of parameters of ANLNS

Description	Symbol	Numerical value
the maximum number of iterations	max_item	1000
the degree of destruction	μ	0.5
the weighted parameter of distance item for $R(i,j)$	α	0.75

in Shaw removal		
the weighted parameter of loading capacity item for $R(i,j)$ in Shaw removal	δ	0.1
the randomness parameters of Shaw removal	P_1	4
the randomness parameters of the worst removal	P_2	4
adaptive adjustment weight parameter	$\sigma_1, \sigma_2, \sigma_3, \sigma_4, \phi$	30, 19, 13, 9， 0.5
correction factor	λ	0.5
start temperature control parameter	t	0.07
cooling rate	β	0.99975
acceptable probability	T_p	0.5

4.3. Case application

Each instance varies in characteristics. Specifically, instances C3 and C12 have identical customer and demand of 100 and 200, respectively, and the distribution of customers is different in both instances. Therefore, this study selects the above instances to represent large-scale and more complex situations, demonstrating the applicability and the solving capability of ANLNS in diverse situations. The vehicle transportation routes of instances 3 and 12 are shown in Figure 2. Meanwhile, this study displays the vehicle transportation routes for the remaining instances in Appendix A.



(a) (b)

Figure 2. (**a**) The vehicle transportation route of instance; (**b**) The vehicle transportation route of instance 12.

It is observed from Figure 2 that the transportation routes using the ANLNS are clear. Meanwhile, Figure 2 highlights that the number of vehicles in instances 3 and 12 is as low as 8 and 10, respectively. Therefore, under varying scales of transportation situations, it is found that the vehicle transportation routes planned by ANLNS are frequently effective. By applying the formula $(C_{actual}/C_{max}) \cdot 100\%$ to calculate the vehicle loading rate of instances 3 and 12, as shown in Figures 3-4. Then, Figures 3-4 indicate that almost all of the loading rate exceeds 0.75. The loading rate of each vehicle in instance 3 is as high as 99.50%, 98.00%, 97.50%, 84.50%, 99.50%, 99.50%, 67.50%, and 83.00%, respectively. The loading capacity rate

of each vehicle in instance 12 is as high as 100.00%, 85.00%, 100.00%, 80.00%, 100%, 100.00%, 95.00%, 75.00%, 85.00%, and 85.00%, respectively. On the other hand, Figures 3-4 reveal that the no-load travel distance is reduced and the efficiency of transportation is improved through ANLNS's excellent goods allocation. Then, the total transportation distance is greatly reduced. In summary, it is observed from Figures 2-4 that ANLNS has significant potential in route planning and goods allocation.

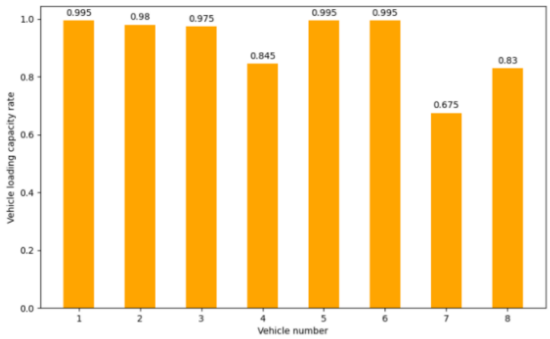


Figure 3. The vehicle loading rate in instance 3.

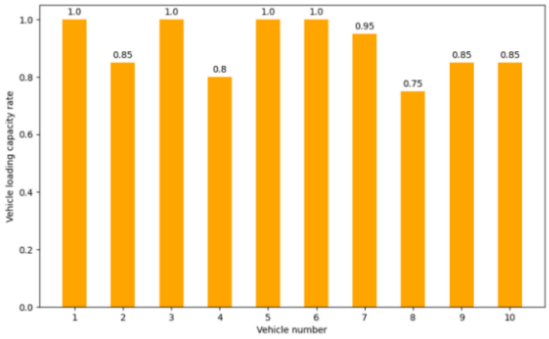
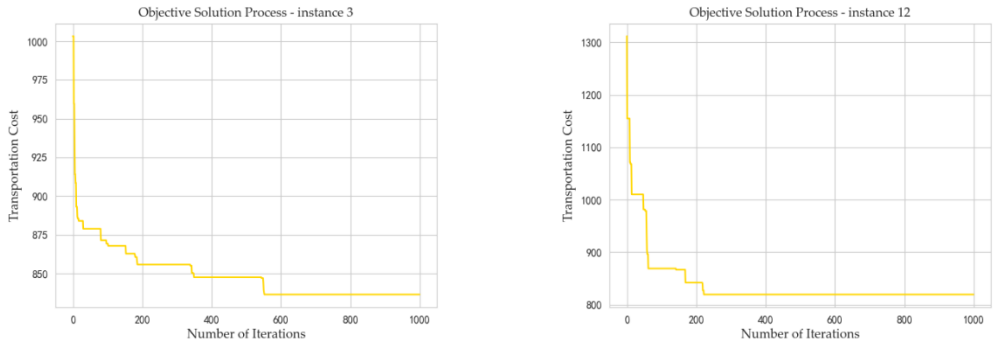


Figure 4. The vehicle loading rate in instance 12.

Figure 5 shows that ANLNS converges quickly after about 450 and 200 iterations, respectively. In the initial stage, ANLNS has a high convergence speed. Therefore, ANLNS effectively converges to the best solution of the ideal area. The current transportation routes are effective and realistic.



(a) (b)

Figure 5. (**a**) The convergence behavior of instance 3; (**b**) The convergence behavior of instance 12.

4.4. Comparative analysis

For convenience, the results of experiments for ANLNS, ALNS, and Akpunar et al. [37] are compared and analyzed to verify the robustness and reliability of the ANLNS. Subsequently, this study uses the degree of destruction, which is 0.3, 0.5 and 0.7, respectively. For all instances, each algorithm runs ten times, and each time contains 1000 iterations. Tables 3-5 below show the comparison results where tables include the benchmark labels in the first column. Within the "ANLNS", "ALNS", and "Akpunar et al.' Hybrid ALNS" columns, the sub-column "BS" represents the best solution from ten algorithm runs. The sub-columns "AS" and "WS" display the average and worst solutions from the same ten runs. The "SD" denotes the standard deviation. Additionally, select the best solutions for each instance from ANLNS, ALNS, and Akpunar et al.' Hybrid ALNS, and highlight them in bold within the BS column.

Table 3. Compared results at a degree of destruction of 0.3.

Instance	ANLNS				ALNS				Akpunar et al.' Hybrid ALNS			
	BS	**AS**	**WS**	**SD**	**BS**	**AS**	**WS**	**SD**	**BS**	**AS**	**WS**	**SD**
C1	**527.94**	533.49	539.90	4.78	528.14	534.82	544.90	4.79	528.14	536.04	546.75	5.80
C2	**863.42**	877.80	886.29	7.51	869.69	882.87	892.87	7.70	864.34	877.45	887.28	6.79
C3	836.43	842.35	854.02	5.02	**830.48**	840.44	849.96	5.56	831.41	838.13	844.65	4.14
C6	526.22	531.81	541.11	4.84	529.43	538.19	541.11	3.16	**524.61**	535.20	542.47	5.74
C7	**857.94**	875.40	884.83	7.86	864.95	882.90	897.10	8.40	866.34	877.86	895.60	7.43
C8	836.20	839.73	843.45	2.16	832.62	840.76	846.19	4.46	**831.98**	839.92	850.89	4.69
C12	819.97	829.08	838.66	6.72	825.38	846.16	874.18	14.62	**819.56**	825.79	844.19	7.75
C14	**819.56**	833.95	854.38	13.93	826.07	843.98	858.30	9.45	819.60	835.39	864.22	14.68

Table 4. Compared results at a degree of destruction of 0.5.

Instance	ANLNS				ALNS				Akpunar et al.' Hybrid ALNS			
	BS	**AS**	**WS**	**SD**	**BS**	**AS**	**WS**	**SD**	**BS**	**AS**	**WS**	**SD**

C1	**529.43**	538.34	543.05	3.57	539.22	542.06	547.67	2.64	535.00	540.48	543.44	2.42
C2	**860.12**	875.46	886.01	7.46	868.98	885.40	898.86	8.24	865.83	876.76	896.02	8.95
C3	**835.72**	849.00	856.70	6.24	844.98	853.66	861.84	4.99	844.26	850.06	857.26	3.68
C6	**528.14**	538.96	545.64	5.25	537.00	543.63	555.33	4.91	**528.14**	538.57	543.31	4.91
C7	**863.95**	877.02	886.75	6.54	870.75	886.62	901.49	8.68	868.51	876.25	886.96	6.83
C8	**836.55**	848.16	856.40	5.48	849.72	854.57	860.73	3.44	841.07	849.57	858.58	6.08
C12	**819.60**	821.36	822.95	1.04	821.92	831.38	847.46	8.23	**819.60**	821.23	823.81	1.37
C14	**819.60**	822.59	829.57	2.79	821.15	831.69	852.10	10.01	**819.60**	821.00	824.21	1.23

Table 5. Compared results at a degree of destruction of 0.7.

Instance	ANLNS				ALNS				Akpunar et al.' Hybrid ALNS			
	BS	**AS**	**WS**	**SD**	**BS**	**AS**	**WS**	**SD**	**BS**	**AS**	**WS**	**SD**
C1	541.31	543.74	548.76	2.74	542.51	547.73	560.77	5.195	**540.47**	545.77	557.29	4.83
C2	**875.95**	907.00	934.81	16.88	884.84	910.26	935.67	16.82	886.17	906.78	929.12	10.95
C3	**837.69**	853.62	865.09	7.82	849.51	857.18	868.53	5.39	840.93	851.23	861.35	5.69
C6	1072.97	1087.61	1102.90	8.14	1067.83	1103.59	1114.37	12.83	**533.63**	542.38	547.90	3.84
C7	1377.23	1404.67	1421.62	12.48	1400.63	1417.16	1436.12	11.70	**875.03**	911.76	941.99	18.00
C8	**536.13**	544.35	552.51	4.77	539.46	544.51	548.28	2.49	842.63	851.91	860.34	5.83
C12	869.94	907.80	929.44	16.07	882.15	917.41	938.27	17.72	**820.48**	821.66	823.71	1.03
C14	846.84	853.05	863.83	4.56	839.02	856.23	870.43	9.09	**819.97**	821.55	823.25	0.87

Tables 3-5 conclude that the best solution obtained by ANLNS is better than the expected result. In the experimental results from 8 instances under three different degrees of destruction, ANLNS obtains 15 best solutions. Specifically, when the degree of destruction is 0.3, 0.5, and 0.7, ANLNS considerably obtains 4(50.0%), 8(100.0%), and 3(37.5%) best solutions respectively. Under the same conditions, the total number of best solutions for ALNS, and Akpunar et al.' Hybrid ALNS is 1 and 11, respectively. Therefore, Tables 3-5 indicate that ANLNS has a robust performance advantage. Meanwhile, it shows that ANLNS possesses a good global search capability and diversifies the search.

In addition, tables 3-5 show the average solution and standard deviation obtained by each algorithm in all instances. it is observed that the average solution and standard deviation obtained by ANLNS are better than the other two algorithms. Even if some instances show that the standard deviation of ANLNS is volatile. However, as a whole, it shows that the ANLNS has a stable and strong ability to handle complex and diverse instances of different scales.

Table 6 below shows the execution time of the ANLNS, ALNS, and Akpunar et al.' Hybrid ALNS algorithm in seconds. Among them, the "DT" indicates the execution time of the algorithm under the degree of destruction is 0.3. The "DF" indicates the execution time of the algorithm under the degree of destruction is 0.5. The "DS" indicates the execution time of the algorithm under the degree of destruction is 0.7.

Table 6. Compared execution time at different degrees of destruction.

Instance	ANLNS			ALNS			Akpunar et al.' HybridALNS		
	DT	DF	DS	DT	DF	DS	DT	DF	DS
C1	9.93	16.05	19.72	9.96	15.98	19.73	10.96	17.07	21.23
C2	23.32	36.82	38.46	23.82	38.77	46.53	25.67	41.80	39.84
C3	62.32	111.5	139.93	62.84	112.26	118.14	68.22	117.48	146.32
C6	9.83	15.74	422.45	10.05	15.67	42.52	11.18	29.50	34.86
C7	23.82	38.23	935.93	23.8	39.88	98.47	26.47	57.93	39.53
C8	63.36	110.62	19.72	63.25	112.02	260.32	124.41	116.96	148.18
C12	56.60	97.62	37.32	57.87	100.43	238.24	106.86	102.96	134.65
C14	56.45	97.79	141.52	57.28	99.12	224.61	61.85	211.02	290.83

Table 6 expresses that ANLNS is more efficient than ALNS and Akpunar et al.' Hybrid ALNS since the run time of ANLNS is significantly smaller than ALNS. Meanwhile, the running time by ANLNS for instance 12 with degree of destruction 0.3 is 56.60 seconds, while the corresponding ALNS takes 57.87 seconds and Akpunar et al.' Hybrid ALNS takes 106.86 seconds. Even if the instance scale and degree of destruction keep increasing, in some cases, ANLNS may be slightly worse than ALNS and Akpunar et al.' Hybrid ALNS. But on the whole, the result shows that the proposed ANLNS is competitive with ALNS and Akpunar et al.' Hybrid ALNS in CVRP.

4.5. Discussion

The traditional ALNS and improved ALNS provide a way of dealing with some practical work. However, these studies do not consider ALNS existing in the fuzziness and uncertainty of assessing and scoring operators. Therefore, this study proposes a novel ANLNS algorithm to solve the above problem. The following details are provided.

Firstly, the NS, which keeps track of the number of times each operator scores in different situations, is integrated into the framework of ALNS. This has been of great help to the ANLNS in evaluating each operator's performance and actual contribution. Secondly, the neutrosophic utility function and score function are proposed in this study, which is helpful to assign scores reasonably within a reasonable interval. Finally, in this study, ANLNS tested three representative instances of varying scales selected from a set of CVRP benchmark instances.

Compared with previous ALNS-related studies, this study found that the ANLNS algorithm not only showed a more detailed and accurate evaluation effect on operator performance but also effectively measured the algorithm's preference for different operators and reduced the existence of subjective factors. Subsequently, in the results section, the experimental results and comparative analysis emphasize the robustness and reliability of ANLNS for solving optimization problems.

However, ANLNS exhibits limitations in exhaustively exploring the solution space. Therefore, future work includes continuing to evaluate whether these methods are retained in the ANLNS algorithm and considering the introduction of artificial intelligence to effectively

improve the algorithm.

5. Conclusion

As described before, in the adaptive weight adjustment stage, the score of operators exists fuzziness and uncertainty. To address this problem, this study proposes an ANLNS algorithm. The main contributions are as follows.

Firstly, this study records the number of times that the operator scores under four different situations, and constructs NSs to quantify these number of times. The application of NSs facilitates the ANLNS to analyze and evaluate operator performance more comprehensively.

Secondly, this study proposes a novel neutrosophic utility function and score function. Then, the proposed function evaluates the ANLNS's preference and assigns the appropriate score for different operators.

Finally, this study verifies a solving ability of ANLNS through CVRP. Moreover, the provided best solution offers a novel way for solving CVRP and other situation optimization problems.

In summary, the effect of this study is that ANLNS can adjust the operator weight to a relatively reasonable level, and make the ANLNS workflow more reasonable and efficient. However, the limitation of this study is that the ANLNS has not thoroughly explored the solution space to find potential high-quality solutions. Therefore, in the future, this research will consider the solution space of exploration and exploitation for potential best solutions. Therefore, combined with the decision tree and attention mechanisms, the research would provide a reasonable, effective, and practical algorithm.

Funding: Fangwei Zhang's work is supported by the Natural Science Foundation of Shandong Province (ZR2021MG003).

Conflicts of Interest: The authors declare no conflict of interest.

Appendix A

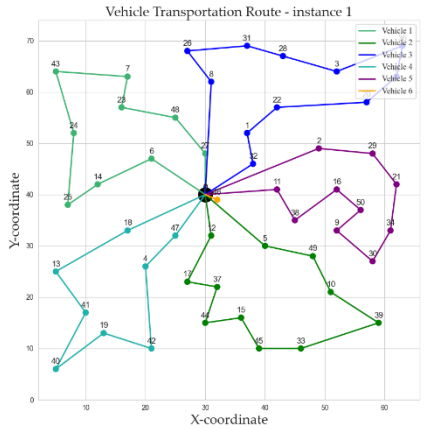


Figure A1

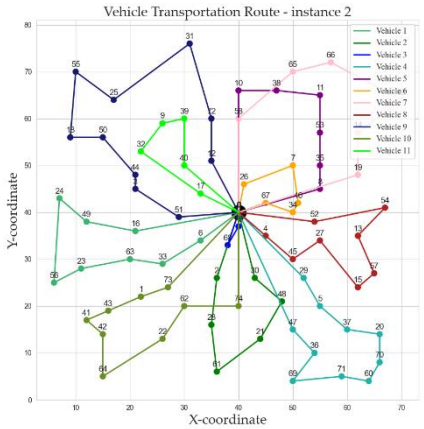


Figure A2

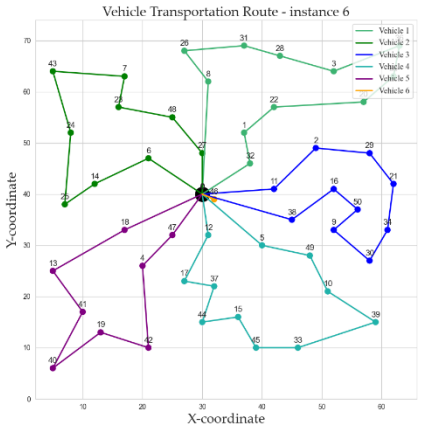


Figure A3

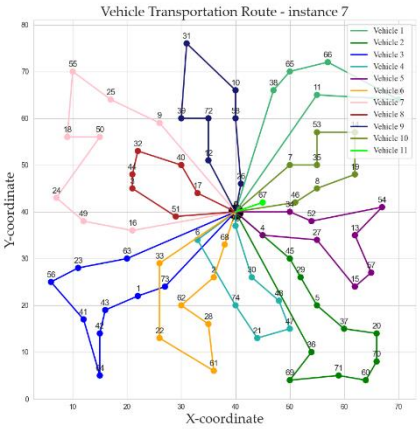


Figure A4

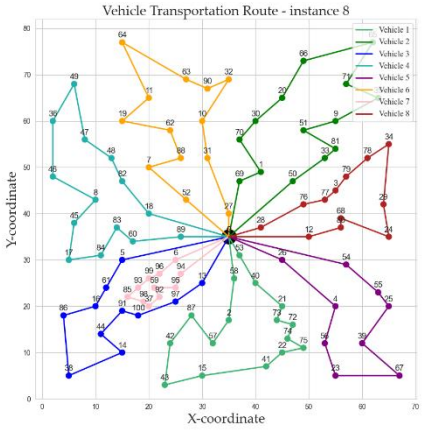


Figure A5

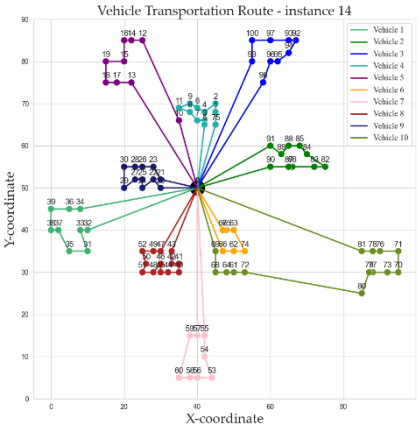


Figure A6

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Received: March 10, 2024. Accepted: July 26, 2024

NSS
Neutrosophic Sets and Systems, Vol. 70, 2024
University of New Mexico

The indefinite integrals of refined neutrosophic trigonometric functions

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Abstract: in this paper, we presented the indefinite integrals of refined neutrosophic trigonometric functions, where we discussed the integrals by distinguishing several cases of these types of refined neutrosophic integrals. Also, we introduced the refined neutrosophic trigonometric identities, which makes it easier for us to find these types of integrals. In addition, to supporting this with appropriate examples.

Keywords: refined; neutrosophic; indefinite integral; indeterminacy; neutrosophic trigonometric.

1. Introduction and Preliminaries

As an alternative to the existing logics, Smarandache proposed the neutrosophic Logic to characterize a mathematical model of uncertainty, vagueness, ambiguity, imprecision, undefined, unknown, incompleteness, inconsistency, redundancy, and contradiction. Smarandache provided the following form of refined neutrosophic numbers: $(a, b_1I_1, b_2I_2, \dots, b_nI_n)$, where:

$a, b_1, b_2, \dots, b_n \in R \text{ or } C$ [1]

Agboola introduced the concept of refined neutrosophic algebraic structures [2]. Also, the refined neutrosophic rings I was studied in paper [3], where it assumed that I splits into two indeterminacies I_1 [contradiction (true (T) and false (F))] and I_2 [ignorance (true (T) or false (F))]. The refined indefinite neutrosophic integral was studied and included the following set of rules and theorem [14]:

Theorem:
If $\int f(x, I_1, I_2)dx = \varphi(x, I_1, I_2)$, then:

$$\int f\big((a + bI_1 + cI_2)x + r + sI_1 + tI_2\big)\, dx$$

$$= \left(\frac{1}{a} + \left[\frac{-ab}{a(a+c)(a+b+c)}\right] I_1 - \left[\frac{c}{a(a+c)}\right] I_2\right) \varphi\big((a+bI_1+cI_2)x + r + sI_1 + tI_2\big) + C$$

where a, b, c, r, s, t are real numbers and $a \neq 0$, $a \neq -c$ and $a \neq -b - c$
C is an indeterminate real constant (i.e. constant of the form $a_0 + a_1 I_1 + a_2 I_2$,and a_0, a_1, a_2 are real numbers, while $I_1, I_2 =$ indeterminacy).

In addition, there are many papers presenting studies on refined neutrosophic numbers [4-5-6-7-8]. Smarandache discussed neutrosophic indefinite integral (Refined Indeterminacy) [11]
Let $g: \mathbb{R} \to \mathbb{R} \cup \{I_1\} \cup \{I_2\} \cup \{I_3\}$, where I_1, I_2, and I_3 are types of sub indeterminacies,

$$g(x) = 7x - 2I_1 + x^2 I_2 + 4x^3 I_3$$

then:

$$F(x) = \int [7x - 2I_1 + x^2 I_2 + 4x^3 I_3] dx$$

$$= \frac{7x^2}{2} - 2xI_1 + \frac{x^3}{3} I_2 + x^4 I_3 + a_0 + a_1 I_1 + a_2 I_2 + a_3 I_3$$

where a_0, a_1, a_2 and a_3 are real constants.

Alhasan also presented several papers on calculus, in which he discussed neutrosophic definite and indefinite integrals. He also presented the most important applications of definite integrals in neutrosophic logic [9-10]. Several studies were also presented in the field of neutrosophic logic in statistics and others [12-13].

2. Main Discussion

We consider C is an indeterminate real constant (i.e. constant of the form $a + bI_1 + cI_2$, where a, b, c are real numbers, while $I_1, I_2 =$ indeterminacy).

Integrating products of neutrosophic trigonometric function:

I. $\int \sin^m\big((a+bI_1+cI_2)x + r + sI_1 + tI_2\big) \cos^n\big((a+bI_1+cI_2)x + r + sI_1 + tI_2\big)\ dx$, where m and n are positive integers.

The two cases below can be distinguished in order to find this integral:

- Case n *is* odd:

- Split of $cos\big((a+bI_1+cI_2)x + r + sI_1 + tI_2\big)$
- Apply $cos^2\big((a+bI_1+cI_2)x + r + sI_1 + tI_2\big) = 1 - sin^2\big((a+bI_1+cI_2)x + r + sI_1 + tI_2\big)$
- We substitute $u = sin\big((a+bI_1+cI_2)x + r + sI_1 + tI_2\big)$

- Case m *is* odd:

- Split of $sin\big((a+bI_1+cI_2)x + r + sI_1 + tI_2\big)$
- Apply $sin^2\big((a+bI_1+cI_2)x + r + sI_1 + tI_2\big) = 1 - cos^2\big((a+bI_1+cI_2)x + r + sI_1 + tI_2\big)$
- We substitute $u = cos\big((a+bI_1+cI_2)x + r + sI_1 + tI_2\big)$

Example 1

Find:

$$\int sin^2\big((5 + I_1 - 3I_2)x + 7I_1\big)\, cos^3\big((5 + I_1 - 3I_2)x + 7I_1\big)\ dx$$

Solution:

$$\int sin^2\big((5+I_1-3I_2)x+7I_1\big)\,cos^3\big((5+I_1-3I_2)x+7I_1\big)\;\;dx$$

$$=\int sin^2\big((5+I_1-3I_2)x+7I_1\big)\,cos^2\big((5+I_1-3I_2)x+7I_1\big)\;\;cos\big((5+I_1-3I_2)x+7I_1\big)\;\;dx$$

$$=\int sin^2\big((5+I_1-3I_2)x+7I_1\big)\big(1-sin^2\big((5+I_1-3I_2)x+7I_1\big)\big)\;\;cos\big((5+I_1-3I_2)x+7I_1\big)\;\;dx$$

$$=\int\big(sin^2\big((5+I_1-3I_2)x+7I_1\big)-sin^4\big((5+I_1-3I_2)x+7I_1\big)\big)\;\;cos\big((5+I_1-3I_2)x+7I_1\big)\;\;dx$$

$$u=sin\big((5+I_1-3I_2)x+7I_1\big)\quad\Longrightarrow\quad\frac{1}{5+I_1-3I_2}\;\;du=cos\big((5+I_1-3I_2)x+7I_1\big)\;\;dx$$

$$\Longrightarrow\;\int sin^2\big((5+I_1-3I_2)x+7I_1\big)\,cos^3\big((5+I_1-3I_2)x+7I_1\big)\;\;dx=\frac{1}{5+I_1-3I_2}\int(u^2-u^4)\,du$$

$$=\left(\frac{1}{5}-\frac{1}{18}I_1+\frac{3}{10}I_2\right)\left(\frac{u^3}{3}-\frac{u^5}{5}\right)+C$$

$$=\left(\frac{1}{5}-\frac{1}{18}I_1+\frac{3}{10}I_2\right)\left(\frac{sin^3\big((5+I_1-3I_2)x+7I_1\big)}{3}-\frac{sin^5\big((5+I_1-3I_2)x+7I_1\big)}{5}\right)+C$$

II. $\int tan^m\big((a+bI_1+cI_2)x+r+sI_1+tI_2\big)\,sec^n\big((a+bI_1+cI_2)x+r+sI_1+tI_2\big)\;\;dx$, where m and n are positive integers.

The two cases below can be distinguished in order to find this integral:

- Case n *is* even:

- Split of $sec^2\big((a+bI_1+cI_2)x+r+sI_1+tI_2\big)$
- Apply $sec^2\big((a+bI_1+cI_2)x+r+sI_1+tI_2\big)=1+tan^2\big((a+bI_1+cI_2)x+r+sI_1+tI_2\big)$
- We substitute $u=tan\big((a+bI_1+cI_2)x+r+sI_1+tI_2\big)$

- Case m *is* odd:

- Split of $sec\big((a+bI_1+cI_2)x+r+sI_1+tI_2\big)tan\big((a+bI_1+cI_2)x+r+sI_1+tI_2\big)$

- Apply $tan^2\big((a+bI_1+cI_2)x+r+sI_1+tI_2\big)=sec^2\big((a+bI_1+cI_2)x+r+sI_1+tI_2\big)-1$
- We substitute $u=sec\big((a+bI_1+cI_2)x+r+sI_1+tI_2\big)$

- Case m even and n odd:

- Apply $tan^2\big((a+bI_1+cI_2)x+r+sI_1+tI_2\big)=sec^2\big((a+bI_1+cI_2)x+r+sI_1+tI_2\big)-1$
- We substitute $u=sec\big((a+bI_1+cI_2)x+r+sI_1+tI_2\big)$ or $u=tan\big((a+bI_1+cI_2)x+r+sI_1+tI_2\big)$, depending on the case.

Example 2

Find:

$$\int tan^2\big((3+I_1+2I_2)x-4+I_2\big)\ \ sec^4\big((3+I_1+2I_2)x-4+I_2\big)\ \ dx$$

Solution:

$$n=4\ \text{(Even)}$$

$$\int tan^2\big((3+I_1+2I_2)x-4+I_2\big)\ \ sec^4\big((3+I_1+2I_2)x-4+I_2\big)\ \ dx$$

$$=\int tan^2\big((3+I_1+2I_2)x-4+I_2\big)\ \ sec^2\big((3+I_1+2I_2)x-4+I_2\big)\quad sec^2\big((3+I_1+2I_2)x-4+I_2\big)\ \ dx$$

$$=\int\Big(tan^2\big((3+I_1+2I_2)x-4+I_2\big)+tan^4\big((3+I_1+2I_2)x-4+I_2\big)\Big)\quad sec^2\big((3+I_1+2I_2)x-4+I_2\big)\ \ dx$$

$$u=tan\big((3+I_1+2I_2)x-4+I_2\big)\quad\Longrightarrow\quad\frac{1}{3+I_1+2I_2}\ du=\ sec^2\big((3+I_1+2I_2)x-4+I_2\big)\ \ dx$$

$$\Longrightarrow\int\Big(tan^2\big((3+I_1+2I_2)x-4+I_2\big)+tan^4\big((3+I_1+2I_2)x-4+I_2\big)\Big)\quad sec^2\big((3+I_1+2I_2)x-4+I_2\big)\ \ dx$$

$$=\ \frac{1}{3+I_1+2I_2}\int(u^2+u^4)\,du$$

$$=\left(\frac{1}{3}-\frac{1}{30}I_1-\frac{2}{15}I_2\right)\left(\frac{u^3}{3}+\frac{u^5}{5}\right)+C$$

$$=\left(\frac{1}{3}-\frac{1}{30}I_1-\frac{2}{15}I_2\right)\left(\frac{tan^3\big((3+I_1+2I_2)x-4+I_2\big)}{3}+\frac{tan^5\big((3+I_1+2I_2)x-4+I_2\big)}{5}\right)+C$$

Example 3

Find:

$$\int tan^3\big((7-4I_1+I_2)x+I_1+I_2\big)\ \ sec^3\big((7-4I_1+I_2)x+I_1+I_2\big)\ \ dx$$

Solution:

$$m=4\ \text{(Odd)}$$

$$\int tan^3\big((7-4I_1+I_2)x+I_1+I_2\big)\ \ sec^3\big((7-4I_1+I_2)x+I_1+I_2\big)\ \ dx$$

$$=\int tan^3\big((7-4I_1+I_2)x+I_1+I_2\big)\ \ sec^2\big((7-4I_1+I_2)x+I_1+I_2\big)\ \ sec\big((7-4I_1+I_2)x+I_1+I_2\big)\ \ tan\big((7-4I_1+I_2)x+I_1+I_2\big)\,dx$$

$$= \int \Big(sec^4\big((7-4I_1+I_2)x+I_1+I_2\big) - sec^2\big((7-4I_1+I_2)x+I_1+I_2\big)\Big)\ sec\big((7-4I_1+I_2)x+I_1+I_2\big)\ tan\big((7-4I_1+I_2)x+I_1+I_2\big)\,dx$$

$$u = sec\big((7-4I_1+I_2)x+I_1+I_2\big)$$

$$\Rightarrow \quad \frac{1}{7-4I_1+I_2}\ du = \ sec\big((7-4I_1+I_2)x+I_1+I_2\big)\ tan\big((7-4I_1+I_2)x+I_1+I_2\big)\ \ dx$$

$$\Rightarrow \int \Big(sec^4\big((7-4I_1+I_2)x+I_1+I_2\big) - sec^2\big((7-4I_1+I_2)x+I_1+I_2\big)\Big)\ sec\big((7-4I_1+I_2)x+I_1+I_2\big)\ tan\big((7-4I_1+I_2)x+I_1+I_2\big)\,dx$$

$$= \frac{1}{7-4I_1+I_2}\int (u^4-u^2)\,du$$

$$= \left(\frac{1}{7}+\frac{1}{8}I_1-\frac{1}{56}I_2\right)\left(\frac{u^5}{5}-\frac{u^3}{3}\right)+C$$

$$= \left(\frac{1}{7}+\frac{1}{8}I_1-\frac{1}{56}I_2\right)\left(\frac{sec^5\big((7-4I_1+I_2)x+I_1+I_2\big)}{5}-\frac{sec^3\big((7-4I_1+I_2)x+I_1+I_2\big)}{3}\right)+C$$

III. $\int cot^m\big((a+bI_1+cI_2)x+r+sI_1+tI_2\big)\,csc^n\big((a+bI_1+cI_2)x+r+sI_1+tI_2\big)\ \ dx$, where m and n are positive integers.

To find this integral, we can distinguish the following cases:

- Case n *is* even:

- Split of $csc^2\big((a+bI_1+cI_2)x+r+sI_1+tI_2\big)$
- Apply $csc^2\big((a+bI_1+cI_2)x+r+sI_1+tI_2\big) = 1+cot^2\big((a+bI_1+cI_2)x+r+sI_1+tI_2\big)$
- We substitute $u = cot\big((a+bI_1+cI_2)x+r+sI_1+tI_2\big)$

- Case m *is* odd:

- Split of $csc\big((a+bI_1+cI_2)x+r+sI_1+tI_2\big)\ cot\big((a+bI_1+cI_2)x+r+sI_1+tI_2\big)$
- Apply $cot^2\big((a+bI_1+cI_2)x+r+sI_1+tI_2\big) = csc^2\big((a+bI_1+cI_2)x+r+sI_1+tI_2\big)-1$
- We substitute $u = csc\big((a+bI_1+cI_2)x+r+sI_1+tI_2\big)$

- Case m even and n odd:

  - Apply $cot^2\big((a+bI_1+cI_2)x+r+sI_1+tI_2\big) = csc^2\big((a+bI_1+cI_2)x+r+sI_1+tI_2\big)-1$

- We substitution $u = csc\big((a + bI_1 + cI_2)x + r + sI_1 + tI_2\big)$ or $u = cot\big((a + bI_1 + cI_2)x + r + sI_1 + tI_2\big)$, depending on the case.

Example 4

Find:

$$\int \sqrt{cot\big((2 + 4I_1 + 6I_2)x - 7 + 5I_1 + 9I_2\big)}\ \ csc^4\big((2 + 4I_1 + 6I_2)x - 7 + 5I_1 + 9I_2\big)\ \ dx$$

Solution:

$n = 4$ (Even)

$$\int \sqrt{cot\big((2 + 4I_1 + 6I_2)x - 7 + 5I_1 + 9I_2\big)}\ \ csc^4\big((2 + 4I_1 + 6I_2)x - 7 + 5I_1 + 9I_2\big)\ \ dx$$

$$= \int cot^{\frac{1}{2}}\big((2 + 4I_1 + 6I_2)x - 7 + 5I_1 + 9I_2\big)\ \ csc^2\big((2 + 4I_1 + 6I_2)x - 7 + 5I_1 + 9I_2\big) csc^2\big((2 + 4I_1 + 6I_2)x - 7 + 5I_1 + 9I_2\big)\ \ dx$$

$$= \int \big(cot^{1/2}\big((2 + 4I_1 + 6I_2)x - 7 + 5I_1 + 9I_2\big) + cot^{3/2}\big((2 + 4I_1 + 6I_2)x - 7 + 5I_1 + 9I_2\big)\big) csc^2\big((2 + 4I_1 + 6I_2)x - 7 + 5I_1 + 9I_2\big)\ \ dx$$

$$u = cot\big((2 + 4I_1 + 6I_2)x - 7 + 5I_1 + 9I_2\big)$$

$$\Longrightarrow \quad \frac{-1}{2 + 4I_1 + 6I_2}\ du = \ csc^2\big((2 + 4I_1 + 6I_2)x - 7 + 5I_1 + 9I_2\big)\ \ dx$$

$$\Longrightarrow \int \Big(cot^{\frac{1}{2}}\big((2 + 4I_1 + 6I_2)x - 7 + 5I_1 + 9I_2\big) + cot^{\frac{3}{2}}\big((2 + 4I_1 + 6I_2)x - 7 + 5I_1 + 9I_2\big)\Big) csc^2\big((2 + 4I_1 + 6I_2)x - 7 + 5I_1 + 9I_2\big)\ \ dx$$

$$= \frac{-1}{2 + 4I_1 + 6I_2} \int \big(u^{1/2} + u^{3/2}\big)\, du$$

$$= -\Big(\frac{1}{2} - \frac{1}{24}I_1 - \frac{3}{8}I_2\Big)\left(\frac{2\ u^{\frac{3}{2}}}{3} + \frac{2\ u^{\frac{5}{2}}}{5}\right) + C$$

$$= \Big(-\frac{1}{2} + \frac{1}{24}I_1 + \frac{3}{8}I_2\Big)\left(\frac{2\ \ cot^{3/2}\big((2 + 4I_1 + 6I_2)x - 7 + 5I_1 + 9I_2\big)}{3} + \frac{2\ \ cot^{5/2}\big((2 + 4I_1 + 6I_2)x - 7 + 5I_1 + 9I_2\big)}{5}\right) + C$$

The refined neutrosophic trigonometric identities:

1) $sin(a + bI_1 + cI_2)x\, cos(r + sI_1 + tI_2)\, x = \frac{1}{2}\ [sin(a + r + (b + s)I_1 + (c + t)I_2)x + sin(a - r + (b - s)I_1 + (c - t)I_2)\, x]$

2) $cos(a + bI_1 + cI_2)x\, sin(r + sI_1 + tI_2)\, x = \frac{1}{2}\ [sin(a + r + (b + s)I_1 + (c + t)I_2)x - sin(a - r + (b - s)I_1 + (c - t)I_2)\, x]$

3) $cos(a+bI_1+cI_2)x\,cos(r+sI_1+tI_2)\,x=\frac{1}{2}\ [cos(a+r+(b+s)I_1+(c+t)I_2)x+cos(a+r+(b+s)I_1+(c+t)I_2)x]$

4) $sin(a+bI_1+cI_2)x\,sin(r+sI_1+tI_2)\,x=\frac{-1}{2}\ [cos(a+r+(b+s)I_1+(c+t)I_2)x-cos(a+r+(b+s)I_1+(c+t)I_2)x]$

Example 5
Find:

1) $\int sin(1-5I_1+6I_2)x\,cos(2+4I_1+I_2)\,x\ dx$

$$=\int\frac{1}{2}\ [sin(3-I_1+7I_2)x+sin(-1-9I_1+5I_2)x]\ dx$$

$$=\frac{1}{2}\left[-\left(\frac{1}{3-I_1+7I_2}\right)cos(3-I_1+7I_2)x-\frac{1}{-1-9I_1+5I_2}cos(-1-9I_1+5I_2)x\right]+C$$

$$=-\left(\frac{1}{6}+\frac{1}{180}I_1-\frac{7}{60}I_2\right)cos(3-I_1+7I_2)x-\left(\frac{1}{2}-\frac{9}{40}I_1+\frac{5}{8}I_2\right)cos(-1-9I_1+5I_2)x+C$$

2) $\int cos(2-I_1+I_2)x\,cos(3+I_1+3I_2)\,x\ dx=\int\frac{1}{2}\ [cos(5+4I_2)x+cos(-1-2I_1-2I_2)\,x]\ dx$

$$=\frac{1}{2}\ \left[\frac{1}{5+4I_2}sin(5+4I_2)x+\frac{1}{-1-2I_1-2I_2}sin(-1-2I_1-2I_2)\,x\right]+C$$

$$=\left(\frac{1}{10}-\frac{2}{45}I_2\right)sin(5+4I_2)x+\left(-\frac{1}{2}+\frac{1}{15}I_1+\frac{1}{3}I_2\right)sin(-1-2I_1-2I_2)x+C$$

3) $\int sin(5+I_2)x\,sin(2+I_1)\,x\ dx=\int\frac{-1}{2}\ [cos(7+I_1+I_2)x-cos(3-I_1+I_2)x]\ dx$

$$=-\frac{1}{2}\left[\left(\frac{1}{7+I_1+I_2}\right)sin(7+I_1+I_2)x-\left(\frac{1}{3-I_1+I_2}\mathrm{I}\right)sin(3-I_1+I_2)x\right]+C$$

$$=\left(-\frac{1}{14}+\frac{1}{144}I_1+\frac{1}{112}I_2\right)sin(7+I_1+I_2)x+\left(\frac{1}{6}+\frac{1}{24}I_1-\frac{1}{24}I_2\right)sin(3-I_1+I_2)x+C$$

3. Conclusions

This paper is consider one of the important papers that presented integrals of refined neutrosophic functions, using several cases of integrals of refined neutrosophic trigonometric functions. Also, in order to make it easier to compute the integrals of refined neutrosophic functions, we developed the refined neutrosophic trigonometric identities, and precise results were obtained.. Furthermore, the paper is thought to be significant for further research.

Acknowledgments "This study is supported via funding from Prince sattam bin Abdulaziz University project number (PSAU/2024/R/1445)".

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Received: March 10, 2024. Accepted: July 27, 2024

Neutrosophic Sets and Systems, Vol.70, 2024
University of New Mexico

Utilizing New Temporal Complex Neutrosophic Aczel-Alsina Aggregation Operators for Multi-Criteria Decision Making: An Application to Tourist Destination Selection

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Abstract: The Temporal Complex Neutrosophic Set (TCNS) is a powerful extension of the Neutrosophic theory, and TCNS has added advantage to represents uncertain, indeterminate, inconsistent and time related factors. In the literature TCNS has been used for multi-criteria decision-making problems, especially in the time-aware systems. In order to tackle effectively decision-making problems in the TCNS environment effectively, it is crucial to make suite aggregation operators that enable decision makers to process preferences in a nuanced manner. In this paper, we introduce novel TCNS aggregation operators based on Aczel and Alsina t-norm and t-conorm using Aczel-Alsina operations, and we propose a multi-criteria decision-making model that utilizes those operators for TCNS-based multi criteria decision making problems (MCDMs). Theorytically, the soundness and completeness of new aggregation operators are proved, and to demonstrate the practicality and effectiveness of our model a real world case study for tourist destination selection in Vietnam is completed along with analysis. The result proved the application potential of the proposal to solve the time-related MCDMs in practical, such as creating a MCDM model for tourist destination selection.

Keywords: Aczel-Alsina operators; Complex Neutrosophic Set; Temporal Complex Neutrosophic Set; MCDM;

1. Introduction

Multi-criteria decision-making (MCDM), a sophisticated and skillful method to assisting human decision-making (DM) in the face of the challenges posed by complex real-world decisions, has applications in a wide range of industries including tourist destination selection, sustainable cement industry, and supply chain management. Recently, the explosion and complexity of information worldwide used in MCDMs have made it increasingly difficult to make the right decision. Among MCDMs challenges, tackling the uncertainty, ambiguity and time-aware context are attracted more attention from researchers and practitioners. The traditional MCDM strategies become outdated and

weak to solve uncertainty and ambiguity of collected information. Therefore, new approaches for MCDMs are required, and appling extension of fuzzy set [1] is among the top.

Fuzzy set theory was introduced by Zadeh [2] in 1965, along with its broader applications, effectively addresses uncertainty and vagueness in data. However, these frameworks lack the capacity to express the partial ambiguity inherent in indeterminate and inconsistent data, particularly its temporal dynamics. Subsequently, Ali and Smarandache's [3] introduction of complex neutrosophic sets (CNSs) provides a new perspective on decision-making within the unit circle in complex space. This strategy is suitable for handling uncertainty, indeterminacy, inconsistency, and temporal cycles. In the field of neutrosophy, there is a widely recognized consensus that t-norm and t-conorm, such as algebraic, Einstein, Hamacher, Acz-Als's t-norm and t-conorm, play crucial roles as integral components of complex neutrosophic sets or neutrosophic systems in decision support.

In the field of complex neutrosophy set, numerous scholars have developed operations in this field that draw on various t-norms and t-conorms to aid decision-making in real-world circumstances. These operations include a wide range of topics, such as operators [4-7, 22-25, 28-29, 30], distance and similarity measures [8-10, 26, 30], and information entropy [5, 11-13]. Aggregation operators are often used in decision-making issues as crutical tools for summarizing information in the complex neutrosophic environment, assisting in decision support. In the complex neutrosophic set environment, several research has acknowledged their importance and presented mean operators. The detail is presented in next section.

Futhermore, the idea of multi-valued complex neutrosophic linguistic sets (MVCNLSs) was put forth by Mahmood et al. [16], leading to the creation of MVCNLSs that represent information of truth, abstinence, falsity, and uncertain. The presentation utilizes complex numbers whose real and imaginary components are constrained to a unit interval. Moreover, this paper provides a flexible framework for presenting Bonferroni mean operators and elaborates on important Dombi laws. Ali et al. [17] explored the notion of complex neutrosophic uncertain linguistic set (CNUL) and formulated valuable hypotheses using Heromian mean (HM) operators. CNUL data represents complex numbers consisting of uncertainty linguistic sets, truth grades, abstinence grades, and falsely grades. Furthermore, they endeavor to present well-known theories that are concerned with this idea. The first group of HM-CNUL aggregation operators based on HM presents such as weighted arithmetic, geometric. All these operators share some common objective: bringing together multiple alternatives into an element that shall be presented as one single thing.

Through a survey of various studies, it becomes evident that aggregation operators play a significant role in MCDM problems within the complex neutrosophic environment. Furthermore, the temporal complex neutrosophic environment proves to be an effective framework for representing the temporal aspects of decision-making problems. However, it is noticeable that there have been few studies that express interest in or have only partially considered the use of average operators for temporal neutrosophic sets employing Aczel and Alsina t-norms or t-conorms.

Take all into account, this paper introduces new aggregation operators for TCNS utilizing Aczel and Alsina t-norm and t-conorm. The following are the specific contributions of this paper:

1. The development of new TCNS aggregation operators based on Aczel and Alsina t-norm and t-conorm.

2. The proposal of a MCDM model grounded in the newly introduced TCNS aggregation operators.

3. The demonstration of the feasibility and soundness of addressing DM problems through the application of the proposed model. This is illustrated with a real-world case study involving the selection of a tourist destination in Vietnam, accompanied by comparative analysis.

The remain of work is presented as follows. The Section 2 presents related works, and required fundamentals are covered in Section 3, that includes the definition of the TCNS and its operators. The new Aczel-Alsina t-norm and t-cornom operations for TCNS is introduced in Section 4. Section 5 provides a comprehensive MCDM model for TCNS environment based on new proposed operators, and that is pertaining to a real-world application presented in Section 6. The application of designed

MCDM for tourist destination selection illustrates the pratical potential of new aggregation operations for TCNS. In Section 7 the summary and discusion are presented.

2. Related works

Zadeh [1] introduced fuzzy set theory in 1965, it successfully handles ambiguity and uncertainty in data, and it has larger uses theoritically and practically as well. These frameworks, however, are unable to capture the partial ambiguity that exists in inconsistent and ambiguous data, especially with regard to its temporal dynamics. Subsequently, the introduction of complex neutrosophic sets (CNSs) by Ali et al. [3] offers a fresh viewpoint on making decisions inside the unit circle in complex space. This approach works well for managing temporal cycles, uncertainty, indeterminacy, and consistency.

It is commonly acknowledged in the field of neutrosophy that t-norm and t-conorm, including algebraic, Einstein, Hamacher, and Acz-Als t-norm and t-conorm, are essential elements of complex neutrosophic sets or neutrosophic systems in decision support. There are many scholars that developed operations draw on various t-norm and t-conorm to aid MCDMs for reality applications. These operations include a wide range of topics, such as operators [4-7, 22-25, 28-30], distance and similarity measures [8-10, 26, 30], and information entropy [5, 11-13]. Aggregation operators are essential tools for combining data in the complex neutrosophic environment and supporting in MCDM situations. Several studies have given mean operators and acknowledged their significance in the complex neutrosophic set environment. Notably, Ajay et al. [12] introduced and investigated the extension of sine trigonometry operational laws to neutrosophic set. They explored the workings and functioning of these extended laws. Furthermore, the research focused on the core of the work, which involved expanding these ST-Ols to complex neutrosophic sets. Mathematical properties were rigorously proven based on the operations. The study delved into basic operations and distance measurements for CNSs, employing the operational laws. Xu et al. [14] presented the single-valued complex neutrosophic set (SVCN) as a valuable method for managing datasets characterized by unpredictability and periodicity. This study presents a SVCN-EDAS for green supplier selection. The paper begins with a concise introduction to the notion of single-valued complex neutrosophic set and their associated operational rules. Subsequently, in order to combine and synthesize the overall information from SVCN, the paper introduces the aggregation operators. These operators are rooted in SVCNs and utilize Einstein product and sum for aggregation. Rahman et al. [15] explore the emerging field of hypersoft sets, aimed at addressing the limitations of previous soft-set-like frameworks by considering and accommodating multi-argument approximate procedures. The authors introduced fresh ideas within the hypersoft set framework, specifically complex intuitionistic fuzzy sets and complex neutrosophic sets, in the context of interval-valued complex neutrosophic set. Two new structures, namely the interval-valued complex intuitionistic hypersoft set and the interval-valued complex neutrosophic hypersoft set, have been created using theoretical, axiomatic, graphical, and computational approaches.

The fundamental operational rules for the single-valued neutrosophic hesitant fuzzy set (SV-NHFS) under Einstein's t-norm and t-conorm were introduced by Kamran et al. [28]. Moreover, SV-NHFS perform key functions for Einstein operators, such as the scalar multiplication, product, and sum. Then, they suggested a list of names for aggregation operators in order to solve the issues in the SV-NHF context, they finally talk about a MCDM method based on the proposed aggregation operators. The authors of the following study [29] talked about fuzzy rough set, probabilistic hesitant fuzzy set, and neutrosophic set as mathematical answers for smart housing communities. In this study, a hybrid structure of these sets, the single valued neutrosophic probabilistic hesitant fuzzy rough set, is presented in a novel generalised sense. This approach works well for handling ambiguity and incomplete data, especially when taking into account several conflicting factors. The suggested decision support system performs better than present tools in terms of choice flexibility, according to the study's comparison of it with other methods. Future research on waste management, public safety diagnostics, housing societies, and hybridization is anticipated to benefit from this study.

The idea of multi-valued complex neutrosophic linguistic sets (MVCNLSs) was put forth by Mahmood et al. [16], leading to the creation of MVCNLSs that represent information of truth, abstinence, falsity, and uncertain. And the way they are presented is with complex numbers having real and imaginary components that are restricted in a unit interval. Moreover, this paper provides a flexible framework for presenting Bonferroni mean operators and elaborates on important Dombi laws. Ali et al. [17] introducted the notion of complex neutrosophic uncertain linguistic (CNUL) and their useful hypotheses based on Heromian mean (HM) operators. CNUL data represents complex numbers consisting of uncertainty linguistic sets, truth grades, abstinence grades, and falsely grades. Furthermore, they endeavor to present well-known theories that are concerned with this idea. The first group of HM-CNUL aggregation operators based on HM presents such as arithmetic, geometric and weighted arithmetic, weighted geometric. All these operators share some common objective: bringing together multiple alternatives into an element that shall be presented as one single thing.

In solving real-world situations and decision-making (DM) scenarios, researchers have shown interest in augmenting the conceptual framework's flexibility through multiple studies, including:

Dat et al. [18] have shown that the most effective means of modeling for decisions problems under uncertainties is through Interval-valued Complex Neutrosophic Set. In the context of DM, linguistic variables have long served as valuable instruments for addressing the issue of sharp neutrosophic membership degrees. This study endeavors to introduce innovative notions, specifically the single-valued and interval-valued linguistic complex neutrosophic set. These new concepts offer greater applicability and adaptability to practical application compared to their predecessors. Al-Quran et al. [19] emphasize that many real-world complicated circumstances frequently entail data in two dimensions characterized by uncertainty, incompleteness, and indeterminacy, which encompass characteristics and the periodicity of problem parameters. Recognizing the requirement for frameworks capable of accommodating such data, neutrosophic sets (NSs) were introduced and subsequently extended to complex neutrosophic sets. The authors presented the concept of Q-complex neutrosophic sets by expanding the variety of membership functions within Q-neutrosophic sets. Broumi et al. [11] note that in the past decade, numerous researchers have directed their focus toward quantifying npredictability and ambiguity in datasets by leveraging the attributes of neutrosophic sets. This section, in particular, is dedicated to the measurement of unpredictability and its variability employing the characteristics of complex neutrosophic sets. Lan et al. [20] proposed Temporal Complex Neutrosophic Sets (TCNSs) to address temporal uncertainty, inconsistency, and uncertainty in data, which is often represented by Complex Neutrosophic Sets. Then, the authors proposed TCNS operations and features. Next, the MCDM method driven by TCNSs is introduced, which extends the ANP method to identify group multi-criteria weights and TOPSIS for ranking options obtained over different time intervals. Furthermore, a case study of selecting a tourism location in Vietnam is given within the TCNSs to prove the viability and potentiality of the suggested approach. A comparison with current methods illustrates the proposed model's availability, consistency and effectiveness. Gamal et al. [29] introduced a framework of MCDM on type-2 neutrosophic numbers. The authors proved that this extension can tackle well the uncertainty resulted from human judgment in decision-making scenarios. Two real-life case studies were discussed in this study to evaluate the proposed method.

Through a survey of various studies, it becomes evident that aggregation operators play a crucial role in MCDM problems within the complex neutrosophic environment. Furthermore, the temporal complex neutrosophic environment proves to be an effective framework for representing the temporal aspects of decision-making problems. However, it is noticeable that there have been few studies that express interest in or have only partially considered the use of average operators for temporal neutrosophic sets employing Aczel and Alsina t-norms or t-conorms.

It is clear from a review of numerous studies that aggregation operators are important when it comes to MCDM issues in the intricate neutrosophic environment. And to built aggregation operators on fuzzy theory, t-norms or t-conorms operators are crucial points. Moreover, it turns out that the temporal complex neutrosophic environment is a useful framework for expressing the temporal

dimensions of decision-making issues. However, it is apparent that the implementation of average operators for temporal neutrosophic sets utilizing t-conorms or t-norms proposed by Alsina and Aczel has not received much attention in research, or has received only limited attention. Therefore, this study aims to introduce novel TCNS aggregation operators according to Aczel and Alsina t-norm and t-conorm using Aczel-Alsina operations, and we propose a MCDM model that utilizes those operators for TCNS-based multi criteria decision making problems.

3. Preliminaries

Definition 1 [20]: Assume there is a universal U, time periods $\tilde{\tau} = \{\tau_1, \tau_2, ..., \tau_p\}$. A Temporal complex neutrosophic set (TCNS) A in U is defined as

$$A(\delta, \tilde{\tau}) = \{\delta, \langle CVT(\delta, \tilde{\tau}), CVI(\delta, \tilde{\tau}), CVF(\delta, \tilde{\tau})\rangle | \delta \in U\} \quad (1)$$

Where $CVT(\delta, \tilde{\tau}) = p(\delta, \tau_S) e^{j\mu(\delta, \tau_s)}$; $CVI(\delta, \tilde{\tau}) = q(\delta, \tau_S) e^{jv(\delta, \tau_s)}$; $CVF(\delta, \tilde{\tau}) = r(\delta, \tau_S) e^{j\eta(\delta, \tau_s)}$; $\tau_S \in \tilde{\tau}$;

In 1982, Aczel and Alsina [21] introducted Aczel-Alsina t-norm and t-conorm, representing a class of flexible aggregation operators adjustable through a parameter.

Definition 2: A mapping $\left(N_\alpha^\theta\right)_{\sigma \in [0,\infty]}$ is a Acz-Als t-norm if

$$\left(TN_\alpha^\theta\right)(a,b) = \begin{cases} TN_D(a,b) & if \quad \theta = 0 \\ \min(a,b) & if \quad \theta = \infty \\ \exp\left(-\left((-\ln a)^\theta + (-\ln b)^\theta\right)^{\frac{1}{\theta}}\right) & otherwise \end{cases} \quad (2)$$

Where $TN_D(a,b)$ is t-norm define in equation (3) and $a,b \in [0,1]$, σ is positive constant

$$\left(TN_D\right)(a,b) = \begin{cases} a & if \quad b = 1 \\ b & if \quad a = 1 \\ 0 & otherwise \end{cases} \quad (3)$$

Definition 3: A mapping $\left(TCN_\alpha^\sigma\right)_{\sigma \in [0,\infty]}$ is a Acz-Als s-norm if

$$\left(TCN_\alpha^\theta\right)(a,b) = \begin{cases} TCN_D(a,b) & if \quad \theta = 0 \\ \max(a,b) & if \quad \theta = \infty \\ \exp\left(-\left((-\ln(1-a))^\theta + (-\ln(1-b))^\theta\right)^{\frac{1}{\theta}}\right) & otherwise \end{cases} \quad (4)$$

Where $TCN_D(a,b)$ is s-norm, define in equation (5) and $a,b \in [0,1]$, σ is positive constant

$$\left(TCN_D\right)(a,b) = \begin{cases} a & when \quad b = 0 \\ b & when \quad a = 0 \\ 0 & otherwise \end{cases} \quad (5)$$

the t-norm $\left(TN_\alpha^\theta\right)_{\theta \in [0,\infty]}, \forall \theta \in [0,\infty]$ and s-norm $\left(TCN_\alpha^\theta\right)_{\theta \in [0,\infty]}, \forall \theta \in [0,\infty]$ are dual to each other

4. Aczel-Alsina-TCNS Aggregation Operators

In this section, we present Acz-Alz operations in relation to TCNS, taking in to account the Acz-Alz t-norm and t-conorm

Definition 4: Suppose θ is positive constant and three TCNS corresponding, $\Gamma(\delta, \tilde{\tau}) = \{\delta, \langle CVT_\Gamma(\delta, \tilde{\tau}), CVI_\Gamma(\delta, \tilde{\tau}), CVF_\Gamma(\delta, \tilde{\tau})\rangle\}$; $\Gamma_1(\delta, \tilde{\tau}) = \{\delta, \langle CVT_{\Gamma_1}(\delta, \tilde{\tau}), CVI_{\Gamma_1}(\delta, \tilde{\tau}), CVF_{\Gamma_1}(\delta, \tilde{\tau})\rangle\}$; $\Gamma_2(\delta, \tilde{\tau}) = \{\delta, \langle CVT_{\Gamma_2}(\delta, \tilde{\tau}), CVI_{\Gamma_2}(\delta, \tilde{\tau}), CVF_{\Gamma_2}(\delta, \tilde{\tau})\rangle\}$. Then Aczel-Alsina-TCNS operations are presented as follows:

i. Addition

$$\Gamma_1 \oplus \Gamma_2 = \{\delta, \langle CVT_{\Gamma_{12}}(\delta, \tilde{\tau}), CVI_{\Gamma_{12}}(\delta, \tilde{\tau}), CVF_{\Gamma_{12}}(\delta, \tilde{\tau}) | \delta \in U\rangle\} \quad (6)$$

Where

$$CVT_{\Gamma_{12}}(\delta,\tilde{\tau})=\left(1-\exp\left(-\left(\left(-\ln\left(1-p_{\Gamma_1}(\delta,\tau_s)\right)\right)^{\theta}+\left(-\ln\left(1-p_{\Gamma_2}(\delta,\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right).e^{j2\pi\left(1-\exp\left(-\left(\left(-\ln\left(1-\mu_{\Gamma_1}(\delta,\tau_s)\right)\right)^{\theta}+\left(-\ln\left(1-\mu_{\Gamma_2}(\delta,\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right)};$$

$$CVI_{\Gamma_{12}}(\delta,\tilde{\tau})=\left(1-\exp\left(-\left(\left(-\ln\left(q_{\Gamma_1}(\delta,\tau_s)\right)\right)^{\theta}+\left(-\ln\left(q_{\Gamma_2}(\delta,\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right).e^{j2\pi\left(1-\exp\left(-\left(\left(-\ln\left(v_{\Gamma_1}(\delta,\tau_s)\right)\right)^{\theta}+\left(-\ln\left(v_{\Gamma_2}(\delta,\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right)};$$

$$CVF_{\Gamma_{12}}(\delta,\tilde{\tau})=\left(1-\exp\left(-\left(\left(-\ln\left(r_{\Gamma_1}(\delta,\tau_s)\right)\right)^{\theta}+\left(-\ln\left(r_{\Gamma_2}(\delta,\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right).e^{j2\pi\left(1-\exp\left(-\left(\left(-\ln\left(\eta_{\Gamma_1}(\delta,\tau_s)\right)\right)^{\theta}+\left(-\ln\left(\eta_{\Gamma_2}(\delta,\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right)}$$

ii. Multiplication

$$\Gamma_1\otimes\Gamma_2=\left\{\delta,\left\langle CVT_{\Gamma_{12}}(\delta,\tilde{\tau}),CVI_{\Gamma_{12}}(\delta,\tilde{\tau}),CVF_{\Gamma_{12}}(\delta,\tilde{\tau})\right\rangle\mid\delta\in U\right\} \tag{7}$$

Where $CVT_{\Gamma_{12}}(\delta,\tilde{\tau})=\left(1-\exp\left(-\left(\left(-\ln\left(p_{\Gamma_1}(\delta,\tau_s)\right)\right)^{\theta}+\left(-\ln\left(p_{\Gamma_2}(\delta,\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right).e^{j2\pi\left(1-\exp\left(-\left(\left(-\ln\left(\mu_{\Gamma_1}(\delta,\tau_s)\right)\right)^{\theta}+\left(-\ln\left(\mu_{\Gamma_2}(\delta,\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right)};$

$$CVI_{\Gamma_{12}}(\delta,\tilde{\tau})=\left(1-\exp\left(-\left(\left(-\ln\left(1-q_{\Gamma_1}(\delta,\tau_s)\right)\right)^{\theta}+\left(-\ln\left(1-q_{\Gamma_2}(\delta,\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right).e^{j2\pi\left(1-\exp\left(-\left(\left(-\ln\left(1-v_{\Gamma_1}(\delta,\tau_s)\right)\right)^{\theta}+\left(-\ln\left(1-v_{\Gamma_2}(\delta,\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right)};$$

$$CVF_{\Gamma_{12}}(\delta,\tilde{\tau})=\left(1-\exp\left(-\left(\left(-\ln\left(1-r_{\Gamma_1}(\delta,\tau_s)\right)\right)^{\theta}+\left(-\ln\left(1-r_{\Gamma_2}(\delta,\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right).e^{j2\pi\left(1-\exp\left(-\left(\left(-\ln\left(1-\eta_{\Gamma_1}(\delta,\tau_s)\right)\right)^{\theta}+\left(-\ln\left(1-\eta_{\Gamma_2}(\delta,\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right)}$$

iii. Scalar Multiplication

$$\zeta.\Gamma=\left\{\delta,\left\langle CVT_{\Gamma}(\delta,\tilde{\tau}),CVI_{\Gamma}(\delta,\tilde{\tau}),CVF_{\Gamma}(\delta,\tilde{\tau})\right\rangle\mid\delta\in U\right\} \tag{8}$$

Where $CVT_{\Gamma}(\delta,\tilde{\tau})=\left(1-\exp\left(-\left(\zeta\left(-\ln\left(1-p_{\Gamma}(\delta,\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right).e^{j2\pi\left(1-\exp\left(-\left(\zeta\left(-\ln\left(1-\mu_{\Gamma}(\delta,\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right)};$

$$CVI_{\Gamma}(\delta,\tilde{\tau})=\left(1-\exp\left(-\left(\zeta\left(-\ln\left(q_{\Gamma}(\delta,\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right).e^{j2\pi\left(1-\exp\left(-\left(\zeta\left(-\ln\left(v_{\Gamma}(\delta,\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right)};$$

$$CVF_{\Gamma}(\delta,\tilde{\tau})=\left(1-\exp\left(-\left(\zeta\left(-\ln\left(r_{\Gamma}(\delta,\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right).e^{j2\pi\left(1-\exp\left(-\left(\zeta\left(-\ln\left(\eta_{\Gamma}(\delta,\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right)}$$

iv. Power

$$\Gamma^{\zeta}=\left\{\delta,\left\langle CVT_{\Gamma}(\delta,\tilde{\tau}),CVI_{\Gamma}(\delta,\tilde{\tau}),CVF_{\Gamma}(\delta,\tilde{\tau})\right\rangle\mid\delta\in U\right\} \tag{9}$$

Where $CVT_{\Gamma}(\delta,\tilde{\tau})=\left(1-\exp\left(-\left(\zeta\left(-\ln\left(p_{\Gamma}(\delta,\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right).e^{j2\pi\left(1-\exp\left(-\left(\zeta\left(-\ln\left(\mu_{\Gamma}(\delta,\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right)};$

$$CVI_{\Gamma}(\delta,\tilde{\tau})=\left(1-\exp\left(-\left(\zeta\left(-\ln\left(1-q_{\Gamma}(\delta,\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right).e^{j2\pi\left(1-\exp\left(-\left(\zeta\left(-\ln\left(1-v_{\Gamma}(\delta,\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right)};$$

$$CVF_{\Gamma}(\delta,\tilde{\tau})=\left(1-\exp\left(-\left(\zeta\left(-\ln\left(1-r_{\Gamma}(\delta,\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right).e^{j2\pi\left(1-\exp\left(-\left(\zeta\left(-\ln\left(1-\eta_{\Gamma}(\delta,\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right)};$$

Theorem 1: The properties for any two TCNS are as follows:

(1) $\Gamma_1 \oplus \Gamma_2 = \Gamma_2 \oplus \Gamma_1$

(2) $\Gamma_1 \otimes \Gamma_2 = \Gamma_2 \otimes \Gamma_1$

(3) $\lambda(\Gamma_1 \oplus \Gamma_2) = \lambda\Gamma_1 \oplus \lambda\Gamma_2$

(4) $(\lambda_1 + \lambda_2)\Gamma = \lambda_1\Gamma_1 \oplus \lambda_2\Gamma$

(5) $(\Gamma_1 \oplus \Gamma_2)^{\lambda} = {\Gamma_1}^{\lambda} \oplus {\Gamma_2}^{\lambda}$

(6) $\Gamma^{\lambda_1} \otimes \Gamma^{\lambda_2} = \Gamma^{\lambda_1+\lambda_2}$

Proof.

(1)
$$\Gamma_1 \oplus \Gamma_2 = \left\langle \begin{array}{l} \left(1-\exp\left(-\left(\left(-\ln\left(1-p_{\Gamma_1}(\delta,\tau_s)\right)\right)^{\rho}+\left(-\ln\left(1-p_{\Gamma_2}(\delta,\tau_s)\right)\right)^{\rho}\right)^{\frac{1}{\theta}}\right)\right).e^{j2\pi\left(1-\exp\left(-\left(\left(-\ln\left(1-\mu_{\Gamma_1}(\delta,\tau_s)\right)\right)^{\rho}+\left(-\ln\left(1-\mu_{\Gamma_2}(\delta,\tau_s)\right)\right)^{\rho}\right)^{\frac{1}{\theta}}\right)\right)}, \\ \left(1-\exp\left(-\left(\left(-\ln\left(1-q_{\Gamma_1}(\delta,\tau_s)\right)\right)^{\rho}+\left(-\ln\left(1-q_{\Gamma_2}(\delta,\tau_s)\right)\right)^{\rho}\right)^{\frac{1}{\theta}}\right)\right).e^{j2\pi\left(1-\exp\left(-\left(\left(-\ln\left(1-v_{\Gamma_1}(\delta,\tau_s)\right)\right)^{\rho}+\left(-\ln\left(1-v_{\Gamma_2}(\delta,\tau_s)\right)\right)^{\rho}\right)^{\frac{1}{\theta}}\right)\right)}, \\ \left(1-\exp\left(-\left(\left(-\ln\left(r_{\Gamma_1}(\delta,\tau_s)\right)\right)^{\rho}+\left(-\ln\left(r_{\Gamma_2}(\delta,\tau_s)\right)\right)^{\rho}\right)^{\frac{1}{\theta}}\right)\right).e^{j2\pi\left(1-\exp\left(-\left(\left(-\ln\left(\eta_{\Gamma_1}(\delta,\tau_s)\right)\right)^{\rho}+\left(-\ln\left(\eta_{\Gamma_2}(\delta,\tau_s)\right)\right)^{\rho}\right)^{\frac{1}{\theta}}\right)\right)} \end{array} \right\rangle$$

$$= \left\langle \begin{array}{l} \left(1-\exp\left(-\left(\left(-\ln\left(1-p_{\Gamma_2}(\delta,\tau_s)\right)\right)^{\rho}+\left(-\ln\left(1-p_{\Gamma_1}(\delta,\tau_s)\right)\right)^{\rho}\right)^{\frac{1}{\theta}}\right)\right).e^{j2\pi\left(1-\exp\left(-\left(\left(-\ln\left(1-\mu_{\Gamma_2}(\delta,\tau_s)\right)\right)^{\rho}+\left(-\ln\left(1-\mu_{\Gamma_1}(\delta,\tau_s)\right)\right)^{\rho}\right)^{\frac{1}{\theta}}\right)\right)}, \\ \left(1-\exp\left(-\left(\left(-\ln\left(1-q_{\Gamma_2}(\delta,\tau_s)\right)\right)^{\rho}+\left(-\ln\left(1-q_{\Gamma_1}(\delta,\tau_s)\right)\right)^{\rho}\right)^{\frac{1}{\theta}}\right)\right).e^{j2\pi\left(1-\exp\left(-\left(\left(-\ln\left(1-v_{\Gamma_2}(\delta,\tau_s)\right)\right)^{\rho}+\left(-\ln\left(1-v_{\Gamma_1}(\delta,\tau_s)\right)\right)^{\rho}\right)^{\frac{1}{\theta}}\right)\right)}, \\ \left(1-\exp\left(-\left(\left(-\ln\left(r_{\Gamma_2}(\delta,\tau_s)\right)\right)^{\rho}+\left(-\ln\left(r_{\Gamma_1}(\delta,\tau_s)\right)\right)^{\rho}\right)^{\frac{1}{\theta}}\right)\right).e^{j2\pi\left(1-\exp\left(-\left(\left(-\ln\left(\eta_{\Gamma_2}(\delta,\tau_s)\right)\right)^{\rho}+\left(-\ln\left(\eta_{\Gamma_1}(\delta,\tau_s)\right)\right)^{\rho}\right)^{\frac{1}{\theta}}\right)\right)} \end{array} \right\rangle$$

$$= \Gamma_2 \oplus \Gamma_1$$

(2)
$$\Gamma_1 \otimes \Gamma_2 = \left\langle \begin{array}{l} \left(1-\exp\left(-\left(\left(-\ln\left(p_{\Gamma_1}(\delta,\tau_s)\right)\right)^{\rho}+\left(-\ln\left(p_{\Gamma_2}(\delta,\tau_s)\right)\right)^{\rho}\right)^{\frac{1}{\theta}}\right)\right).e^{j2\pi\left(1-\exp\left(-\left(\left(-\ln\left(\mu_{\Gamma_1}(\delta,\tau_s)\right)\right)^{\rho}+\left(-\ln\left(\mu_{\Gamma_2}(\delta,\tau_s)\right)\right)^{\rho}\right)^{\frac{1}{\theta}}\right)\right)}, \\ \left(1-\exp\left(-\left(\left(-\ln\left(1-q_{\Gamma_1}(\delta,\tau_s)\right)\right)^{\rho}+\left(-\ln\left(1-q_{\Gamma_2}(\delta,\tau_s)\right)\right)^{\rho}\right)^{\frac{1}{\theta}}\right)\right).e^{j2\pi\left(1-\exp\left(-\left(\left(-\ln\left(1-v_{\Gamma_1}(\delta,\tau_s)\right)\right)^{\rho}+\left(-\ln\left(1-v_{\Gamma_2}(\delta,\tau_s)\right)\right)^{\rho}\right)^{\frac{1}{\theta}}\right)\right)}, \\ \left(1-\exp\left(-\left(\left(-\ln\left(1-r_{\Gamma_1}(\delta,\tau_s)\right)\right)^{\rho}+\left(-\ln\left(1-r_{\Gamma_2}(\delta,\tau_s)\right)\right)^{\rho}\right)^{\frac{1}{\theta}}\right)\right).e^{j2\pi\left(1-\exp\left(-\left(\left(-\ln\left(1-\eta_{\Gamma_1}(\delta,\tau_s)\right)\right)^{\rho}+\left(-\ln\left(1-\eta_{\Gamma_2}(\delta,\tau_s)\right)\right)^{\rho}\right)^{\frac{1}{\theta}}\right)\right)} \end{array} \right\rangle$$

$$= \left\langle \begin{array}{l} \left(1-\exp\left(-\left(\left(-\ln\left(p_{\Gamma_2}(\delta,\tau_s)\right)\right)^{\theta}+\left(-\ln\left(p_{\Gamma_1}(\delta,\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right).e^{j2\pi\left(1-\exp\left(-\left(\left(-\ln\left(\mu_{\Gamma_2}(\delta,\tau_s)\right)\right)^{\theta}+\left(-\ln\left(\mu_{\Gamma_1}(\delta,\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right)}, \\ \left(1-\exp\left(-\left(\left(-\ln\left(1-q_{\Gamma_2}(\delta,\tau_s)\right)\right)^{\theta}+\left(-\ln\left(1-q_{\Gamma_1}(\delta,\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right).e^{j2\pi\left(1-\exp\left(-\left(\left(-\ln\left(1-v_{\Gamma_2}(\delta,\tau_s)\right)\right)^{\theta}+\left(-\ln\left(1-v_{\Gamma_1}(\delta,\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right)}, \\ \left(1-\exp\left(-\left(\left(-\ln\left(1-r_{\Gamma_2}(\delta,\tau_s)\right)\right)^{\theta}+\left(-\ln\left(1-r_{\Gamma_1}(\delta,\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right).e^{j2\pi\left(1-\exp\left(-\left(\left(-\ln\left(1-\eta_{\Gamma_2}(\delta,\tau_s)\right)\right)^{\theta}+\left(-\ln\left(1-\eta_{\Gamma_1}(\delta,\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right)} \end{array} \right\rangle$$

$$= \Gamma_2 \otimes \Gamma_1$$

(3) $$\lambda(\Gamma_1 \oplus \Gamma_2) = \lambda \left\langle \begin{array}{l} \left(1-\exp\left(-\left(\left(-\ln\left(1-p_{\Gamma_1}(\delta,\tau_s)\right)\right)^{\theta}+\left(-\ln\left(1-p_{\Gamma_2}(\delta,\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right).e^{j2\pi\left(1-\exp\left(-\left(\left(-\ln\left(1-\mu_{\Gamma_1}(\delta,\tau_s)\right)\right)^{\theta}+\left(-\ln\left(1-\mu_{\Gamma_2}(\delta,\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right)}, \\ \left(1-\exp\left(-\left(\left(-\ln\left(1-q_{\Gamma_1}(\delta,\tau_s)\right)\right)^{\theta}+\left(-\ln\left(1-q_{\Gamma_2}(\delta,\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right).e^{j2\pi\left(1-\exp\left(-\left(\left(-\ln\left(1-v_{\Gamma_1}(\delta,\tau_s)\right)\right)^{\theta}+\left(-\ln\left(1-v_{\Gamma_2}(\delta,\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right)}, \\ \left(1-\exp\left(-\left(\left(-\ln\left(r_{\Gamma_1}(\delta,\tau_s)\right)\right)^{\theta}+\left(-\ln\left(r_{\Gamma_2}(\delta,\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right).e^{j2\pi\left(1-\exp\left(-\left(\left(-\ln\left(\eta_{\Gamma_1}(\delta,\tau_s)\right)\right)^{\theta}+\left(-\ln\left(\eta_{\Gamma_2}(\delta,\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right)} \end{array} \right\rangle$$

$$= \left\langle \begin{array}{l} \left(1-\exp\left(-\left(\lambda\left(-\ln\left(1-p_{\Gamma_1}(\delta,\tau_s)\right)\right)^{\theta}+\lambda\left(-\ln\left(1-p_{\Gamma_2}(\delta,\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right).e^{j2\pi\left(1-\exp\left(-\left(\lambda\left(-\ln\left(1-\mu_{\Gamma_1}(\delta,\tau_s)\right)\right)^{\theta}+\lambda\left(-\ln\left(1-\mu_{\Gamma_2}(\delta,\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right)}, \\ \left(1-\exp\left(-\left(\lambda\left(-\ln\left(1-q_{\Gamma_1}(\delta,\tau_s)\right)\right)^{\theta}+\lambda\left(-\ln\left(1-q_{\Gamma_2}(\delta,\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right).e^{j2\pi\left(1-\exp\left(-\left(\lambda\left(-\ln\left(1-v_{\Gamma_1}(\delta,\tau_s)\right)\right)^{\theta}+\lambda\left(-\ln\left(1-v_{\Gamma_2}(\delta,\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right)}, \\ \left(1-\exp\left(-\left(\lambda\left(-\ln\left(r_{\Gamma_1}(\delta,\tau_s)\right)\right)^{\theta}+\lambda\left(-\ln\left(r_{\Gamma_2}(\delta,\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right).e^{j2\pi\left(1-\exp\left(-\left(\lambda\left(-\ln\left(\eta_{\Gamma_1}(\delta,\tau_s)\right)\right)^{\theta}+\lambda\left(-\ln\left(\eta_{\Gamma_2}(\delta,\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right)} \end{array} \right\rangle$$

$$= \lambda\Gamma_1 \oplus \lambda\Gamma_2$$

(4) $$(\lambda_1+\lambda_2)\Gamma = \lambda_1\Gamma \oplus \lambda_2\Gamma$$

$$\lambda_1\Gamma \oplus \lambda_2\Gamma = \left\langle \begin{array}{l} \left(1-\exp\left(-\left(\lambda_1\left(-\ln\left(1-p_{\Gamma}(\delta,\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right).e^{j2\pi\left(1-\exp\left(-\left(\lambda_1\left(-\ln\left(1-\mu_{\Gamma}(\delta,\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right)}, \\ \left(1-\exp\left(-\left(\lambda_1\left(-\ln\left(q_{\Gamma}(\delta,\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right).e^{j2\pi\left(1-\exp\left(-\left(\lambda_1\left(-\ln\left(v_{\Gamma}(\delta,\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right)}, \\ \left(1-\exp\left(-\left(\lambda_1\left(-\ln\left(r_{\Gamma}(\delta,\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right).e^{j2\pi\left(1-\exp\left(-\left(\lambda_1\left(-\ln\left(\eta_{\Gamma}(\delta,\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right)} \end{array} \right\rangle$$

$$\oplus\left\langle\begin{array}{l}\left(1-\exp\left(-\left(\lambda_2\left(-\ln\left(1-p_\Gamma(\delta,\tau_s)\right)\right)^\theta\right)^{\frac{1}{\theta}}\right)\right).e^{j2\pi\left(1-\exp\left(-\left(\lambda_2\left(-\ln\left(1-\mu_\Gamma(\delta,\tau_s)\right)\right)^\theta\right)^{\frac{1}{\theta}}\right)\right)},\\ \left(1-\exp\left(-\left(\lambda_2\left(-\ln\left(q_\Gamma(\delta,\tau_s)\right)\right)^\theta\right)^{\frac{1}{\theta}}\right)\right).e^{j2\pi\left(1-\exp\left(-\left(\lambda_2\left(-\ln\left(v_\Gamma(\delta,\tau_s)\right)\right)^\theta\right)^{\frac{1}{\theta}}\right)\right)},\\ \left(1-\exp\left(-\left(\lambda_2\left(-\ln\left(r_\Gamma(\delta,\tau_s)\right)\right)^\theta\right)^{\frac{1}{\theta}}\right)\right).e^{j2\pi\left(1-\exp\left(-\left(\lambda_2\left(-\ln\left(\eta_\Gamma(\delta,\tau_s)\right)\right)^\theta\right)^{\frac{1}{\theta}}\right)\right)}\end{array}\right\rangle$$

$$=\left\langle\begin{array}{l}\left(1-\exp\left(-\left(\left(\lambda_1+\lambda_2\right)\left(-\ln\left(1-p_\Gamma(\delta,\tau_s)\right)\right)^\theta\right)^{\frac{1}{\theta}}\right)\right).e^{j2\pi\left(1-\exp\left(-\left(\left(\lambda_1+\lambda_2\right)\left(-\ln\left(1-\mu_\Gamma(\delta,\tau_s)\right)\right)^\theta\right)^{\frac{1}{\theta}}\right)\right)},\\ \left(1-\exp\left(-\left(\left(\lambda_1+\lambda_2\right)\left(-\ln\left(q_\Gamma(\delta,\tau_s)\right)\right)^\theta\right)^{\frac{1}{\theta}}\right)\right).e^{j2\pi\left(1-\exp\left(-\left(\left(\lambda_1+\lambda_2\right)\left(-\ln\left(v_\Gamma(\delta,\tau_s)\right)\right)^\theta\right)^{\frac{1}{\theta}}\right)\right)},\\ \left(1-\exp\left(-\left(\left(\lambda_1+\lambda_2\right)\left(-\ln\left(r_\Gamma(\delta,\tau_s)\right)\right)^\theta\right)^{\frac{1}{\theta}}\right)\right).e^{j2\pi\left(1-\exp\left(-\left(\left(\lambda_1+\lambda_2\right)\left(-\ln\left(\eta_\Gamma(\delta,\tau_s)\right)\right)^\theta\right)^{\frac{1}{\theta}}\right)\right)}\end{array}\right\rangle$$

$$=\left(\lambda_1+\lambda_2\right)\Gamma$$

(5) $\left(\Gamma_1\oplus\Gamma_2\right)^\lambda=\Gamma_1{}^\lambda\oplus\Gamma_2{}^\lambda$

It is obvious as (3)

(6) $\Gamma^{\lambda_1}\otimes\Gamma^{\lambda_2}=\Gamma^{\lambda_1+\lambda_2}$

It is obvious as (4)

Definition 5: The Temporal Commplex Neutrosophic Weighted Averaging Operator (TCN-A) has a computational form:

$$TCN-A\left(\Gamma_1,\Gamma_2,\Gamma_3,...,\Gamma_n\right)=\oplus_{k=1}^n\left(w_k\Gamma_k\right)=\left\langle\overline{CVT}\left(\delta,\tilde{\tau}\right),\overline{CVI}\left(\delta,\tilde{\tau}\right),\overline{CVF}\left(\delta,\tilde{\tau}\right)\right\rangle \tag{10}$$

Where $\overline{CVT}\left(\delta,\tilde{\tau}\right)=\left(1-\exp\left(-\left(\sum_{k=1}^n w_k\left(-\ln\left(1-p_{\Gamma_k}(\delta,\tau_s)\right)\right)^\theta\right)^{\frac{1}{\theta}}\right)\right).e^{j2\pi\left(1-\exp\left(-\left(\sum_{k=1}^n w_k\left(-\ln\left(1-\mu_{\Gamma_k}(\delta,\tau_s)\right)\right)^\theta\right)^{\frac{1}{\theta}}\right)\right)}$

$$\overline{CVI}\left(\delta,\tilde{\tau}\right)=\left(1-\exp\left(-\left(\sum_{k=1}^n w_k\left(-\ln\left(q_{\Gamma_k}(\delta,\tau_s)\right)\right)^\theta\right)^{\frac{1}{\theta}}\right)\right).e^{j2\pi\left(1-\exp\left(-\left(\sum_{k=1}^n w_k\left(-\ln\left(v_{\Gamma_k}(\delta,\tau_s)\right)\right)^\theta\right)^{\frac{1}{\theta}}\right)\right)}$$

$$\overline{CVF}\left(\delta,\tilde{\tau}\right)=\left(1-\exp\left(-\left(\sum_{k=1}^n w_i\left(-\ln\left(r_{\Gamma_k}(\delta,\tau_s)\right)\right)^\theta\right)^{\frac{1}{\theta}}\right)\right).e^{j2\pi\left(1-\exp\left(-\left(\sum_{k=1}^n w_i\left(-\ln\left(\eta_{\Gamma_k}(\delta,\tau_s)\right)\right)^\theta\right)^{\frac{1}{\theta}}\right)\right)}$$

Theorem 2: Let a set of TCNS; $\Gamma=\Gamma_1,\Gamma_2,...,\Gamma_n$, then the result obtained by utilizing TCN-A operator is still a TCNS.

Proof.

Using Mathematical induction. Equation (10) holds because it simplifies the outcome to the trivial one, which is obviously TCNS when $n=1$,

$$TCN-A\left(\Gamma_1\right)=\left\langle \begin{array}{l} \left(1-\exp\left(-\left(w_1\left(-\ln\left(1-p_{\Gamma_1}(\delta,\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right).e^{j2\pi\left(1-\exp\left(-\left(w_1\left(-\ln\left(1-\mu_{\Gamma_1}(\delta,\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right)}, \\ \left(1-\exp\left(-\left(w_1\left(-\ln\left(q_{\Gamma_1}(\delta,\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right).e^{j2\pi\left(1-\exp\left(-\left(w_1\left(-\ln\left(v_{\Gamma_1}(\delta,\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right)}, \\ \left(1-\exp\left(-\left(w_1\left(-\ln\left(r_{\Gamma_1}(\delta,\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right).e^{j2\pi\left(1-\exp\left(-\left(w_1\left(-\ln\left(\eta_{\Gamma_1}(\delta,\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right)} \end{array}\right\rangle$$

Assume that equation (10) is true for $n=m$,

$$TCN-A\left(\Gamma_1,\Gamma_2,...,\Gamma_m\right)=$$

$$\left\langle \begin{array}{l} \left(1-\exp\left(-\left(\sum_{k=1}^{m}w_k\left(-\ln\left(1-p_{\Gamma_k}(\delta,\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right).e^{j2\pi\left(1-\exp\left(-\left(\sum_{k=1}^{m}w_k\left(-\ln\left(1-\mu_{\Gamma_k}(\delta,\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right)}, \\ \left(1-\exp\left(-\left(\sum_{k=1}^{m}w_k\left(-\ln\left(q_{\Gamma_k}(\delta,\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right).e^{j2\pi\left(1-\exp\left(-\left(\sum_{k=1}^{m}w_k\left(-\ln\left(v_{\Gamma_k}(\delta,\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right)}, \\ \left(1-\exp\left(-\left(\sum_{k=1}^{m}w_k\left(-\ln\left(r_{\Gamma_k}(\delta,\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right).e^{j2\pi\left(1-\exp\left(-\left(\sum_{k=1}^{m}w_k\left(-\ln\left(\eta_{\Gamma_k}(\delta,\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right)} \end{array}\right\rangle$$

When $n=m+1$

$$TCN-A\left(\Gamma_1,\Gamma_2,...,\Gamma_n\right)=\left\langle \begin{array}{l} \left(1-\exp\left(-\left(\sum_{k=1}^{m}w_k\left(-\ln\left(1-p_{\Gamma_k}(\delta,\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right).e^{j2\pi\left(1-\exp\left(-\left(\sum_{k=1}^{m}w_k\left(-\ln\left(1-\mu_{\Gamma_k}(\delta,\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right)}, \\ \left(1-\exp\left(-\left(\sum_{k=1}^{m}w_k\left(-\ln\left(q_{\Gamma_k}(\delta,\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right).e^{j2\pi\left(1-\exp\left(-\left(\sum_{k=1}^{m}w_k\left(-\ln\left(v_{\Gamma_k}(\delta,\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right)}, \\ \left(1-\exp\left(-\left(\sum_{k=1}^{m}w_k\left(-\ln\left(r_{\Gamma_k}(\delta,\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right).e^{j2\pi\left(1-\exp\left(-\left(\sum_{k=1}^{m}w_k\left(-\ln\left(\eta_{\Gamma_k}(\delta,\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right)} \end{array}\right\rangle$$

$$\oplus\left\langle\begin{array}{l}\left(1-\exp\left(-\left(w_{m+1}\left(-\ln\left(1-p_{\Gamma_{m+1}}(\delta,\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right).e^{j2\pi\left(1-\exp\left(-\left(w_{m+1}\left(-\ln\left(1-\mu_{\Gamma_{m+1}}(\delta,\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right)},\\ \left(1-\exp\left(-\left(w_{m+1}\left(-\ln\left(q_{\Gamma_{m+1}}(\delta,\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right).e^{j2\pi\left(1-\exp\left(-\left(w_{m+1}\left(-\ln\left(v_{\Gamma_{m+1}}(\delta,\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right)},\\ \left(1-\exp\left(-\left(w_{m+1}\left(-\ln\left(r_{\Gamma_{m+1}}(\delta,\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right).e^{j2\pi\left(1-\exp\left(-\left(w_{m+1}\left(-\ln\left(\eta_{\Gamma_{m+1}}(\delta,\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right)}\end{array}\right\rangle$$

$$=\left\langle\begin{array}{l}\left(1-\exp\left(-\left(\sum_{k=1}^{n}w_k\left(-\ln\left(1-p_{\Gamma_k}(\delta,\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right).e^{j2\pi\left(1-\exp\left(-\left(\sum_{k=1}^{n}w_k\left(-\ln\left(1-\mu_{\Gamma_k}(\delta,\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right)},\\ \left(1-\exp\left(-\left(\sum_{k=1}^{n}w_k\left(-\ln\left(q_{\Gamma_k}(\delta,\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right).e^{j2\pi\left(1-\exp\left(-\left(\sum_{k=1}^{n}w_k\left(-\ln\left(v_{\Gamma_k}(\delta,\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right)},\\ \left(1-\exp\left(-\left(\sum_{k=1}^{n}w_k\left(-\ln\left(r_{\Gamma_k}(\delta,\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right).e^{j2\pi\left(1-\exp\left(-\left(\sum_{k=1}^{n}w_k\left(-\ln\left(\eta_{\Gamma_k}(\delta,\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right)}\end{array}\right\rangle\square$$

Definition 6: The Temporal Commplex Neutrosophic Weighted Geometric Operator (TCN-G) has a computational form:

$$TCN-G\left(\Gamma_1,\Gamma_2,...,\Gamma_n\right)=\otimes_{k=1}^{n}\left(w_k\Gamma_k\right)=\left\langle\overline{CVT}(\delta,\tilde{\tau}),\overline{CVI}(\delta,\tilde{\tau}),\overline{CVF}(\delta,\tilde{\tau})\right\rangle \tag{11}$$

Where $\overline{CVT}(\delta,\tilde{\tau})=\left(1-\exp\left(-\left(\sum_{k=1}^{n}w_k\left(-\ln\left(p_{\Gamma_k}(\delta,\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right).e^{j2\pi\left(1-\exp\left(-\left(\sum_{k=1}^{n}w_k\left(-\ln\left(\mu_{\Gamma_k}(\delta,\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right)}$

$$\overline{CVI}(\delta,\tilde{\tau})=\left(1-\exp\left(-\left(\sum_{k=1}^{n}w_k\left(-\ln\left(1-q_{\Gamma_k}(\delta,\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right).e^{j2\pi\left(1-\exp\left(-\left(\sum_{k=1}^{n}w_k\left(-\ln\left(1-v_{\Gamma_k}(\delta,\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right)}$$

$$\overline{CVF}(\delta,\tilde{\tau})=\left(1-\exp\left(-\left(\sum_{k=1}^{n}w_k\left(-\ln\left(1-r_{\Gamma_k}(\delta,\tau_s)\right)\right)^{\sigma}\right)^{\frac{1}{\sigma}}\right)\right).e^{j2\pi\left(1-\exp\left(-\left(\sum_{k=1}^{n}w_k\left(-\ln\left(1-\eta_{\Gamma_k}(\delta,\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right)}$$

Theorem 3: Let a set of TCNS; $\Gamma=\Gamma_1,\Gamma_2...,\Gamma_n$, then the result obtained by utilizing TCN-G is still a TCNS.

Proof.

Using Mathematical induction. Equation (11) holds because it simplifies the outcome to the trivial one, which is obviously TCNS when $n=1$,

$$TCN-A\left(\Gamma_1\right)=\left\langle \begin{array}{l} \left(1-\exp\left(-\left(w_1\left(-\ln\left(p_{\Gamma_1}(\delta,\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right).e^{j2\pi\left(1-\exp\left(-\left(w_1\left(-\ln\left(\mu_{\Gamma_1}(\delta,\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right)}, \\ \left(1-\exp\left(-\left(w_1\left(-\ln\left(1-q_{\Gamma_1}(\delta,\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right).e^{j2\pi\left(1-\exp\left(-\left(w_1\left(-\ln\left(1-v_{\Gamma_1}(\delta,\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right)}, \\ \left(1-\exp\left(-\left(w_1\left(-\ln\left(1-r_{\Gamma_1}(\delta,\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right).e^{j2\pi\left(1-\exp\left(-\left(w_i\left(-\ln\left(1-\eta_{\Gamma_1}(\delta,\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right)} \end{array}\right\rangle$$

Assume that equation (11) is true for $n=m$,

$$TCN-A\left(\Gamma_1,\Gamma_2,...,\Gamma_m\right)=\left\langle \begin{array}{l} \left(1-\exp\left(-\left(\sum_{k=1}^{m}w_k\left(-\ln\left(p_{\Gamma_k}(\delta,\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right).e^{j2\pi\left(1-\exp\left(-\left(\sum_{k=1}^{m}w_k\left(-\ln\left(\mu_{\Gamma_k}(\delta,\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right)}, \\ \left(1-\exp\left(-\left(\sum_{k=1}^{m}w_k\left(-\ln\left(1-q_{\Gamma_k}(\delta,\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right).e^{j2\pi\left(1-\exp\left(-\left(\sum_{k=1}^{m}w_k\left(-\ln\left(1-v_{\Gamma_k}(\delta,\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right)}, \\ \left(1-\exp\left(-\left(\sum_{k=1}^{m}w_k\left(-\ln\left(1-r_{\Gamma_k}(\delta,\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right).e^{j2\pi\left(1-\exp\left(-\left(\sum_{k=1}^{m}w_k\left(-\ln\left(1-\eta_{\Gamma_k}(\delta,\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right)} \end{array}\right\rangle$$

When $n=m+1$

$$TCN-A\left(\Gamma_1,\Gamma_2,...,\Gamma_n\right)=\left\langle \begin{array}{l} \left(1-\exp\left(-\left(\sum_{k=1}^{m}w_k\left(-\ln\left(p_{\Gamma_k}(\delta,\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right).e^{j2\pi\left(1-\exp\left(-\left(\sum_{k=1}^{m}w_k\left(-\ln\left(\mu_{\Gamma_k}(\delta,\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right)}, \\ \left(1-\exp\left(-\left(\sum_{k=1}^{m}w_k\left(-\ln\left(1-q_{\Gamma_k}(\delta,\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right).e^{j2\pi\left(1-\exp\left(-\left(\sum_{k=1}^{m}w_k\left(-\ln\left(1-v_{\Gamma_k}(\delta,\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right)}, \\ \left(1-\exp\left(-\left(\sum_{k=1}^{m}w_k\left(-\ln\left(1-r_{\Gamma_k}(\delta,\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right).e^{j2\pi\left(1-\exp\left(-\left(\sum_{k=1}^{m}w_k\left(-\ln\left(1-\eta_{\Gamma_k}(\delta,\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right)} \end{array}\right\rangle$$

$$\otimes\left\langle\begin{array}{l}\left(1-\exp\left(-\left(w_{m+1}\left(-\ln\left(p_{\Gamma_{m+1}}(\delta,\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right).e^{j2\pi\left(1-\exp\left(-\left(w_{m+1}\left(-\ln\left(\mu_{\Gamma_{m+1}}(\delta,\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right)},\\ \left(1-\exp\left(-\left(w_{m+1}\left(-\ln\left(1-q_{\Gamma_{m+1}}(\delta,\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right).e^{j2\pi\left(1-\exp\left(-\left(w_{m+1}\left(-\ln\left(1-v_{\Gamma_{m+1}}(\delta,\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right)},\\ \left(1-\exp\left(-\left(w_{m+1}\left(-\ln\left(1-r_{\Gamma_{m+1}}(\delta,\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right).e^{j2\pi\left(1-\exp\left(-\left(w_{m+1}\left(-\ln\left(1-\eta_{\Gamma_{m+1}}(\delta,\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right)}\end{array}\right\rangle$$

$$=\left\langle\begin{array}{l}\left(1-\exp\left(-\left(\sum_{k=1}^{n}w_k\left(-\ln\left(p_{\Gamma_k}(\delta,\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right).e^{j2\pi\left(1-\exp\left(-\left(\sum_{k=1}^{n}w_k\left(-\ln\left(\mu_{\Gamma_k}(\delta,\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right)},\\ \left(1-\exp\left(-\left(\sum_{k=1}^{n}w_k\left(-\ln\left(1-q_{\Gamma_k}(\delta,\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right).e^{j2\pi\left(1-\exp\left(-\left(\sum_{k=1}^{n}w_k\left(-\ln\left(1-v_{\Gamma_k}(\delta,\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right)},\\ \left(1-\exp\left(-\left(\sum_{k=1}^{n}w_k\left(-\ln\left(1-r_{\Gamma_k}(\delta,\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right).e^{j2\pi\left(1-\exp\left(-\left(\sum_{k=1}^{n}w_k\left(-\ln\left(1-\eta_{\Gamma_k}(\delta,\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right)}\end{array}\right\rangle\ \square$$

5. MCDM based on Aggregation Operators of TCNS

In this section, the paper presents an approach to MCDM using the proposed Aczel-Alsina TCNS operators. The MCDM problem on TCNS is defined as follow.

Assume $\tilde{D}=\left\{D_1,D_2,...,D_{n_D}\right\}$, $\tilde{C}=\left\{C_1,C_2,...,C_{n_C}\right\}$ and $\tilde{A}=\left\{A_1,A_2,...,A_{n_A}\right\}$ are sets of decision makers, criteria and alternatives. The evaluation characteristic of an alternatives A_{i_a}; $i_a=1,2,...,n_A$ on an criteria C_{i_c}; $i_c=1,2,...,n_C$ in time period τ_s; $s=1,2,...,n_\tau$ by decision maker D_{i_d}; $i_d=1,2,...,n_D$, is determined by matrix $X^{i_d}(\tilde{\tau})=\left(\theta^{i_d}_{i_ai_c}(\tilde{\tau})\right)_{n_A\times n_C}$. where $\theta^{i_d}_{i_ai_c}(\tau_l)$ taken language value of CNS on a time period τ_l.

Let $\theta^{i_d}_{i_ai_c}(\tau_s)=\left\langle p^{i_d}_{i_ai_c}(\tau_s)e^{j\mu^{i_d}_{i_ai_c}(\tau_l)},q^{i_d}_{i_ai_c}(\tau_s)e^{jv^{i_d}_{i_ai_c}(\tau_l)},r^{i_d}_{i_ai_c}(\tau_s)e^{j\eta^{i_d}_{i_ai_c}(\tau_s)}\right\rangle$

A model for MCDM on TCNS based on TOPSIS method and the proposed Aczel-Alsina TCNS operators includes 8 consequent steps. This MCDM model can be clearly described as follow.

Step 1. Normalize the evaluation matrix $X^{i_d}(\tau_s)$ to $\tilde{X}^{i_d}(\tau_s)$,

For benefit-type attributes C_{i_c}, $\psi^{i_d}_{i_ai_c}(\tau_s)=\left\langle p^{i_d}_{i_ai_c}(\tau_s)e^{j\mu^{i_d}_{i_ai_c}(\tau_s)},q^{i_d}_{i_ai_c}(\tau_s)e^{jv^{i_d}_{i_ai_c}(\tau_s)},r^{i_d}_{i_ai_c}(\tau_s)e^{j\eta^{i_d}_{i_ai_c}(\tau_s)}\right\rangle$

For cost-type attributes C_{i_c},

$\psi^{i_d}_{i_ai_c}(\tau_s)=\left\langle r^{i_d}_{i_ai_c}(\tau_s)e^{j\left(2\pi-\eta^{i_d}_{i_ai_c}(\tau_s)\right)},\left(1-q^{i_d}_{i_ai_c}(\tau_s)\right)e^{j\left(2\pi-v^{i_d}_{i_ai_c}(\tau_s)\right)},p^{i_d}_{i_ai_c}(\tau_s)e^{j\left(2\pi-\mu^{i_d}_{i_ai_c}(\tau_s)\right)}\right\rangle$

Step 2. According $\tilde{X}^{i_d}(\tau_l)$ matrix and the weight vector $w^d=\left\{w^d_1,w^d_2,...,w^d_{n_D}\right\}$ of decision makers, the aggregated decision making matrix $\bar{X}(\tau_l)$ of decision makers denoted as (utilizing it within the working space).

$$\overline{X}(\tau_l)=\left(\theta_{i_a i_c}(\tau_s)\right)_{m\times n}=\left\langle p_{i_a i_c}(\tau_s)e^{j\mu_{i_a i_c}(\tau_s)},q_{i_a i_c}(\tau_s)e^{j\nu_{i_a i_c}(\tau_s)},r_{i_a i_c}(\tau_s)e^{j\eta_{i_a i_c}(\tau_s)}\right\rangle \tag{12}$$

Step 3. Determines the average solution $AvS=\left(AvS_j\right)_{1\times n_C}$ with $AvS_j=\frac{1}{n_A}\bigoplus_{i_a=1}^{n_A}\psi_{i_a i_c}(\tau_l)$.

Based on Definition 4, 5, we can get

$$AvS_j(\tilde{\tau})=\frac{1}{n_A}\bigoplus_{i_a=1}^{n_A}\theta_{i_a i_c}(\tau_s)=\left\langle \overline{T}_{AvS_j},\overline{I}_{AvS_j},\overline{F}_{AvS_j}\right\rangle \tag{13}$$

$$\overline{T}_{AvS_j}=\left(1-\exp\left(-\left(\sum_{i_a=1}^{n_A}\frac{1}{n_A}\left(-\ln\left(1-p_{i_a}(\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right).e^{j2\pi\left(1-\exp\left(-\left(\sum_{i_a=1}^{n_A}\frac{1}{n_A}\left(-\ln\left(1-\mu_{i_a}(\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right)}$$

$$\overline{I}_{AvS_j}=\left(1-\exp\left(-\left(\sum_{i_a=1}^{n_A}\frac{1}{n_A}\left(-\ln\left(q_{i_a}(\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right).e^{j2\pi\left(1-\exp\left(-\left(\sum_{i_a=1}^{n_A}\frac{1}{n_A}\left(-\ln\left(v_{i_a}(\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right)}$$

$$\overline{F}_{AvS_j}=\left(1-\exp\left(-\left(\sum_{i_a=1}^{n_A}\frac{1}{n_A}\left(-\ln\left(r_{i_a}(\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right).e^{j2\pi\left(1-\exp\left(-\left(\sum_{i_a=1}^{n_A}\frac{1}{n_A}\left(-\ln\left(\eta_{\Gamma_i}(\tau_s)\right)\right)^{\theta}\right)^{\frac{1}{\theta}}\right)\right)}$$

Step 4. Compute $PDA=\left(P_{i_a i_c}\right)_{n_A\times n_C}$ and $NDA=\left(N_{i_a i_c}\right)_{n_A\times n_C}$ are positive/negative distance from average

$$PDA_{i_A i_C}=\frac{\max\left\{0,s\left(\theta_{i_A i_C}(\tilde{\tau})\right)-s\left(AvS_j(\tilde{\tau})\right)\right\}}{s\left(AvS_j(\tilde{\tau})\right)};\quad NDA_{i_A i_C}=\frac{\max\left\{0,s\left(AvS_j(\tilde{\tau})\right)-s\left(\theta_{i_A i_C}(\tilde{\tau})\right)\right\}}{s\left(AvS_j(\tilde{\tau})\right)}; \tag{14}$$

Where $s\left(\theta_{i_A i_C}(\tilde{\tau})\right)$ and $s\left(AvS_j(\tilde{\tau})\right)$ are score functions,

Step 5. Compute weighted sums *SP* and *SN* from *PDA* and *NDA*.

$$SP_{i_A}=\sum_{i_C=1}^{n_C}w_{i_C}PDA_{i_A i_C};\quad SN_{i_A}=\sum_{i_C=1}^{n_C}w_{i_C}NDA_{i_A i_C}; \tag{15}$$

Where w_{i_C} is the weight of the creteria i_C

Step 6. Normalize SP_{i_A}, SN_{i_A} to obtain NSP_{i_A}, NSN_{i_A}

$$NSP_{i_A}=\frac{SP_{i_A}}{\max\limits_{i_A}\left\{SP_{i_A}\right\}};\ NSN_{i_A}=\frac{SN_{i_A}}{\max\limits_{i_A}\left\{SN_{i_A}\right\}}; \tag{16}$$

Step 7. Compute the score values S_{i_A}; $i_{A=1,2,3,\ldots,m}$ for all alternatives.

$$S_{i_A}=\frac{1}{2}\left(NSP_{i_A}+NSN_{i_A}\right); \tag{17}$$

Step 8. Ranking the alternatives according to the values of S_{i_A}

6. Application

6.1. The case study

In this study, we adopt an illustrative case based on Lan et al.'s work (2023) [20] to exemplify the practicality and efficiency of our proposed MCDM. The scenario we examine involves a company's decision process in selecting a tourist destination for the next summer within Vietnam. After initial pre-evaluation, the company selected set of candidates that includes five potential locations in

example for evaluation. Any destination is judged according to a set of features that present the acttractiveness of a destination to traveller. These features are considered as criteria for decision maker in MCDM. To enhance the precision of the assessment, we identify the top twenty crucial criteria, that were proved to be best to Vietnamese tourist, which are as follows in Figure 1.

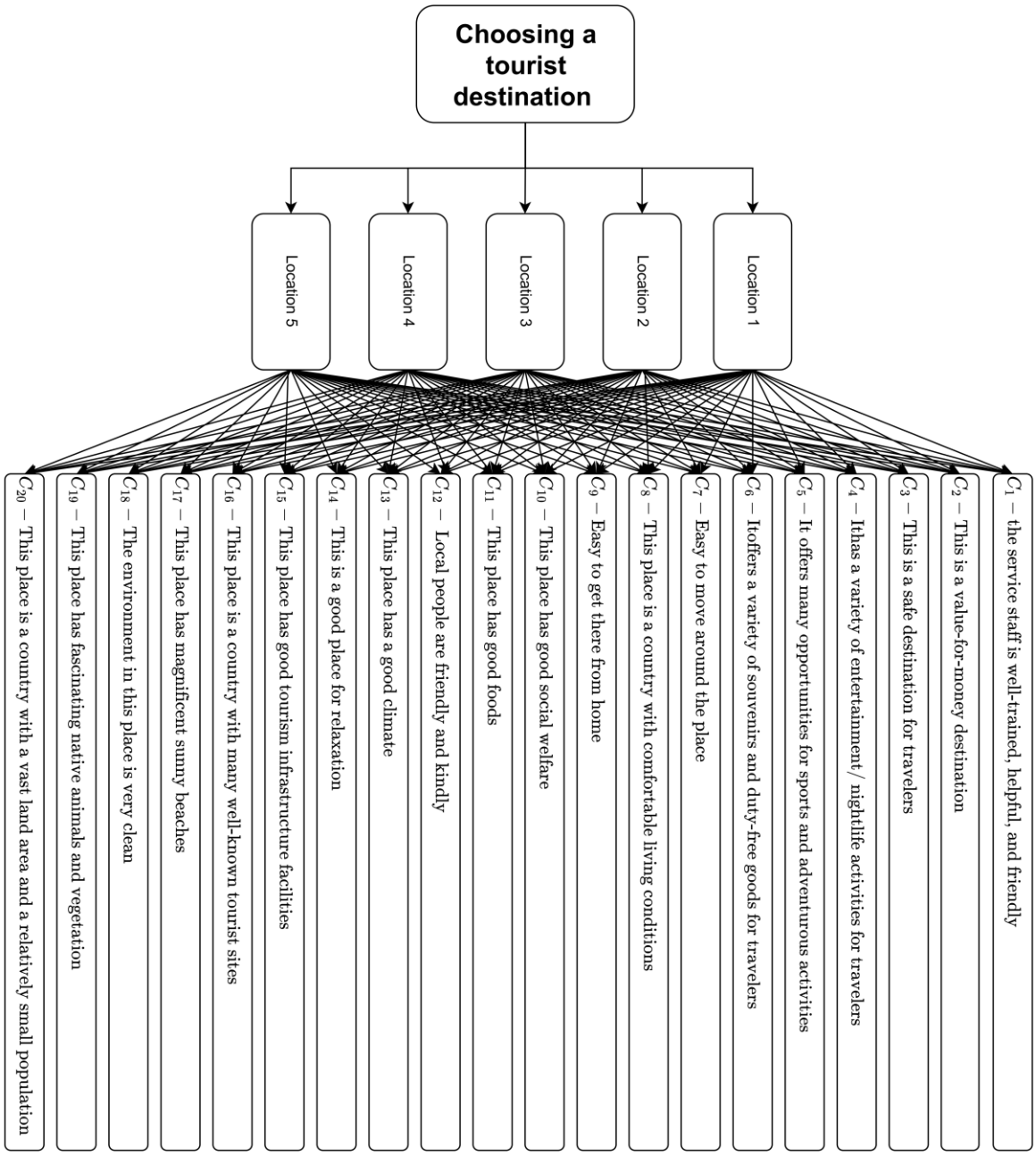
Choosing a tourist destination
Location 5
Location 4
Location 3
Location 2
Location 1
C_{20} – This place is a country with a vast land area and a relatively small population
C_{19} – This place has fascinating native animals and vegetation
C_{18} – The environment in this place is very clean
C_{17} – This place has magnificent sunny beaches
C_{16} – This place is a country with many well-known tourist sites
C_{15} – This place has good tourism infrastructure facilities
C_{14} – This is a good place for relaxation
C_{13} – This place has a good climate
C_{12} – Local people are friendly and kindly
C_{11} – This place has good foods
C_{10} – This place has good social welfare
C_9 – Easy to get there from home
C_8 – This place is a country with comfortable living conditions
C_7 – Easy to move around the place
C_6 – Itoffers a variety of souvenirs and duty-free goods for travelers
C_5 – It offers many opportunities for sports and adventurous activities
C_4 – Ithas a variety of entertainment/ nightlife activities for travelers
C_3 – This is a safe destination for travelers
C_2 – This is a value-for-money destination
C_1 – the service staff is well-trained, helpful, and friendly

Figure 1. Tourist Destination Choice Problem

6.2. Address MCDM

The input data of this study is similar to the work presented in [20], and the new MCDM on TCNS has some remarkable steps as follow.

(1) We consider 20 criteria as benefit criteria within the experimental scope of the article

(2) Compute the value of the average solution using equation (12) with $\theta = 5$, the results of AvS shows in Table 1.

Table 1. The results of AvS

Criteria	AvS	Criteria	AvS	Criteria	AvS	Criteria	AvS
C_1	0.85544	C_6	0.86562	C_{11}	0.86694	C_{16}	0.87568
C_2	0.86356	C_7	0.85077	C_{12}	0.85654	C_{17}	0.87558
C_3	0.87216	C_8	0.86154	C_{13}	0.86779	C_{18}	0.86487
C_4	0.85742	C_9	0.84721	C_{14}	0.84973	C_{19}	0.86283
C_5	0.84606	C_{10}	0.85732	C_{15}	0.85001	C_{20}	0.86655

(3) Compute the positive/negative distance from average using equation (14). Table 2 shows the results of PDA and NDA for 5 locations.

Table 2. The results of PDA and NDA

Criteria	L_1		L_2		L_3		L_4		L_5	
	PDA	*NDA*	*PDA*	*NDA*	*PDA*	*NDA*	*PDA*	*NDA*	*PDA*	*NDA*
C_1	0.03178	0.00000	0.01413	0.00000	0.03154	0.00000	0.03019	0.00000	0.03019	0.00000
C_2	0.03004	0.00000	0.01474	0.00000	0.02207	0.00000	0.02050	0.00000	0.00735	0.00000
C_3	0.01200	0.00000	0.00000	0.00349	0.00674	0.00000	0.00000	0.00007	0.01634	0.00000
C_4	0.00413	0.00000	0.00000	0.04281	0.02970	0.00000	0.02780	0.00000	0.02780	0.00000
C_5	0.02819	0.00000	0.02819	0.00000	0.04161	0.00000	0.00000	0.02652	0.04161	0.00000
C_6	0.01964	0.00000	0.00000	0.04852	0.01765	0.00000	0.01109	0.00000	0.01807	0.00000
C_7	0.03174	0.00000	0.00590	0.00000	0.03541	0.00000	0.02874	0.00000	0.02874	0.00000
C_8	0.02288	0.00000	0.00806	0.00000	0.02247	0.00000	0.02288	0.00000	0.02288	0.00000
C_9	0.05082	0.00000	0.05317	0.00000	0.03776	0.00000	0.03232	0.00000	0.02937	0.00000
C_{10}	0.03109	0.00000	0.01435	0.00000	0.02792	0.00000	0.03774	0.00000	0.03774	0.00000
C_{11}	0.01652	0.00000	0.00068	0.00000	0.02202	0.00000	0.00955	0.00000	0.01652	0.00000
C_{12}	0.02886	0.00000	0.02886	0.00000	0.02886	0.00000	0.02886	0.00000	0.02886	0.00000
C_{13}	0.01116	0.00000	0.00496	0.00000	0.01709	0.00000	0.00856	0.00000	0.01552	0.00000
C_{14}	0.04206	0.00000	0.04206	0.00000	0.03711	0.00000	0.03000	0.00000	0.03000	0.00000
C_{15}	0.03598	0.00000	0.02341	0.00000	0.03966	0.00000	0.02966	0.00000	0.02966	0.00000
C_{16}	0.00000	0.00659	0.00000	0.00052	0.01245	0.00000	0.00637	0.00000	0.00000	0.00659
C_{17}	0.01130	0.00000	0.00000	0.00040	0.00081	0.00000	0.00649	0.00000	0.00000	0.00647
C_{18}	0.02382	0.00000	0.01160	0.00000	0.01320	0.00000	0.01124	0.00000	0.01895	0.00000
C_{19}	0.02015	0.00000	0.00807	0.00000	0.01560	0.00000	0.02452	0.00000	0.02452	0.00000
C_{20}	0.01860	0.00000	0.00929	0.00000	0.02665	0.00000	0.01001	0.00000	0.01698	0.00000

(4) Calculate the values of SP_{i_A} and SN_{i_A} using equation (15). Table 3 shows results of SP and SN values for 5 locations.

Table 3. The results of SP and SN

Location	SP	SN
L_1	0.02354	0.00033
L_2	0.01337	0.00479
L_3	0.02432	0.00000
L_4	0.01883	0.00133
L_5	0.02205	0.00065

(5) Normalize SP_{i_A} and SN_{i_A} values to obtain NSP_{i_A} and NSN_{i_A} using equations (16). Using eqaution (17) to calculate score values and the results shows in Table 5.

Table 5. The results of NSP and NSN and S

Location	NSP	NSN	S	Ranking
L_1	0.96802	0.06886	0.51844	4
L_2	0.55002	1	0.77501	1
L_3	1	0	0.5	5

L_4	0.77425	0.27770	0.52598	2
L_5	0.90700	0.13648	0.52174	3

6.3. Comparison

Herein, to illustrate the impact of our novel operators, we compare our model's performance to Lan et al. [20]. Table 6 shows that the ranking of 5 locations are $L_2 \succ L_1 \succ L_3 \succ L_5 \succ L_4$ and L_2 is best location. Meanwhile, the result of the our model gives the raking $L_2 \succ L_4 \succ L_5 \succ L_1 \succ L_3$ and L_2 is best location. This result shows that the proposed model uses the Aczel-Alsina aggregation operators which works well in temporal complex neutrosophic environments.

Table 6. Compare table at ANP - TCNS - TOPSIS

Location	Score	Ranking
L_1	0.166891	2
L_2	0.175318	1
L_3	0.157247	3
L_4	0.145068	5
L_5	0.157223	4

7. Conclusions

Among other t-norm and t-conorm, the Aczel-Alsina t-norm and t-conorm have the advantage of being adjustable by changing a parameter. To take this advantage, this paper presents some novel aggregation operators of Temporal complex neutrosophic set (TCNS) based on Aczel-Alsina t-norm and t-conorm. A MCDM model utilizing the proposed TCNS aggregate operator is presented, this model is suitable for the domains in which time effect is matter and adjustable parameter of Aczel-Alsina t-norm and t-conorm can give benefit. Along with this theoretical model, to demonstration of the feasibility and soundness of addressing MCDM problems through the application of the proposed model, a real-world case study involving the selection of a tourist destination in Vietnam, accompanied by comparative analysis. The result of case study showed that proposed aggregation operators according to Aczel-Alsina t-norm and t-conorm for TCNS has pratical potential. However, in practical this result is limited in a case study and it can be evaluating in a larger scale. The future work of this study will focus on this aspect and will design a mechanism for tourist destination selection for wide range.

Funding: This research is funded by Thuongmai University, Hanoi, Vietnam.

Acknowledgments: The authors would like to thank Professor Florentin Smarandache for valuable comments and suggestions.

Conflicts of Interest: The authors declare no conflict of interest.

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Received: March 20, 2024. Accepted: July 28, 2024

Neutrosophic Sets and Systems, Vol. 70, 2024
University of New Mexico

Fixed Point Results in Neutrosophic b- Metric Like Spaces

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Abstract: In this manuscript we introduce the idea of Neutrosophic b-metric like spaces along with numerous significant illustrations. Moreover, we present numerous fixed point results in Neutrosophic b-metric like space and we gave some examples to support our main results.

Keywords: b-metric like spaces; Neutrosophic b-metric like spaces; Fixed point; Contractive map; Unique solution.

1. Introduction

Zadeh [20], laid the foundation for fuzzy mathematics in 1965. Kramosil and Michalek[11] initially brought up the idea of fuzzy metric space and then refitted by George and Veeramani[4]. Harandi [5] imported the perception of metric-like spaces, which nicely broden the concept of metric spaces.*b*-MetricLike Spaces [*bMLS*] were first discussed by Alghamdiet al.[2] using the idea of metric like spaces. In this manner, Fuzzy Metric Like Space [*FMLS*] were developed by Shukla and Abbas [15] they also developed the concept of metric like space. Park [13] proposed the idea of intuitionistic fuzzy metric spaces.Konwar [10] developed the concept of intuitionistic fuzzy *b*-metric space . We see [1, 6, 12, 19, 16, 17] for certain required definitions. In the framework of *b*-metric spaces,Delfani et al. [3] demonstrated several fixed point results.In 1998, Smarandache[18] developed the ideas of neutrosophic logic and Neutrosophic Set [*NS*]. Kirisci and Simsek[9] founded the concept of Neutrosophic Metric Spaces [*NMS*]which addresess membership, non-membership and neutralness.

We introduce the notion of Netrosophic*b*-Metric Spaces [*NbMS*]in order to generalise the notion of *NbMS* and demonstrate several fixed point findings in this framework.We also furnish this work with examples.

2. Preliminaries

Definition 2.1. [2]A *bMLS* on a set $\mathfrak{S} \neq \emptyset$ is a function $\varphi : \mathfrak{S} \times \mathfrak{S} \to [0, +\infty)$ such that for all $\hbar, \mathfrak{q}, \mathfrak{r} \in \mathfrak{S}$ and $b \geq 1$, if it enjoys the conditions listed below :

1. If $\varphi(\hbar, \mathfrak{q}) = 0$, then $\hbar = \mathfrak{q}$;
2. $\varphi(\hbar, \mathfrak{q}) = \varphi(\mathfrak{q}, \hbar)$;
3. $\varphi(\hbar, \mathfrak{q}) \leq b[\varphi(\hbar, \mathfrak{r}) + \varphi(\mathfrak{r}, \mathfrak{q})]$.

The pair $(\mathfrak{S}, \varphi)$ is said to be *bMLS*.

Example 2.2.[2] Let$\mathfrak{S} = [0, \infty)$. Define $\varphi : \mathfrak{S} \times \mathfrak{S} \to [0, +\infty)$by $\varphi(\hbar, \mathfrak{q}) = (\hbar + \mathfrak{q})^2$. Then $(\mathfrak{S}, \varphi)$ is a *bMLS* with $b = 1$.

Definition 2.3. [5]A 3-tuple $(\mathfrak{S}, \mathfrak{A}, *)$ is named to be a*FMLS* if $\mathfrak{S} \neq \emptyset$ is a random set, $*$ is a *CTN* and and $\mathfrak{A}$ is a *FS* on $\mathfrak{S} \times \mathfrak{S} \times (0, \infty)$ such that for all $\hbar, \mathfrak{q}, \mathfrak{r} \in \mathfrak{S}$, $\vartheta, \omega > 0$,
$FL1).\ \mathfrak{A}(\hbar, \mathfrak{q}, \vartheta) > 0$;
$FL2).$ If $\mathfrak{A}(\hbar, \mathfrak{q}, \vartheta) = 1$, then $\hbar = \mathfrak{q}$;
$FL3).\ \mathfrak{A}(\hbar, \mathfrak{q}, \vartheta) = \mathfrak{A}(\mathfrak{q}, \hbar, \vartheta)$;
$FL4).\ \mathfrak{A}(\hbar, \mathfrak{r}, \vartheta + \omega) \geq \mathfrak{A}(\hbar, \mathfrak{q}, \vartheta) * \mathfrak{A}(\mathfrak{q}, \mathfrak{r}, \omega)$;
$FL5).\ \mathfrak{A}(\hbar, \mathfrak{q}, \cdot): (0, \infty) \to [0,1]$is continuous.

Example 2.4.[15] Let $\mathfrak{S} = \mathbb{R}^+, \rho \in \mathbb{R}^+$and $m > 0$. Define CTN by $\hbar * \mathfrak{q} = \hbar\mathfrak{q}$ and $\mathfrak{A}$ by $\mathfrak{A}(\hbar, \mathfrak{q}, \vartheta) = \frac{\rho\vartheta}{\rho\vartheta + m(\max\{\hbar, \mathfrak{q}\})}$ for all $\hbar, \mathfrak{q} \in \mathfrak{S}, \vartheta > 0$. Then $(\mathfrak{S}, \mathfrak{A}, *)$ is an *FMLS*.

Definition 2.5. Let $\mathfrak{S} \neq \emptyset$. For a sixtuple $(\mathfrak{S}, \mathfrak{A}_{b\ell}, \mathfrak{B}_{b\ell}, \mathfrak{C}_{b\ell}, *, \diamond)$, where $*$ is a *CTN*, $\diamond$ is a *CTCN*, $b \geq 1$ and$\mathfrak{A}_{b\ell}, \mathfrak{B}_{b\ell}, \mathfrak{C}_{b\ell}$ are FS on $\mathfrak{S} \times \mathfrak{S} \times (0, \infty)$, if$(\mathfrak{S}, \mathfrak{A}_{b\ell}, \mathfrak{B}_{b\ell}, \mathfrak{C}_{b\ell}, *, \diamond)$ satisfies the following, for all $\hbar, \mathfrak{q} \in \mathfrak{S}$ and $\vartheta, \omega > 0$,

1. $0 \leq \mathfrak{A}_{b\ell}(\hbar, \mathfrak{q}, \vartheta) \leq 1; 0 \leq \mathfrak{B}_{b\ell}(\hbar, \mathfrak{q}, \vartheta) \leq 1; 0 \leq \mathfrak{C}_{b\ell}(\hbar, \mathfrak{q}, \vartheta) \leq 1$;
2. $\mathfrak{A}_{b\ell}(\hbar, \mathfrak{q}, \vartheta) + \mathfrak{B}_{b\ell}(\hbar, \mathfrak{q}, \vartheta) + \mathfrak{C}_{b}(\hbar, \mathfrak{q}, \vartheta) \leq 3$;
3. $\mathfrak{A}_{b\ell}(\hbar, \mathfrak{q}, \vartheta) = 1 \Leftrightarrow \hbar = \mathfrak{q}$;
4. $\mathfrak{A}_{b\ell}(\hbar, \mathfrak{q}, \vartheta) = \mathfrak{A}_{b\ell}(\mathfrak{q}, \hbar, \vartheta)$;
5. $\mathfrak{A}_{b\ell}(\hbar, \mathfrak{r}, b(\vartheta + \omega)) \geq \mathfrak{A}_{b\ell}(\hbar, \mathfrak{q}, \vartheta) * \mathfrak{A}_{b\ell}(\mathfrak{q}, \mathfrak{r}, \omega)$;
6. $\mathfrak{A}_{b\ell}(\hbar, \mathfrak{q}, .) : [0, \infty) \to [0,1]$ is neutrosophic continuous;
7. $\lim_{\vartheta \to \infty} \mathfrak{A}_{b\ell}(\hbar, \mathfrak{q}, \vartheta) = 1$;
8. $\mathfrak{B}_{b\ell}(\hbar, \mathfrak{q}, \vartheta) = 0 \Leftrightarrow \hbar = \mathfrak{q}$;
9. $\mathfrak{B}_{b\ell}(\hbar, \mathfrak{q}, \vartheta) = \mathfrak{B}_{b\ell}(\mathfrak{q}, \hbar, \vartheta)$;
10. $\mathfrak{B}_{b\ell}(\hbar, \mathfrak{r}, b(\vartheta + \omega)) \leq \mathfrak{B}_{b\ell}(\hbar, \mathfrak{q}, \vartheta) \diamond \mathfrak{B}_{b\ell}(\mathfrak{q}, \mathfrak{r}, \omega)$;
11. $\mathfrak{B}_{b\ell}(\hbar, \mathfrak{q}, .) : [0, \infty) \to [0,1]$ is neutrosophic continuous;
12. $\lim_{\vartheta \to \infty} \mathfrak{B}_{b\ell}(\hbar, \mathfrak{q}, \vartheta) = 0$;
13. $\mathfrak{C}_{b\ell}(\hbar, \mathfrak{q}, \vartheta) = 0 \Leftrightarrow \hbar = \mathfrak{q}$;
14. $\mathfrak{C}_{b\ell}(\hbar, \mathfrak{q}, \vartheta) = \mathfrak{C}_{b\ell}(\mathfrak{q}, \hbar, \vartheta)$;
15. $\mathfrak{C}_{b\ell}(\hbar, \mathfrak{r}, b(\vartheta + \omega)) \leq \mathfrak{C}_{b\ell}(\hbar, \mathfrak{q}, \vartheta) \diamond \mathfrak{C}_{b\ell}(\mathfrak{q}, \mathfrak{r}, \omega)$;
16. $\mathfrak{C}_{b\ell}(\hbar, \mathfrak{q}, .) : [0, \infty) \to [0,1]$ is neutrosophic continuous;
17. $\lim_{\vartheta \to \infty} \mathfrak{C}_{b\ell}(\hbar, \mathfrak{q}, \vartheta) = 0$;
18. $\vartheta < 0$ then $\mathfrak{A}_{b\ell}(\hbar, \mathfrak{q}, \vartheta) = 0, \mathfrak{B}_{b\ell}(\hbar, \mathfrak{q}, \vartheta) = 1, \mathfrak{C}_{b\ell}(\hbar, \mathfrak{q}, \vartheta) = 1$.

Then $(\mathfrak{S}, \mathfrak{A}_{b\ell}, \mathfrak{B}_{b\ell}, \mathfrak{C}_{b\ell}, *, \diamond)$ is called a*NbMLS*.

Remark 2.6. In the above definition, assume that a set $\mathfrak{S}$ is a *NbMLS* with a *CTN*$(*)$ and *CTCN*$(\diamond)$. Then the *NbMLS*$\mathfrak{S}$ does not satisfy (3),(8) and (13) of *NbMS*, that is, the self-distance may not be equal to 1 and 0, i.e., $\mathfrak{A}_{b\ell}(\hbar, \hbar, \vartheta) \neq 1$, $\mathfrak{B}_{b\ell}(\hbar, \hbar, \vartheta) \neq 0$ and $\mathfrak{C}_{b\ell}(\hbar, \hbar, \vartheta) \neq 0$ for all $\vartheta > 0$ or may be for all $\vartheta \in \mathfrak{S}$. But all other conditions are the same.

Example 2.7. Let $\mathfrak{S} = (0, \infty)$. Define a *CTN* by $\mathfrak{u} * \mathfrak{v} = \mathfrak{u}\mathfrak{v}$ and a *CTCN* by $\mathfrak{u} \diamond \mathfrak{v} = \max\{\mathfrak{u}, \mathfrak{v}\}$ and also define $\mathfrak{A}_{b\ell}, \mathfrak{B}_{b\ell}$ and $\mathfrak{C}_{b\ell}$by

$$\mathfrak{A}_{b\ell}(\hbar, \mathfrak{q}, \vartheta) = \left[e^{\frac{(\hbar+\mathfrak{q})^2}{\vartheta}}\right]^{-1}, \mathfrak{B}_{b\ell}(\hbar, \mathfrak{q}, \vartheta) = 1 - \left[e^{\frac{(\hbar+\mathfrak{q})^2}{\vartheta}}\right]^{-1} \text{ and } \mathfrak{C}_{b\ell}(\hbar, \mathfrak{q}, \vartheta) = \left[e^{\frac{(\hbar+\mathfrak{q})^2}{\vartheta}}\right] - 1$$

for all $\hbar, \mathfrak{q} \in \mathfrak{S}, \vartheta > 0$. Then it is a*NbMLS*. But it is not a*NbMS*.

Remark 2.8 The above example shows that *NbMLS* need not be an *NbMS*. Also every *NbMS* must be an *NbMLS*.
The following example shows that *NbMLS* need not be continuous.

Example 2.9 Let $= [0,\infty)$, $\mathfrak{A}_{\mathcal{b}\ell}(\hbar,\mathfrak{q},\vartheta) = e^{-\frac{\varphi(\hbar+\mathfrak{q})}{\vartheta}}$, $\mathfrak{B}_{\mathcal{b}\ell}(\hbar,\mathfrak{q},\vartheta) = 1-\left(e^{-\frac{\varphi(\hbar+\mathfrak{q})}{\vartheta}}\right)$ and $\mathfrak{C}_{\mathcal{b}\ell}(\hbar,\mathfrak{q},\vartheta) = e^{\frac{\varphi(\hbar+\mathfrak{q})}{\vartheta}} - 1$ for all $\hbar,\mathfrak{q} \in \mathfrak{S}, \vartheta > 0$ and $\varphi(\hbar,\mathfrak{q}) = \begin{cases} 0, & \text{if } \hbar = \mathfrak{q} \\ 2(\hbar+\mathfrak{q})^2, & \text{if } \hbar,\mathfrak{q} \in [0,1] \\ \frac{1}{2}(\hbar+\mathfrak{q})^2, & \text{otherwise.} \end{cases}$

Define a *CTN* by $\mathfrak{u} * \mathfrak{v} = \mathfrak{u}\mathfrak{v}$ and a *CTCN*by $\mathfrak{u} \diamond \mathfrak{v} = \max\{\mathfrak{u},\mathfrak{v}\}$. Then $(\mathfrak{S},\mathfrak{A}_{\mathcal{b}\ell},\mathfrak{B}_{\mathcal{b}\ell},\mathfrak{C}_{\mathcal{b}\ell},*,\diamond)$ is a *NbMLS* with a coefficient *b*=4. To illustrate the discontinuity, we have

$$\lim_{ï\to\infty}\mathfrak{A}_{\mathcal{b}\ell}\left(0,\frac{1}{ï},\vartheta\right) = \lim_{ï\to\infty}e^{-2(1-\frac{1}{ï})^2} = \mathrm{e}^{-2} = \mathfrak{A}_{\mathcal{b}\ell}(0,1,\vartheta)$$
$$\lim_{ï\to\infty}\mathfrak{B}_{\mathcal{b}\ell}\left(0,\frac{1}{ï},\vartheta\right) = 1-\lim_{ï\to\infty}e^{-2(1-\frac{1}{ï})^2} = 1-\mathrm{e}^{-2} = \mathfrak{B}_{\mathcal{b}\ell}(0,1,\vartheta)\text{and}$$
$$\lim_{ï\to\infty}\mathfrak{C}_{\mathcal{b}\ell}\left(0,\frac{1}{ï},\vartheta\right)=\lim_{ï\to\infty}e^{2(1-\frac{1}{ï})^2} - 1 = \mathrm{e}^2 - 1 = \mathfrak{C}_{\mathcal{b}\ell}(0,1,\vartheta)$$

Since,

$$\lim_{ï\to\infty}\mathfrak{A}_{\mathcal{b}\ell}\left(1,1-\frac{1}{ï},\vartheta\right) = \lim_{ï\to\infty}e^{-2(2-\frac{1}{ï})^2} = \mathrm{e}^{-8} \neq 1 = \mathfrak{A}_{\mathcal{b}\ell}(1,1,\vartheta)$$
$$\lim_{ï\to\infty}\mathfrak{B}_{\mathcal{b}\ell}\left(1,1-\frac{1}{ï},\vartheta\right) = 1-\lim_{ï\to\infty}e^{-2(2-\frac{1}{ï})^2} = 1-\mathrm{e}^{-8} \neq 0 = \mathfrak{B}_{\mathcal{b}\ell}(1,1,\vartheta)$$
$$\lim_{ï\to\infty}\mathfrak{C}_{\mathcal{b}\ell}\left(1,1-\frac{1}{ï},\vartheta\right)=\lim_{ï\to\infty}e^{2(2-\frac{1}{ï})^2} - 1 = \mathrm{e}^8 - 1 \neq 0 = \mathfrak{C}_{\mathcal{b}\ell}(0,1,\vartheta)$$

$\mathfrak{A}_{\mathcal{b}\ell}(\hbar,\mathfrak{q},\vartheta)$, $\mathfrak{B}_{\mathcal{b}\ell}(\hbar,\mathfrak{q},\vartheta)$ and $\mathfrak{C}_{\mathcal{b}\ell}(\hbar,\mathfrak{q},\vartheta)$ are not continuous.

Definition 2.10. A sequence $\{\hbar_ï\}$ in an *NbMLS*$(\mathfrak{S},\mathfrak{A}_{\mathcal{b}\ell},\mathfrak{B}_{\mathcal{b}\ell},\mathfrak{C}_{\mathcal{b}\ell},*,\diamond)$ is converge to$\hbar \in \mathfrak{S}$if $\lim_{ï\to\infty}\mathfrak{A}_{\mathcal{b}\ell}(\hbar_ï,\hbar,\vartheta) = \mathfrak{A}_{\mathcal{b}\ell}(\hbar,\hbar,\vartheta)$,$\lim_{ï\to\infty}\mathfrak{B}_{\mathcal{b}\ell}(\hbar_ï,\hbar,\vartheta) = \mathfrak{B}_{\mathcal{b}\ell}(\hbar,\hbar,\vartheta)$ and $\lim_{ï\to\infty}\mathfrak{C}_{\mathcal{b}\ell}(\hbar_ï,\hbar,\vartheta) = \mathfrak{C}_{\mathcal{b}\ell}(\hbar,\hbar,\vartheta)$ for all $\vartheta > 0$.

Definition 2.11.A sequence $\{\hbar_ï\}$ in an*NbMLS*$(\mathfrak{S},\mathfrak{A}_{\mathcal{b}\ell},\mathfrak{B}_{\mathcal{b}\ell},\mathfrak{C}_{\mathcal{b}\ell},*,\diamond)$ is named to be a Cauchy sequence if $\lim_{ï\to\infty}\mathfrak{A}_{\mathcal{b}\ell}(\hbar_ï,\hbar_{ï+p},\vartheta)$,$\lim_{ï\to\infty}\mathfrak{B}_{\mathcal{b}\ell}(\hbar_ï,\hbar_{ï+p},\vartheta)$ and $\lim_{ï\to\infty}\mathfrak{C}_{\mathcal{b}\ell}(\hbar_ï,\hbar_{ï+p},\vartheta)$exist and are finite for all $\vartheta \geq 0, p \geq 1$.

Definition 2.12. An *NbMLS*$(\mathfrak{S},\mathfrak{A}_{\mathcal{b}\ell},\mathfrak{B}_{\mathcal{b}\ell},\mathfrak{C}_{\mathcal{b}\ell},*,\diamond)$ is named to be complete if every Cauchy sequence$\{\hbar_ï\}$ in $\mathfrak{S}$ tends to some $\hbar \in \mathfrak{S}$ such that $\lim_{ï\to\infty}\mathfrak{A}_{\mathcal{b}\ell}(\hbar_ï,\hbar,\vartheta) = \mathfrak{A}_{\mathcal{b}\ell}(\hbar,\hbar,\vartheta) = \lim_{ï\to\infty}\mathfrak{A}_{\mathcal{b}\ell}(\hbar_ï,\hbar_{ï+p},\vartheta)$
$\lim_{ï\to\infty}\mathfrak{B}_{\mathcal{b}\ell}(\hbar_ï,\hbar,\vartheta) = \mathfrak{B}_{\mathcal{b}\ell}(\hbar,\hbar,\vartheta) = \lim_{ï\to\infty}\mathfrak{B}_{\mathcal{b}\ell}(\hbar_ï,\hbar_{ï+p},\vartheta)$and
$\lim_{ï\to\infty}\mathfrak{C}_{\mathcal{b}\ell}(\hbar_ï,\hbar,\vartheta) = \mathfrak{C}_{\mathcal{b}\ell}(\hbar,\hbar,\vartheta) = \lim_{ï\to\infty}\mathfrak{C}_{\mathcal{b}\ell}(\hbar_ï,\hbar_{ï+p},\vartheta)$for all $\vartheta \geq 0, p \geq 1$.

3. Main Results.

Theorem 3.1. Let $(\mathfrak{S},\mathfrak{A}_{\mathcal{b}\ell},\mathfrak{B}_{\mathcal{b}\ell},\mathfrak{C}_{\mathcal{b}\ell},*,\diamond)$be a complete *NbMLS* such that $\lim_{\vartheta\to\infty}\mathfrak{A}_{\mathcal{b}\ell}(\hbar,\mathfrak{q},\vartheta) = 1, \lim_{\vartheta\to\infty}\mathfrak{B}_{\mathcal{b}\ell}(\hbar,\mathfrak{q},\vartheta) = 0$ and $\lim_{\vartheta\to\infty}\mathfrak{C}_{\mathcal{b}\ell}(\hbar,\mathfrak{q},\vartheta) = 0$for all $\hbar,\mathfrak{q} \in \mathfrak{S}, \vartheta > 0$ and $\mathcal{J}: \mathfrak{S} \to \mathfrak{S}$ be a mapping satisfying the conditions
$\mathfrak{A}_{\mathcal{b}\ell}(\mathcal{J}\hbar,\mathcal{J}\mathfrak{q},\tau\vartheta) \geq \mathfrak{A}_{\mathcal{b}\ell}(\hbar,\mathfrak{q},\vartheta)$, $\mathfrak{B}_{\mathcal{b}\ell}(\mathcal{J}\hbar,\mathcal{J}\mathfrak{q},\tau\vartheta) \leq \mathfrak{B}_{\mathcal{b}\ell}(\hbar,\mathfrak{q},\vartheta)and\mathfrak{C}_{\mathcal{b}\ell}(\mathcal{J}\hbar,\mathcal{J}\mathfrak{q},\tau\vartheta) \leq \mathfrak{C}_{\mathcal{b}\ell}(\hbar,\mathfrak{q},\vartheta)$(3.1.1)
for all $\hbar,\mathfrak{q} \in \mathfrak{S}, \vartheta > 0$, $\tau \in (0,1)$. Then $\mathfrak{S}$ has a unique fixed point $\mathfrak{d} \in \mathfrak{S}$ and $\mathfrak{A}_{\mathcal{b}\ell}(\mathfrak{d},\mathfrak{d},\vartheta) = 1, \mathfrak{B}_{\mathcal{b}\ell}(\mathfrak{d},\mathfrak{d},\vartheta) = 0$ and $\mathfrak{C}_{\mathcal{b}\ell}(\mathfrak{d},\mathfrak{d},\vartheta) = 0$ for all $\vartheta > 0$.

Proof: Let $(\mathfrak{S},\mathfrak{A}_{\mathcal{b}\ell},\mathfrak{B}_{\mathcal{b}\ell},\mathfrak{C}_{\mathcal{b}\ell},*,\diamond)$ be a complete *NbMLS*. For a given element $\hbar_0 \in \mathfrak{S}$, define a sequence $\{\hbar_ï\}$ in $\mathfrak{S}$ by

$$\hbar_1 = \mathcal{J}\hbar_0, \hbar_2 = \mathcal{J}^2\hbar = \mathcal{J}\hbar_1, \ldots, \hbar_ï = \mathcal{J}^ï\hbar_0 = \mathcal{J}\hbar_{ï-1} \text{ for each } ï \in \mathrm{N}$$

If $\hbar_ï = \hbar_{ï-1}$ for some $ï \in \mathrm{N}$, then $\hbar_ï$ is a fixed point of $\mathcal{J}$. We consider that $\hbar_ï \neq \hbar_{ï-1}$ for each$ï \in \mathrm{N}$. For $\vartheta > 0$ and $ï \in \mathrm{N}$, we get from (3.1.1) that

$$\mathfrak{A}_{\mathcal{b}\ell}(\hbar_ï,\hbar_{ï+1},\vartheta) \geq \mathfrak{A}_{\mathcal{b}\ell}(\hbar_{ï+1},\hbar_ï,\tau\vartheta) = \mathfrak{A}_{\mathcal{b}\ell}(\mathcal{J}\hbar_ï,\mathcal{J}\hbar_{ï-1},\tau\vartheta) \geq \mathfrak{A}_{\mathcal{b}\ell}(\hbar_ï,\hbar_{ï-1},\vartheta)$$

$\mathfrak{B}_{\mathcal{b}\ell}(\hbar_{\ddot{\imath}},\hbar_{\ddot{\imath}+1},\vartheta)\leq\mathfrak{B}_{\mathcal{b}\ell}(\hbar_{\ddot{\imath}+1},\hbar_{\ddot{\imath}},\tau\vartheta)=\mathfrak{B}_{\mathcal{b}\ell}(\mathcal{J}\hbar_{\ddot{\imath}},\mathcal{J}\hbar_{\ddot{\imath}-1},\tau\vartheta)\leq\mathfrak{B}_{\mathcal{b}\ell}(\hbar_{\ddot{\imath}},\hbar_{\ddot{\imath}-1},\vartheta)$and

$\mathfrak{C}_{\mathcal{b}\ell}(\hbar_{\ddot{\imath}},\hbar_{\ddot{\imath}+1},\vartheta)\leq\mathfrak{C}_{\mathcal{b}\ell}(\hbar_{\ddot{\imath}+1},\hbar_{\ddot{\imath}},\tau\vartheta)=\mathfrak{C}_{\mathcal{b}\ell}(\mathcal{J}\hbar_{\ddot{\imath}},\mathcal{J}\hbar_{\ddot{\imath}-1},\tau\vartheta)\leq\mathfrak{C}_{\mathcal{b}\ell}(\hbar_{\ddot{\imath}},\hbar_{\ddot{\imath}-1},\vartheta)$

for all $\ddot{\imath}\in\mathrm{N}$ and $\vartheta>0$. Thus, by using the above expression we can deduce that

$$\begin{aligned}\mathfrak{A}_{\mathcal{b}\ell}(\hbar_{\ddot{\imath}+1},\hbar_{\ddot{\imath}},\vartheta)&\geq\mathfrak{A}_{\mathcal{b}\ell}(\hbar_{\ddot{\imath}+1},\hbar_{\ddot{\imath}},\tau\vartheta)=\mathfrak{A}_{\mathcal{b}\ell}(\mathcal{J}\hbar_{\ddot{\imath}},\mathcal{J}\hbar_{\ddot{\imath}-1},\tau\vartheta)\geq\mathfrak{A}_{\mathcal{b}\ell}(\hbar_{\ddot{\imath}},\hbar_{\ddot{\imath}-1},\vartheta)\qquad(3.1.2)\\&=\mathfrak{A}_{\mathcal{b}\ell}(\mathcal{J}\hbar_{\ddot{\imath}-1},\mathcal{J}\hbar_{\ddot{\imath}-2},\vartheta)\geq\mathfrak{A}_{\mathcal{b}\ell}\left(\hbar_{\ddot{\imath}-1},\hbar_{\ddot{\imath}-2},\frac{\vartheta}{\tau}\right)\geq\cdots\geq\mathfrak{A}_{\mathcal{b}\ell}\left(\hbar_1,\hbar_0,\frac{\vartheta}{\tau^{\ddot{\imath}}}\right)\\\mathfrak{B}_{\mathcal{b}\ell}(\hbar_{\ddot{\imath}+1},\hbar_{\ddot{\imath}},\vartheta)&\leq\mathfrak{B}_{\mathcal{b}\ell}(\hbar_{\ddot{\imath}+1},\hbar_{\ddot{\imath}},\tau\vartheta)=\mathfrak{B}_{\mathcal{b}\ell}(\mathcal{J}\hbar_{\ddot{\imath}},\mathcal{J}\hbar_{\ddot{\imath}-1},\tau\vartheta)\leq\mathfrak{B}_{\mathcal{b}\ell}(\hbar_{\ddot{\imath}},\hbar_{\ddot{\imath}-1},\vartheta)(3.1.3)\\&=\mathfrak{B}_{\mathcal{b}\ell}(\mathcal{J}\hbar_{\ddot{\imath}-1},\mathcal{J}\hbar_{\ddot{\imath}-2},\vartheta)\leq\mathfrak{B}_{\mathcal{b}\ell}\left(\hbar_{\ddot{\imath}-1},\hbar_{\ddot{\imath}-2},\frac{\vartheta}{\tau}\right)\leq\cdots\leq\mathfrak{B}_{\mathcal{b}\ell}\left(\hbar_1,\hbar_0,\frac{\vartheta}{\tau^{\ddot{\imath}}}\right)\quad\text{and}\\\mathfrak{C}_{\mathcal{b}\ell}(\hbar_{\ddot{\imath}+1},\hbar_{\ddot{\imath}},\vartheta)&\leq\mathfrak{C}_{\mathcal{b}\ell}(\hbar_{\ddot{\imath}+1},\hbar_{\ddot{\imath}},\tau\vartheta)=\mathfrak{C}_{\mathcal{b}\ell}(\mathcal{J}\hbar_{\ddot{\imath}},\mathcal{J}\hbar_{\ddot{\imath}-1},\tau\vartheta)\leq\mathfrak{C}_{\mathcal{b}\ell}(\hbar_{\ddot{\imath}},\hbar_{\ddot{\imath}-1},\vartheta)\qquad(3.1.4)\\&=\mathfrak{C}_{\mathcal{b}\ell}(\mathcal{J}\hbar_{\ddot{\imath}-1},\mathcal{J}\hbar_{\ddot{\imath}-2},\vartheta)\leq\mathfrak{C}_{\mathcal{b}\ell}\left(\hbar_{\ddot{\imath}-1},\hbar_{\ddot{\imath}-2},\frac{\vartheta}{\tau}\right)\leq\cdots\leq\mathfrak{C}_{\mathcal{b}\ell}\left(\hbar_1,\hbar_0,\frac{\vartheta}{\tau^{\ddot{\imath}}}\right)\end{aligned}$$

foreach $\ddot{\imath}\in\mathrm{N},p\geq1$ and $\vartheta>0$. Thus we have

$$\begin{aligned}\mathfrak{A}_{\mathcal{b}\ell}(\hbar_{\ddot{\imath}},\hbar_{\ddot{\imath}+p},\vartheta)&\geq\mathfrak{A}_{\mathcal{b}\ell}\left(\hbar_{\ddot{\imath}},\hbar_{\ddot{\imath}+1},\frac{\vartheta}{b}\right)*\mathfrak{A}_{\mathcal{b}\ell}\left(\hbar_{\ddot{\imath}+1},\hbar_{\ddot{\imath}+p},\frac{\vartheta}{b}\right)\\\mathfrak{B}_{\mathcal{b}\ell}(\hbar_{\ddot{\imath}},\hbar_{\ddot{\imath}+p},\vartheta)&\leq\mathfrak{B}_{\mathcal{b}\ell}\left(\hbar_{\ddot{\imath}},\hbar_{\ddot{\imath}+1},\frac{\vartheta}{b}\right)\diamond\mathfrak{B}_{\mathcal{b}\ell}\left(\hbar_{\ddot{\imath}+1},\hbar_{\ddot{\imath}+p},\frac{\vartheta}{b}\right)\\\mathfrak{C}_{\mathcal{b}\ell}(\hbar_{\ddot{\imath}},\hbar_{\ddot{\imath}+p},\vartheta)&\leq\mathfrak{C}_{\mathcal{b}\ell}\left(\hbar_{\ddot{\imath}},\hbar_{\ddot{\imath}+1},\frac{\vartheta}{b}\right)\diamond\mathfrak{C}_{\mathcal{b}\ell}\left(\hbar_{\ddot{\imath}+1},\hbar_{\ddot{\imath}+p},\frac{\vartheta}{b}\right)\end{aligned}$$

Continuing in this way, we get

$$\begin{aligned}\mathfrak{A}_{\mathcal{b}\ell}(\hbar_{\ddot{\imath}},\hbar_{\ddot{\imath}+p},\vartheta)&\geq\mathfrak{A}_{\mathcal{b}\ell}\left(\hbar_{\ddot{\imath}},\hbar_{\ddot{\imath}+1},\frac{\vartheta}{b}\right)*\mathfrak{A}_{\mathcal{b}\ell}\left(\hbar_{\ddot{\imath}+1},\hbar_{\ddot{\imath}+2},\frac{\vartheta}{b^2}\right)*\cdots*\mathfrak{A}_{\mathcal{b}\ell}\left(\hbar_{\ddot{\imath}+p-1},\hbar_{\ddot{\imath}+p},\frac{\vartheta}{b^{p-1}}\right)\\\mathfrak{B}_{\mathcal{b}\ell}(\hbar_{\ddot{\imath}},\hbar_{\ddot{\imath}+p},\vartheta)&\leq\mathfrak{B}_{\mathcal{b}\ell}\left(\hbar_{\ddot{\imath}},\hbar_{\ddot{\imath}+1},\frac{\vartheta}{b}\right)\diamond\mathfrak{B}_{\mathcal{b}\ell}\left(\hbar_{\ddot{\imath}+1},\hbar_{\ddot{\imath}+2},\frac{\vartheta}{b^2}\right)\diamond\cdots\diamond\mathfrak{B}_{\mathcal{b}\ell}\left(\hbar_{\ddot{\imath}+p-1},\hbar_{\ddot{\imath}+p},\frac{\vartheta}{b^{p-1}}\right)\text{and}\\\mathfrak{C}_{\mathcal{b}\ell}(\hbar_{\ddot{\imath}},\hbar_{\ddot{\imath}+p},\vartheta)&\leq\mathfrak{C}_{\mathcal{b}\ell}\left(\hbar_{\ddot{\imath}},\hbar_{\ddot{\imath}+1},\frac{\vartheta}{b}\right)\mathfrak{C}_{\mathcal{b}\ell}\left(\hbar_{\ddot{\imath}+1},\hbar_{\ddot{\imath}+2},\frac{\vartheta}{b^2}\right)\diamond\cdots\diamond\mathfrak{C}_{\mathcal{b}\ell}\left(\hbar_{\ddot{\imath}+p-1},\hbar_{\ddot{\imath}+p},\frac{\vartheta}{b^{p-1}}\right)\end{aligned}$$

Using (3.1.2), (3.1.3) and (3.1.4) in the above inequality, we have

$$\begin{aligned}\mathfrak{A}_{\mathcal{b}\ell}(\hbar_{\ddot{\imath}},\hbar_{\ddot{\imath}+p},\vartheta)&\geq\mathfrak{A}_{\mathcal{b}\ell}\left(\hbar_0,\hbar_1,\frac{\vartheta}{b\tau^{\ddot{\imath}}}\right)*\mathfrak{A}_{\mathcal{b}\ell}\left(\hbar_0,\hbar_1,\frac{\vartheta}{b^2\tau^{\ddot{\imath}+1}}\right)*\cdots*\mathfrak{A}_{\mathcal{b}\ell}\left(\hbar_0,\hbar_1,\frac{\vartheta}{b^{p-1}\tau^{\ddot{\imath}+p-1}}\right)(3.1.5)\\\mathfrak{B}_{\mathcal{b}\ell}(\hbar_{\ddot{\imath}},\hbar_{\ddot{\imath}+p},\vartheta)&\leq\mathfrak{B}_{\mathcal{b}\ell}\left(\hbar_0,\hbar_1,\frac{\vartheta}{b\tau^{\ddot{\imath}}}\right)\diamond\mathfrak{B}_{\mathcal{b}\ell}\left(\hbar_0,\hbar_1,\frac{\vartheta}{b^2\tau^{\ddot{\imath}+1}}\right)\diamond\cdots\diamond\mathfrak{B}_{\mathcal{b}\ell}\left(\hbar_0,\hbar_1,\frac{\vartheta}{b^{p-1}\tau^{\ddot{\imath}+p-1}}\right)\qquad(3.1.6)\end{aligned}$$

and

$$\mathfrak{C}_{\mathcal{b}\ell}(\hbar_{\ddot{\imath}},\hbar_{\ddot{\imath}+p},\vartheta)\leq\mathfrak{C}_{\mathcal{b}\ell}\left(\hbar_0,\hbar_1,\frac{\vartheta}{b\tau^{\ddot{\imath}}}\right)\diamond\mathfrak{C}_{\mathcal{b}\ell}\left(\hbar_0,\hbar_1,\frac{\vartheta}{b^2\tau^{\ddot{\imath}+1}}\right)\diamond\cdots\diamond\mathfrak{C}_{\mathcal{b}\ell}\left(\hbar_0,\hbar_1,\frac{\vartheta}{b^{p-1}\tau^{\ddot{\imath}+p-1}}\right)(3.1.7)$$

Here b is a random positive integer.

We know that $\lim_{\vartheta\to\infty}\mathfrak{A}_{\mathcal{b}\ell}(\hbar,\mathfrak{q},\vartheta)=1,\lim_{\vartheta\to\infty}\mathfrak{B}_{\mathcal{b}\ell}(\hbar,\mathfrak{q},\vartheta)=0$and $\lim_{\vartheta\to\infty}\mathfrak{C}_{\mathcal{b}\ell}(\hbar,\mathfrak{q},\vartheta)=0$ for all $\hbar,\mathfrak{q}\in\mathfrak{S}$ and $\vartheta>0,\tau\in(0,1)$. It follows from (3.1.5) ,(3.1.6) and (3.1.7) that for all $\vartheta>0,p\geq1$

$\lim_{\ddot{\imath}\to\infty}\mathfrak{A}_{\mathcal{b}\ell}(\hbar_{\ddot{\imath}},\hbar_{\ddot{\imath}+p},\vartheta)=1*1*\cdots*1=1$,

$\lim_{\ddot{\imath}\to\infty}\mathfrak{B}_{\mathcal{b}\ell}(\hbar_{\ddot{\imath}},\hbar_{\ddot{\imath}+p},\vartheta)=0\diamond0\diamond\cdots\diamond0=0$ and

$\lim_{\ddot{\imath}\to\infty}\mathfrak{C}_{\mathcal{b}\ell}(\hbar_{\ddot{\imath}},\hbar_{\ddot{\imath}+p},\vartheta)=0\diamond0\diamond\cdots\diamond0=0$

Hence $\{\hbar_{\ddot{\imath}}\}$ is a Cauchy sequence. The completeness of the $NbMLS$ $(\mathfrak{S},\mathfrak{A}_{\mathcal{b}\ell},\mathfrak{B}_{\mathcal{b}\ell},\mathfrak{C}_{\mathcal{b}\ell},*,\diamond)$ ensures that there exists $\mathfrak{d}\in\mathfrak{S}$ such thatfor all $\vartheta>0,p\geq1$

$$\lim_{\ddot{\imath}\to\infty}\mathfrak{A}_{\mathcal{b}\ell}(\hbar_{\ddot{\imath}},\mathfrak{d},\vartheta)=\lim_{\ddot{\imath}\to\infty}\mathfrak{A}_{\mathcal{b}\ell}(\hbar_{\ddot{\imath}},\hbar_{\ddot{\imath}+p},\vartheta)=\mathfrak{A}_{\mathcal{b}\ell}(\mathfrak{d},\mathfrak{d},\vartheta)=1\quad(3.1.8)$$

$$\lim_{\ddot{\imath}\to\infty}\mathfrak{B}_{\mathcal{b}\ell}(\hbar_{\ddot{\imath}},\mathfrak{d},\vartheta)=\lim_{\ddot{\imath}\to\infty}\mathfrak{B}_{\mathcal{b}\ell}(\hbar_{\ddot{\imath}},\hbar_{\ddot{\imath}+p},\vartheta)=\mathfrak{B}_{\mathcal{b}\ell}(\mathfrak{d},\mathfrak{d},\vartheta)=0\qquad(3.1.9)$$

$$\lim_{\ddot{\imath}\to\infty}\mathfrak{C}_{\mathcal{b}\ell}(\hbar_{\ddot{\imath}},\mathfrak{d},\vartheta)=\lim_{\ddot{\imath}\to\infty}\mathfrak{C}_{\mathcal{b}\ell}(\hbar_{\ddot{\imath}},\hbar_{\ddot{\imath}+p},\vartheta)=\mathfrak{C}_{\mathcal{b}\ell}(\mathfrak{d},\mathfrak{d},\vartheta)=0.\qquad(3.1.10)$$

Now, we show that $\mathfrak{d}\in\mathfrak{S}$ is a fixed point of $\mathcal{J}$. We have

$$\begin{aligned}\mathfrak{A}_{\mathcal{b}\ell}(\mathfrak{d},\mathcal{J}\mathfrak{d},\vartheta)&\geq\mathfrak{A}_{\mathcal{b}\ell}\left(\mathfrak{d},\hbar_{\ddot{\imath}+1},\frac{\vartheta}{2b}\right)*\mathfrak{A}_{\mathcal{b}\ell}\left(\hbar_{\ddot{\imath}+1},\mathcal{J}\mathfrak{d},\frac{\vartheta}{2b}\right)\\&=\mathfrak{A}_{\mathcal{b}\ell}\left(\mathfrak{d},\hbar_{\ddot{\imath}+1},\frac{\vartheta}{2b}\right)*\mathfrak{A}_{\mathcal{b}\ell}\left(\mathcal{J}\hbar_{\ddot{\imath}},\mathcal{J}\mathfrak{d},\frac{\vartheta}{2b}\right)\\&\geq\mathfrak{A}_{\mathcal{b}\ell}\left(\mathfrak{d},\hbar_{\ddot{\imath}+1},\frac{\vartheta}{2b}\right)*\mathfrak{A}_{\mathcal{b}\ell}\left(\hbar_{\ddot{\imath}},\mathfrak{d},\frac{\vartheta}{2b\tau}\right)\end{aligned}$$

$$\begin{aligned}
\mathfrak{B}_{\mathfrak{b}\ell}(\mathfrak{d},\mathcal{J}\mathfrak{d},\vartheta) &\leq \mathfrak{B}_{\mathfrak{b}\ell}\left(\mathfrak{d},\hbar_{\ddot{\imath}+1},\frac{\vartheta}{2b}\right)\diamond\mathfrak{B}_{\mathfrak{b}\ell}\left(\hbar_{\ddot{\imath}+1},\mathcal{J}\mathfrak{d},\frac{\vartheta}{2b}\right)\\
&= \mathfrak{B}_{\mathfrak{b}\ell}\left(\mathfrak{d},\hbar_{\ddot{\imath}+1},\frac{\vartheta}{2b}\right)\diamond\mathfrak{B}_{\mathfrak{b}\ell}\left(\mathcal{J}\hbar_{\ddot{\imath}},\mathcal{J}\mathfrak{d},\frac{\vartheta}{2b}\right)\text{ and}\\
&\leq \mathfrak{B}_{\mathfrak{b}\ell}\left(\mathfrak{d},\hbar_{\ddot{\imath}+1},\frac{\vartheta}{2b}\right)\diamond\mathfrak{B}_{\mathfrak{b}\ell}\left(\hbar_{\ddot{\imath}},\mathfrak{d},\frac{\vartheta}{2b\tau}\right)\\
\mathfrak{C}_{\mathfrak{b}\ell}(\mathfrak{d},\mathcal{J}\mathfrak{d},\vartheta) &\leq \mathfrak{C}_{\mathfrak{b}\ell}\left(\mathfrak{d},\hbar_{\ddot{\imath}+1},\frac{\vartheta}{2b}\right)\diamond\mathfrak{C}_{\mathfrak{b}\ell}\left(\hbar_{\ddot{\imath}+1},\mathcal{J}\mathfrak{d},\frac{\vartheta}{2b}\right)\\
&= \mathfrak{C}_{\mathfrak{b}\ell}\left(\mathfrak{d},\hbar_{\ddot{\imath}+1},\frac{\vartheta}{2b}\right)\diamond\mathfrak{C}_{\mathfrak{b}\ell}\left(\mathcal{J}\hbar_{\ddot{\imath}},\mathcal{J}\mathfrak{d},\frac{\vartheta}{2b}\right)\\
&\leq \mathfrak{C}_{\mathfrak{b}\ell}\left(\mathfrak{d},\hbar_{\ddot{\imath}+1},\frac{\vartheta}{2b}\right)\diamond\mathfrak{C}_{\mathfrak{b}\ell}\left(\hbar_{\ddot{\imath}},\mathfrak{d},\frac{\vartheta}{2b\tau}\right)
\end{aligned}$$

for each$\vartheta > 0$. Taking the limit as $\ddot{\imath} \to +\infty$, and by (3.1.8),(3.1.9) and (3.1.10), we get
$\mathfrak{A}_{\mathfrak{b}\ell}(\mathfrak{d},\mathcal{J}\mathfrak{d},\vartheta) = 1 * 1 = 1and\mathfrak{B}_{\mathfrak{b}\ell}(\mathfrak{d},\mathcal{J}\mathfrak{d},\vartheta) = 0 \diamond 0 = 0$ and $\mathfrak{C}_{\mathfrak{b}\ell}(\mathfrak{d},\mathcal{J}\mathfrak{d},\vartheta) = 0 \diamond 0 = 0$
Therefore, $\mathfrak{d}$ is a fixed point of $\mathcal{J}$and $\mathfrak{A}_{\mathfrak{b}\ell}(\mathfrak{d},\mathfrak{d},\vartheta) = 1$,$\mathfrak{B}_{\mathfrak{b}\ell}(\mathfrak{d},\mathfrak{d},\vartheta) = 0$and $\mathfrak{C}_{\mathfrak{b}\ell}(\mathfrak{d},\mathfrak{d},\vartheta) = 0$for all $\vartheta > 0$.
Now, we examine the uniqueness of fixed point. For this, assume that $\dot{\mathfrak{h}}$ and $\mathfrak{d}$ are two distinct fixed points of $\mathcal{J}$. Then by (3.1.1), we have

$$\mathfrak{A}_{\mathfrak{b}\ell}(\mathfrak{d},\dot{\mathfrak{h}},\vartheta) = \mathfrak{A}_{\mathfrak{b}\ell}(\mathcal{J}\mathfrak{d},\mathcal{J}\dot{\mathfrak{h}},\vartheta) \geq \mathfrak{A}_{\mathfrak{b}\ell}\left(\mathfrak{d},\dot{\mathfrak{h}},\frac{\vartheta}{\tau}\right)$$

$$\mathfrak{B}_{\mathfrak{b}\ell}(\mathfrak{d},\dot{\mathfrak{h}},\vartheta) = \mathfrak{B}_{\mathfrak{b}\ell}(\mathcal{J}\mathfrak{d},\mathcal{J}\dot{\mathfrak{h}},\vartheta) \leq \mathfrak{B}_{\mathfrak{b}\ell}\left(\mathfrak{d},\dot{\mathfrak{h}},\frac{\vartheta}{\tau}\right)\text{and}$$

$$\mathfrak{C}_{\mathfrak{b}\ell}(\mathfrak{d},\dot{\mathfrak{h}},\vartheta) = \mathfrak{C}_{\mathfrak{b}\ell}(\mathcal{J}\mathfrak{d},\mathcal{J}\dot{\mathfrak{h}},\vartheta) \leq \mathfrak{C}_{\mathfrak{b}\ell}\left(\mathfrak{d},\dot{\mathfrak{h}},\frac{\vartheta}{\tau}\right)$$

for all $\vartheta > 0$. Thus we obtain

$$\mathfrak{A}_{\mathfrak{b}\ell}(\mathfrak{d},\dot{\mathfrak{h}},\vartheta) \geq \mathfrak{A}_{\mathfrak{b}\ell}\left(\mathfrak{d},\dot{\mathfrak{h}},\frac{\vartheta}{\tau^{\ddot{\imath}}}\right)\text{ for all }\ddot{\imath}\in \mathrm{N}$$

$$\mathfrak{B}_{\mathfrak{b}\ell}(\mathfrak{d},\dot{\mathfrak{h}},\vartheta) \leq \mathfrak{B}_{\mathfrak{b}\ell}\left(\mathfrak{d},\dot{\mathfrak{h}},\frac{\vartheta}{\tau^{\ddot{\imath}}}\right)\text{and}\mathfrak{C}_{\mathfrak{b}\ell}(\mathfrak{d},\dot{\mathfrak{h}},\vartheta) \leq \mathfrak{C}_{\mathfrak{b}\ell}\left(\mathfrak{d},\dot{\mathfrak{h}},\frac{\vartheta}{\tau^{\ddot{\imath}}}\right)$$

Letting the limit as $\ddot{\imath} \to +\infty$ and using $\lim_{\vartheta\to\infty}\mathfrak{A}_{\mathfrak{b}\ell}(\hbar,\mathfrak{q},\vartheta) = 1, \lim_{\vartheta\to\infty}\mathfrak{B}_{\mathfrak{b}\ell}(\hbar,\mathfrak{q},\vartheta) = 0$ and $\lim_{\vartheta\to\infty}\mathfrak{C}_{\mathfrak{b}\ell}(\hbar,\mathfrak{q},\vartheta) = 0$ we get $\mathfrak{d} = \dot{\mathfrak{h}}$. Thus the fixed point is unique.

Example 3.2. Let $\mathfrak{S} = [0,1]$ and the *CTN* and *CTCN*respectively defined by $\mathfrak{u} * \mathfrak{v} = \mathfrak{u}\mathfrak{v}$ and $\mathfrak{u} \diamond \mathfrak{v} = \max\{\mathfrak{u},\mathfrak{v}\}$. Also, $\mathfrak{A}_{\mathfrak{b}\ell}, \mathfrak{B}_{\mathfrak{b}\ell}$ and $\mathfrak{C}_{\mathfrak{b}\ell}$are defined by $\mathfrak{u} * \mathfrak{v} = \min\{\mathfrak{u},\mathfrak{v}\}$and $\mathfrak{u} \diamond \mathfrak{v} = \max\{\mathfrak{u},\mathfrak{v}\}$.
$\mathfrak{A}_{\mathfrak{b}\ell}(\hbar,\mathfrak{q},\vartheta) = e^{\frac{-(\hbar+\mathfrak{q})^2}{\vartheta}}$,$\mathfrak{B}_{\mathfrak{b}\ell}(\hbar,\mathfrak{q},\vartheta) = 1 - e^{\frac{-(\hbar+\mathfrak{q})^2}{\vartheta}}$ and $\mathfrak{C}_{\mathfrak{b}\ell}(\hbar,\mathfrak{q},\vartheta) = e^{\frac{(\hbar+\mathfrak{q})^2}{\vartheta}} - 1$ for all $\hbar,\mathfrak{q}\in\mathfrak{S},\vartheta > 0$.
Then$(\mathfrak{S},\mathfrak{A}_{\mathfrak{b}\ell},\mathfrak{B}_{\mathfrak{b}\ell},\mathfrak{C}_{\mathfrak{b}\ell},*,\diamond)$ is a complete *NMLS.* Define $\mathcal{J}:\mathfrak{S}\to\mathfrak{S}$ by $\mathcal{J}\hbar = \begin{cases}0, & \hbar\in\left[0,\frac{1}{2}\right]\\ \frac{\hbar}{6}, & \hbar\in\left(\frac{1}{2},1\right]\end{cases}$
Then

$$\lim_{\vartheta\to\infty}\mathfrak{A}_{\mathfrak{b}\ell}(\hbar,\mathfrak{q},\vartheta) = \lim_{\vartheta\to\infty}e^{\frac{-(\hbar+\mathfrak{q})^2}{\vartheta}} = 1\ ,\lim_{\vartheta\to\infty}\mathfrak{B}_{\mathfrak{b}\ell}(\hbar,\mathfrak{q},\vartheta) = \lim_{\vartheta\to\infty}\left(1-e^{\frac{-(\hbar+\mathfrak{q})^2}{\vartheta}}\right) = 0\text{and}$$

$$\lim_{\vartheta\to\infty}\mathfrak{C}_{\mathfrak{b}\ell}(\hbar,\mathfrak{q},\vartheta) = \lim_{\vartheta\to\infty}\left(e^{\frac{(\hbar+\mathfrak{q})^2}{\vartheta}}-1\right) = 0$$

For $\tau\in\left[\frac{1}{2},1\right)$, we have four cases:
Case 1) If $\hbar,\mathfrak{q}\in\left[0,\frac{1}{2}\right]$, then $\mathcal{J}\hbar = \mathcal{J}\mathfrak{q} = 0$.
Case 2) If $\hbar\in\left[0,\frac{1}{2}\right]$and $\mathfrak{q}\in\left(\frac{1}{2},1\right]$, then $\mathcal{J}\hbar = 0$ and $\mathcal{J}\mathfrak{q} = \frac{\mathfrak{q}}{6}$.
Case 3) If $\hbar,\mathfrak{q}\in\left(\frac{1}{2},1\right]$, then $\mathcal{J}\hbar = \frac{\hbar}{6}$ and $\mathcal{J}\mathfrak{q} = \frac{\mathfrak{q}}{6}$.
Case 4) If $\hbar\in\left(\frac{1}{2},1\right]$and $\mathfrak{q}\in\left[0,\frac{1}{2}\right]$, then $\mathcal{J}\hbar = \frac{\hbar}{6}$ and $\mathcal{J}\mathfrak{q} = 0$.
From all four cases, we obtain that
$\mathfrak{A}_{\mathfrak{b}\ell}(\mathcal{J}\hbar,\mathcal{J}\mathfrak{q},\tau\vartheta) \geq \mathfrak{A}_{\mathfrak{b}\ell}(\hbar,\mathfrak{q},\vartheta)$,$\mathfrak{B}_{\mathfrak{b}\ell}(\mathcal{J}\hbar,\mathcal{J}\mathfrak{q},\tau\vartheta) \leq \mathfrak{B}_{\mathfrak{b}\ell}(\hbar,\mathfrak{q},\vartheta)$ and$\mathfrak{C}_{\mathfrak{b}\ell}(\mathcal{J}\hbar,\mathcal{J}\mathfrak{q},\tau\vartheta) \leq \mathfrak{C}_{\mathfrak{b}\ell}(\hbar,\mathfrak{q},\vartheta)$.
Thus all the requirements of Theorem (3.1) are met and 0 is the unique fixed point of $\mathcal{J}$. Also,

$$\mathfrak{A}_{\mathfrak{b}\ell}(\mathfrak{d},\mathfrak{d},\vartheta) = \mathfrak{A}_{\mathfrak{b}\ell}(0,0,\vartheta) = e^0 = 1, \mathfrak{B}_{\mathfrak{b}\ell}(\mathfrak{d},\mathfrak{d},\vartheta) = \mathfrak{B}_{\mathfrak{b}\ell}(0,0,\vartheta) = 1 - e^0 = 0$$

and $\mathfrak{C}_{\mathfrak{b}\ell}(\mathfrak{d},\mathfrak{d},\vartheta) = \mathfrak{C}_{\mathfrak{b}\ell}(0,0,\vartheta) = e^0 - 1 = 0$ for all $\vartheta > 0$.

Definition 3.3. Let$(\mathfrak{S}, \mathfrak{A}_{\mathfrak{b}\ell}, \mathfrak{B}_{\mathfrak{b}\ell}, \mathfrak{C}_{\mathfrak{b}\ell}, *, \diamond)$ be an *NbMLS.* A mapping $\mathcal{J} : \mathfrak{S} \to \mathfrak{S}$ is named to be *NbML* contractive if there exists $\rho \in (0,1)$ such that

$\frac{1}{\mathfrak{A}_{\mathfrak{b}\ell}(\mathcal{J}\hbar, \mathcal{J}\mathfrak{q}, \vartheta)} - 1 \leq \rho\left[\frac{1}{\mathfrak{A}_{\mathfrak{b}\ell}(\hbar, \mathfrak{q}, \vartheta)} - 1\right]$, $\mathfrak{B}_{\mathfrak{b}\ell}(\mathcal{J}\hbar, \mathcal{J}\mathfrak{q}, \vartheta) \leq \rho\mathfrak{B}_{\mathfrak{b}\ell}(\hbar, \mathfrak{q}, \vartheta)$ and $\mathfrak{C}_{\mathfrak{b}\ell}(\mathcal{J}\hbar, \mathcal{J}\mathfrak{q}, \vartheta) \leq \rho\mathfrak{C}_{\mathfrak{b}\ell}(\hbar, \mathfrak{q}, \vartheta)$ (I)

for all $\hbar, \mathfrak{q} \in \mathfrak{S}$ and $\vartheta > 0$. Here ρ is called the *NbML*contractive constant of $\mathcal{J}$.

Theorem 3.4. Let$(\mathfrak{S}, \mathfrak{A}_{\mathfrak{b}\ell}, \mathfrak{B}_{\mathfrak{b}\ell}, \mathfrak{C}_{\mathfrak{b}\ell}, *, \diamond)$ be a complete *NbMLS* and $\mathcal{J} : \mathfrak{S} \to \mathfrak{S}$ be a *NbML* contractive mapping with an *NbML* contractive constant ρ. Then $\mathcal{J}$ has a unique fixed point $\mathfrak{d} \in \mathfrak{S}$ such that $\mathfrak{A}_{\mathfrak{b}\ell}(\mathfrak{d}, \mathfrak{d}, \vartheta) = 1$, $\mathfrak{B}_{\mathfrak{b}\ell}(\mathfrak{d}, \mathfrak{d}, \vartheta) = 0$and $\mathfrak{C}_{\mathfrak{b}\ell}(\mathfrak{d}, \mathfrak{d}, \vartheta) = 0$for all $\vartheta > 0$.

Proof: Let$(\mathfrak{S}, \mathfrak{A}_{\mathfrak{b}\ell}, \mathfrak{B}_{\mathfrak{b}\ell}, \mathfrak{C}_{\mathfrak{b}\ell}, *, \diamond)$be a complete *NbMLS*. For a given element $\mathfrak{p}_0 \in \mathfrak{S}$, define a sequence $\{\hbar_{ï}\}$ in $\mathfrak{S}$ by

$$\hbar_1 = \mathcal{J}\hbar_0, \hbar_2 = \mathcal{J}^2\hbar_0 = \mathcal{J}\hbar_1, \dots, \hbar_{ï} = \mathcal{J}^{ï}\hbar_0 = \mathcal{J}\hbar_{ï-1} \text{ for all } ï \in \mathrm{N}$$

If $\hbar_{ï} = \hbar_{ï-1}$ for some $ï \in \mathrm{N}$, then $\hbar_{ï}$ is a fixed point of $\mathcal{J}$. We assume that $\hbar_{ï} \neq \hbar_{ï-1}$ for all $ï \in \mathrm{N}$. For $\vartheta > 0$and $ï \in \mathrm{N}$, we get from (I) that

$$\frac{1}{\mathfrak{A}_{\mathfrak{b}\ell}(\hbar_{ï}, \hbar_{ï+1}, \vartheta)} - 1 = \frac{1}{\mathfrak{A}_{\mathfrak{b}\ell}(\mathcal{J}\hbar_{ï-1}, \mathcal{J}\hbar_{ï}, \vartheta)} - 1 \leq \rho\left[\frac{1}{\mathfrak{A}_{\mathfrak{b}\ell}(\hbar_{ï-1}, \hbar_{ï}, \vartheta)} - 1\right]$$

Then we have

$$\frac{1}{\mathfrak{A}_{\mathfrak{b}\ell}(\hbar_{ï}, \hbar_{ï+1}, \vartheta)} \leq \frac{\rho}{\mathfrak{A}_{\mathfrak{b}\ell}(\mathcal{J}\hbar_{ï-1}, \mathcal{J}\hbar_{ï}, \vartheta)} + (1-\rho)$$
$$= \frac{\rho}{\mathfrak{A}_{\mathfrak{b}\ell}(\mathcal{J}\hbar_{ï-2}, \mathcal{J}\hbar_{ï-1}, \vartheta)} + (1-\rho) \leq \frac{\rho^2}{\mathfrak{A}_{\mathfrak{b}\ell}(\hbar_{ï-2}, \hbar_{ï-1}, \vartheta)} + \rho(1-\rho) + (1-\rho)$$

for all $\vartheta > 0$. Continuing in this way, we get

$$\begin{aligned}\frac{1}{\mathfrak{A}_{\mathfrak{b}\ell}(\hbar_{ï}, \hbar_{ï+1}, \vartheta)} &\leq \frac{\rho^{ï}}{\mathfrak{A}_{\mathfrak{b}\ell}(\hbar_0, \hbar_1, \vartheta)} + \rho^{ï-1}(1-\rho) + \rho^{ï-2}(1-\rho) + \cdots + \rho(1-\rho) + (1-\rho)\\ &\leq \frac{\rho^{ï}}{\mathfrak{A}_{\mathfrak{b}\ell}(\hbar_0, \hbar_1, \vartheta)} + (\rho^{ï-1} + \rho^{ï-2} + \cdots + 1)(1-\rho)\\ &\leq \frac{\rho^{ï}}{\mathfrak{A}_{\mathfrak{b}\ell}(\hbar_0, \hbar_1, \vartheta)} + (1-\rho^{ï})\end{aligned}$$

Thus$\frac{1}{\frac{\rho^{ï}}{\mathfrak{A}_{\mathfrak{b}\ell}(\hbar_0, \hbar_1, \vartheta)} + (1-\rho^{ï})} \leq \mathfrak{A}_{\mathfrak{b}\ell}(\hbar_{ï}, \hbar_{ï+1}, \vartheta)$ for all $\vartheta > 0, ï \in \mathrm{N}$ (3.4.1)

$$\begin{aligned}\mathfrak{B}_{\mathfrak{b}\ell}(\hbar_{ï}, \hbar_{ï+1}, \vartheta) &= \mathfrak{B}_{\mathfrak{b}\ell}(\mathcal{J}\hbar_{ï-1}, \mathcal{J}\hbar_{ï}, \vartheta) \leq \rho\mathfrak{B}_{\mathfrak{b}\ell}(\hbar_{ï-1}, \hbar_{ï}, \vartheta) = \rho\mathfrak{B}_{\mathfrak{b}\ell}(\mathcal{J}\hbar_{ï-2}, \mathcal{J}\hbar_{ï-1}, \vartheta)\\ &\leq \rho^2\mathfrak{B}_{\mathfrak{b}\ell}(\hbar_{ï-2}, \hbar_{ï-1}, \vartheta) \leq \cdots \leq \rho^{ï}\mathfrak{B}_{\mathfrak{b}\ell}(\hbar_0, \hbar_1, \vartheta)\end{aligned} \quad (3.4.2)$$

and

$$\begin{aligned}\mathfrak{C}_{\mathfrak{b}\ell}(\hbar_{ï}, \hbar_{ï+1}, \vartheta) &= \mathfrak{C}_{\mathfrak{b}\ell}(\mathcal{J}\hbar_{ï-1}, \mathcal{J}\hbar_{ï}, \vartheta) \leq \rho\mathfrak{C}_{\mathfrak{b}\ell}(\hbar_{ï-1}, \hbar_{ï}, \vartheta) = \rho\mathfrak{C}_{\mathfrak{b}\ell}(\mathcal{J}\hbar_{ï-2}, \mathcal{J}\hbar_{ï-1}, \vartheta)\\ &\leq \rho^2\mathfrak{C}_{\mathfrak{b}\ell}(\hbar_{ï-2}, \hbar_{ï-1}, \vartheta) \leq \cdots \leq \rho^{ï}\mathfrak{C}_{\mathfrak{b}\ell}(\hbar_0, \hbar_1, \vartheta).\end{aligned} \quad (3.4.3)$$

Now, for $p \geq 1$and $ï \in \mathrm{N}$, we have

$$\begin{aligned}\mathfrak{A}_{\mathfrak{b}\ell}(\hbar_{ï}, \hbar_{ï+1}, \vartheta) &\geq \mathfrak{A}_{\mathfrak{b}\ell}\left(\hbar_{ï}, \hbar_{ï+1}, \frac{\vartheta}{b}\right) * \mathfrak{A}_{\mathfrak{b}\ell}\left(\hbar_{ï+1}, \hbar_{ï+p}, \frac{\vartheta}{b}\right)\\ &\geq \mathfrak{A}_{\mathfrak{b}\ell}\left(\hbar_{ï}, \hbar_{ï+1}, \frac{\vartheta}{b}\right) * \mathfrak{A}_{\mathfrak{b}\ell}\left(\hbar_{ï+1}, \hbar_{ï+2}, \frac{\vartheta}{b^2}\right) * \mathfrak{A}_{\mathfrak{b}\ell}\left(\hbar_{ï+2}, \hbar_{ï+p}, \frac{\vartheta}{b^2}\right).\end{aligned}$$

Continuing in this way, we get

$$\mathfrak{A}_{\mathfrak{b}\ell}(\hbar_{ï}, \hbar_{ï+p}, \vartheta) \geq \mathfrak{A}_{\mathfrak{b}\ell}\left(\hbar_{ï}, \hbar_{ï+1}, \frac{\vartheta}{b}\right) * \mathfrak{A}_{\mathfrak{b}\ell}\left(\hbar_{ï+1}, \hbar_{ï+2}, \frac{\vartheta}{b^2}\right) * \cdots * \mathfrak{A}_{\mathfrak{b}\ell}\left(\hbar_{ï+p-1}, \hbar_{ï+p}, \frac{\vartheta}{b^{p-1}}\right)$$

$$\begin{aligned}\mathfrak{B}_{\mathfrak{b}\ell}(\hbar_{ï}, \hbar_{ï+p}, \vartheta) &\leq \mathfrak{B}_{\mathfrak{b}\ell}\left(\hbar_{ï}, \hbar_{ï+1}, \frac{\vartheta}{b}\right) \diamond \mathfrak{B}_{\mathfrak{b}\ell}\left(\hbar_{ï+1}, \hbar_{ï+p}, \frac{\vartheta}{b}\right)\\ &\leq \mathfrak{B}_{\mathfrak{b}\ell}\left(\hbar_{ï}, \hbar_{ï+1}, \frac{\vartheta}{b}\right) \diamond \mathfrak{B}_{\mathfrak{b}\ell}\left(\hbar_{ï+1}, \hbar_{ï+2}, \frac{\vartheta}{b^2}\right) \diamond \mathfrak{B}_{\mathfrak{b}\ell}\left(\hbar_{ï+2}, \hbar_{ï+p}, \frac{\vartheta}{b^2}\right)\end{aligned}$$

Continuing in this way, we get

$$\mathfrak{B}_{\mathfrak{b}\ell}(\hbar_{ï}, \hbar_{ï+p}, \vartheta) \leq \mathfrak{B}_{\mathfrak{b}\ell}\left(\hbar_{ï}, \hbar_{ï+1}, \frac{\vartheta}{b}\right) \diamond \mathfrak{B}_{\mathfrak{b}\ell}\left(\hbar_{ï+1}, \hbar_{ï+2}, \frac{\vartheta}{b^2}\right) \diamond \cdots \diamond \mathfrak{B}_{\mathfrak{b}\ell}\left(\hbar_{ï+p-1}, \hbar_{ï+p}, \frac{\vartheta}{b^{p-1}}\right)$$

and

$$\begin{aligned}\mathfrak{C}_{\mathcal{b}\ell}(\hbar_{\ddot{\imath}},\hbar_{\ddot{\imath}+p},\vartheta) &\leq \mathfrak{C}_{\mathcal{b}\ell}\left(\hbar_{\ddot{\imath}},\hbar_{\ddot{\imath}+1},\frac{\vartheta}{b}\right)\diamond\mathfrak{C}_{\mathcal{b}\ell}\left(\hbar_{\ddot{\imath}+1},\hbar_{\ddot{\imath}+p},\frac{\vartheta}{b}\right)\\ &\leq \mathfrak{C}_{\mathcal{b}\ell}\left(\hbar_{\ddot{\imath}},\hbar_{\ddot{\imath}+1},\frac{\vartheta}{b}\right)\diamond\mathfrak{C}_{\mathcal{b}\ell}\left(\hbar_{\ddot{\imath}+1},\hbar_{\ddot{\imath}+2},\frac{\vartheta}{b^2}\right)\diamond\mathfrak{C}_{\mathcal{b}\ell}\left(\hbar_{\ddot{\imath}+2},\hbar_{\ddot{\imath}+p},\frac{\vartheta}{b^2}\right)\end{aligned}$$

Continuing in this way, we get

$$\mathfrak{C}_{\mathcal{b}\ell}(\hbar_{\ddot{\imath}},\hbar_{\ddot{\imath}+p},\vartheta)\leq\mathfrak{C}_{\mathcal{b}\ell}\left(\hbar_{\ddot{\imath}},\hbar_{\ddot{\imath}+1},\frac{\vartheta}{b}\right)\diamond\mathfrak{C}_{\mathcal{b}\ell}\left(\hbar_{\ddot{\imath}+1},\hbar_{\ddot{\imath}+2},\frac{\vartheta}{b^2}\right)\diamond\cdots\diamond\mathfrak{C}_{\mathcal{b}\ell}\left(\hbar_{\ddot{\imath}+p-1},\hbar_{\ddot{\imath}+p},\frac{\vartheta}{b^{p-1}}\right)$$

Using (3.4.1), (3.4.2) and (3.4.3) in the above inequality, we have

$$\begin{aligned}\mathfrak{A}_{\mathcal{b}\ell}(\hbar_{\ddot{\imath}},\hbar_{\ddot{\imath}+p},\vartheta) &\geq \frac{1}{\frac{\rho^{\ddot{\imath}}}{\mathfrak{A}_{\mathcal{b}\ell}\left(\hbar_0,\hbar_1,\frac{\vartheta}{b}\right)}+(1-\rho^{\ddot{\imath}})}*\frac{1}{\frac{\rho^{\ddot{\imath}+1}}{\mathfrak{A}_{\mathcal{b}\ell}\left(\hbar_0,\hbar_1,\frac{\vartheta}{b^2}\right)}+(1-\rho^{\ddot{\imath}+1})}*\cdots*\frac{1}{\frac{\rho^{\ddot{\imath}+p-1}}{\mathfrak{A}_{\mathcal{b}\ell}\left(\hbar_0,\hbar_1,\frac{\vartheta}{b^{p-1}}\right)}+(1-\rho^{\ddot{\imath}+p-1})}\\ &\geq \frac{1}{\frac{\rho^{\ddot{\imath}}}{\mathfrak{A}_{\mathcal{b}\ell}\left(\hbar_0,\hbar_1,\frac{\vartheta}{b}\right)}+1}*\frac{1}{\frac{\rho^{\ddot{\imath}}}{\mathfrak{A}_{\mathcal{b}\ell}\left(\hbar_0,\hbar_1,\frac{\vartheta}{b^2}\right)}+1}*\cdots*\frac{\rho^{\ddot{\imath}+p-1}}{\frac{\rho^{\ddot{\imath}}}{\mathfrak{A}_{\mathcal{b}\ell}\left(\hbar_0,\hbar_1,\frac{\vartheta}{b^{p-1}}\right)}+1}\end{aligned}$$

$$\mathfrak{B}_{\mathcal{b}\ell}(\hbar_{\ddot{\imath}},\hbar_{\ddot{\imath}+p},\vartheta)\leq\rho^{\ddot{\imath}}\mathfrak{B}_{\mathcal{b}\ell}\left(\hbar_0,\hbar_1,\frac{\vartheta}{b}\right)\diamond\rho^{\ddot{\imath}+1}\mathfrak{B}_{\mathcal{b}\ell}\left(\hbar_0,\hbar_1,\frac{\vartheta}{b^2}\right)\diamond\cdots\diamond\rho^{\ddot{\imath}+p-1}\mathfrak{B}_{\mathcal{b}\ell}\left(\hbar_0,\hbar_1,\frac{\vartheta}{b^{p-1}}\right)\text{ and}$$

$$\mathfrak{C}_{\mathcal{b}\ell}(\hbar_{\ddot{\imath}},\hbar_{\ddot{\imath}+p},\vartheta)\leq\rho^{\ddot{\imath}}\mathfrak{C}_{\mathcal{b}\ell}\left(\hbar_0,\hbar_1,\frac{\vartheta}{b}\right)\diamond\rho^{\ddot{\imath}+1}\mathfrak{C}_{\mathcal{b}\ell}\left(\hbar_0,\hbar_1,\frac{\vartheta}{b^2}\right)\diamond\cdots\diamond\rho^{\ddot{\imath}+p-1}\mathfrak{C}_{\mathcal{b}\ell}\left(\hbar_0,\hbar_1,\frac{\vartheta}{b^{p-1}}\right)$$

Where$b$ is a random positive integer and $\rho\in(0,1)$. So we deduce from the above expression that$\{\hbar_{\ddot{\imath}}\}$ is a Cauchy sequence in$(\mathfrak{S},\mathfrak{A}_{\mathcal{b}\ell},\mathfrak{B}_{\mathcal{b}\ell},\mathfrak{C}_{\mathcal{b}\ell},*,\diamond)$. By the completeness of$(\mathfrak{S},\mathfrak{A}_{\mathcal{b}\ell},\mathfrak{B}_{\mathcal{b}\ell},\mathfrak{C}_{\mathcal{b}\ell},*,\diamond)$, there is $\mathfrak{d}\in\mathfrak{S}$ such that

$$\lim_{\ddot{\imath}\to\infty}\mathfrak{A}_{\mathcal{b}\ell}(\hbar_{\ddot{\imath}},\mathfrak{d},\vartheta)=\lim_{\ddot{\imath}\to\infty}\mathfrak{A}_{\mathcal{b}\ell}(\hbar_{\ddot{\imath}},\hbar_{\ddot{\imath}+p},\vartheta)=\lim_{\ddot{\imath}\to\infty}\mathfrak{A}_{\mathcal{b}\ell}(\mathfrak{d},\mathfrak{d},\vartheta)=1 \tag{3.4.4}$$

$$\lim_{\ddot{\imath}\to\infty}\mathfrak{B}_{\mathcal{b}\ell}(\hbar_{\ddot{\imath}},\mathfrak{d},\vartheta)=\lim_{\ddot{\imath}\to\infty}\mathfrak{B}_{\mathcal{b}\ell}(\hbar_{\ddot{\imath}},\hbar_{\ddot{\imath}+p},\vartheta)=\lim_{\ddot{\imath}\to\infty}\mathfrak{B}_{\mathcal{b}\ell}(\mathfrak{d},\mathfrak{d},\vartheta)=0 \tag{3.4.5}$$

$$\lim_{\ddot{\imath}\to\infty}\mathfrak{C}_{\mathcal{b}\ell}(\hbar_{\ddot{\imath}},\mathfrak{d},\vartheta)=\lim_{\ddot{\imath}\to\infty}\mathfrak{C}_{\mathcal{b}\ell}(\hbar_{\ddot{\imath}},\hbar_{\ddot{\imath}+p},\vartheta)=\lim_{n\to\infty}\mathfrak{C}_{\mathcal{b}\ell}(\mathfrak{d},\mathfrak{d},\vartheta)=0 \tag{3.4.6}$$

for all $\vartheta>0,p\geq1$.

We now establish that $\mathfrak{d}$ is a fixed point for $\mathcal{J}$. We determine this from (I) that

$$\begin{aligned}\frac{1}{\mathfrak{A}_{\mathcal{b}\ell}(\mathcal{J}\hbar_{\ddot{\imath}},\mathcal{J}\mathfrak{d},\vartheta)}-1 &\leq\rho\left[\frac{1}{\mathfrak{A}_{\mathcal{b}\ell}(\hbar_{\ddot{\imath}},\mathfrak{d},\vartheta)}-1\right]=\frac{\rho}{\mathfrak{A}_{\mathcal{b}\ell}(\hbar_{\ddot{\imath}},\mathfrak{d},\vartheta)}-\rho,\\ \frac{1}{\mathfrak{A}_{\mathcal{b}\ell}(\hbar_{\ddot{\imath}},\mathfrak{d},\vartheta)}+1-\rho &\leq\mathfrak{A}_{\mathcal{b}\ell}(\mathcal{J}\hbar_{\ddot{\imath}},\mathcal{J}\mathfrak{d},\vartheta).\end{aligned}$$

Utilize the above inequality, we get

$$\begin{aligned}&\mathfrak{A}_{\mathcal{b}\ell}(\mathfrak{d},\mathcal{J}\mathfrak{d},\vartheta)\geq\mathfrak{A}_{\mathcal{b}\ell}\left(\mathfrak{d},\hbar_{\ddot{\imath}+1},\frac{\vartheta}{2b}\right)*\mathfrak{A}_{\mathcal{b}\ell}\left(\hbar_{\ddot{\imath}+1},\mathcal{J}\mathfrak{d},\frac{\vartheta}{2b}\right)\\ &=\mathfrak{A}_{\mathcal{b}\ell}\left(\mathfrak{d},\hbar_{\ddot{\imath}+1},\frac{\vartheta}{2b}\right)*\mathfrak{A}_{\mathcal{b}\ell}\left(\mathcal{J}\hbar_{\ddot{\imath}},\mathcal{J}\mathfrak{d},\frac{\vartheta}{2b}\right)\\ &\geq\mathfrak{A}_{\mathcal{b}\ell}\left(\mathfrak{d},\hbar_{\ddot{\imath}+1},\frac{\vartheta}{2b}\right)*\frac{1}{\frac{\rho}{\mathfrak{A}_{\mathcal{b}\ell}\left(\hbar_{\ddot{\imath}},\mathfrak{d},\frac{\vartheta}{2b}\right)}+1-\rho}\end{aligned}$$

$$\begin{aligned}\mathfrak{B}_{\mathcal{b}\ell}(\mathfrak{d},\mathcal{J}\mathfrak{d},\vartheta) &\leq\mathfrak{B}_{\mathcal{b}\ell}\left(\mathfrak{d},\hbar_{\ddot{\imath}+1},\frac{\vartheta}{2b}\right)\diamond\mathfrak{B}_{\mathcal{b}\ell}\left(\hbar_{\ddot{\imath}+1},\mathcal{J}\mathfrak{d},\frac{\vartheta}{2b}\right)=\mathfrak{B}_{\mathcal{b}\ell}\left(\mathfrak{d},\hbar_{\ddot{\imath}+1},\frac{\vartheta}{2b}\right)\diamond\mathfrak{B}_{\mathcal{b}\ell}\left(\mathcal{J}\hbar_{\ddot{\imath}},\mathcal{J}\mathfrak{d},\frac{\vartheta}{2b}\right)\\ &\leq\mathfrak{B}_{\mathcal{b}\ell}\left(\mathfrak{d},\hbar_{\ddot{\imath}+1},\frac{\vartheta}{2b}\right)\diamond\rho\mathfrak{B}_{\mathcal{b}\ell}\left(\hbar_{\ddot{\imath}},\mathfrak{d},\frac{\vartheta}{2b}\right)\end{aligned}$$

$$\begin{aligned}\mathfrak{C}_{\mathcal{b}\ell}(\mathfrak{d},\mathcal{J}\mathfrak{d},\vartheta) &\leq\mathfrak{C}_{\mathcal{b}\ell}\left(\mathfrak{d},\hbar_{\ddot{\imath}+1},\frac{\vartheta}{2b}\right)\diamond\mathfrak{C}_{\mathcal{b}\ell}\left(\hbar_{\ddot{\imath}+1},\mathcal{J}\mathfrak{d},\frac{\vartheta}{2b}\right)=\mathfrak{C}_{\mathcal{b}\ell}\left(\mathfrak{d},\hbar_{\ddot{\imath}+1},\frac{\vartheta}{2b}\right)\diamond\mathfrak{C}_{\mathcal{b}\ell}\left(\mathcal{J}\hbar_{\ddot{\imath}},\mathcal{J}\mathfrak{d},\frac{\vartheta}{2b}\right)\\ &\leq\mathfrak{C}_{\mathcal{b}\ell}\left(\mathfrak{d},\hbar_{\ddot{\imath}+1},\frac{\vartheta}{2b}\right)\diamond\rho\mathfrak{C}_{\mathcal{b}\ell}\left(\hbar_{\ddot{\imath}},\mathfrak{d},\frac{\vartheta}{2b}\right)\end{aligned}$$

Letting the limit as $\ddot{\imath}\to\infty$ and applying (3.4.4.), (3.4.5) and (3.4.6) in the above expression, we obtain that $\mathfrak{A}_{\mathcal{b}\ell}(\mathfrak{d},\mathcal{J}\mathfrak{d},\vartheta)=1,\mathfrak{B}_{\mathcal{b}\ell}(\mathfrak{d},\mathcal{J}\mathfrak{d},\vartheta)=0$and $\mathfrak{C}_{\mathcal{b}\ell}(\mathfrak{d},\mathcal{J}\mathfrak{d},\vartheta)=0$,that is, $\mathcal{J}\mathfrak{d}=\mathfrak{d}$. Therefore, $\mathfrak{d}$ is a fixed point of $\mathcal{J}$and $\mathfrak{A}_{\mathcal{b}\ell}(\mathfrak{d},\mathfrak{d},\vartheta)=1,\mathfrak{B}_{\mathcal{b}\ell}(\mathfrak{d},\mathfrak{d},\vartheta)=0$and $\mathfrak{C}_{\mathcal{b}\ell}(\mathfrak{d},\mathfrak{d},\vartheta)=0$ for all $\vartheta>0$.

Now we demonstrate the uniqueness of the fixed point $\mathfrak{d}$ of $\mathcal{J}$. Let $\dot{\mathfrak{h}}$ be a fixed point of $\mathcal{J}$different from $\mathfrak{d}$such that $\mathfrak{A}_{\mathfrak{b}\ell}(\mathfrak{d},\dot{\mathfrak{h}},\vartheta) \neq 1\mathfrak{B}_{\mathfrak{b}\ell}(\mathfrak{d},\dot{\mathfrak{h}},\vartheta) \neq 0$and $\mathfrak{C}_{\mathfrak{b}\ell}(\mathfrak{d},\dot{\mathfrak{h}},\vartheta) \neq 0$for some $\vartheta > 0$. It follows from (3.4.1) that

$$\frac{1}{\mathfrak{A}_{\mathfrak{b}\ell}(\mathfrak{d},\dot{\mathfrak{h}},\vartheta)} - 1 = \frac{1}{\mathfrak{A}_{\mathfrak{b}\ell}(\mathcal{J}\mathfrak{d},\mathcal{J}\dot{\mathfrak{h}},\vartheta)} - 1 \leq \rho\left[\frac{1}{\mathfrak{A}_{\mathfrak{b}\ell}(\mathfrak{d},\dot{\mathfrak{h}},\vartheta)} - 1\right] < \frac{1}{\mathfrak{A}_{\mathfrak{b}\ell}(\mathfrak{d},\dot{\mathfrak{h}},\vartheta)} - 1$$

$\mathfrak{B}_{\mathfrak{b}\ell}(\mathfrak{d},\dot{\mathfrak{h}},\vartheta) = \mathfrak{B}_{\mathfrak{b}\ell}(\mathcal{J}\mathfrak{d},\mathcal{J}\dot{\mathfrak{h}},\vartheta) \leq \rho\mathfrak{B}_{\mathfrak{b}\ell}(\mathfrak{d},\dot{\mathfrak{h}},\vartheta) < \mathfrak{B}_{\mathfrak{b}\ell}(\mathfrak{d},\dot{\mathfrak{h}},\vartheta)$and

$\mathfrak{C}_{\mathfrak{b}\ell}(\mathfrak{d},\dot{\mathfrak{h}},\vartheta) = \mathfrak{C}_{\mathfrak{b}\ell}(\mathcal{J}\mathfrak{d},\mathcal{J}\dot{\mathfrak{h}},\vartheta) \leq \rho\mathfrak{C}_{\mathfrak{b}\ell}(\mathfrak{d},\dot{\mathfrak{h}},\vartheta) < \mathfrak{C}_{\mathfrak{b}\ell}(\mathfrak{d},\dot{\mathfrak{h}},\vartheta)$this is not possible.

$\therefore$we have $\mathfrak{A}_{\mathfrak{b}\ell}(\mathfrak{d},\dot{\mathfrak{h}},\vartheta) = 1, \mathfrak{B}_{\mathfrak{b}\ell}(\mathfrak{d},\dot{\mathfrak{h}},\vartheta) = 0$and $\mathfrak{C}_{\mathfrak{b}\ell}(\mathfrak{d},\dot{\mathfrak{h}},\vartheta) = 0$for all $\vartheta > 0$, and hence $\mathfrak{d} = \dot{\mathfrak{h}}$.

Corollary 3.5. Let $(\mathfrak{S},\mathfrak{A}_{\mathfrak{b}\ell},\mathfrak{B}_{\mathfrak{b}\ell},\mathfrak{C}_{\mathfrak{b}\ell},*,\diamond)$ be a complete *NbMLS* and $\mathcal{J}:\mathfrak{S}\to\mathfrak{S}$ be a mapping satisfying $\frac{1}{\mathfrak{A}_{\mathfrak{b}\ell}(\mathcal{J}^{\ddot{\imath}}\hbar,\mathcal{J}^{\ddot{\imath}}\mathfrak{q},\vartheta)} - 1 \leq \rho\left[\frac{1}{\mathfrak{A}_{\mathfrak{b}\ell}(\hbar,\mathfrak{q},\vartheta)} - 1\right], \mathfrak{B}_{\mathfrak{b}\ell}(\mathcal{J}^{\ddot{\imath}}\hbar,\mathcal{J}^{\ddot{\imath}}\mathfrak{q},\vartheta) \leq \rho\mathfrak{B}_{\mathfrak{b}\ell}(\hbar,\mathfrak{q},\vartheta)$and

$\mathfrak{C}_{\mathfrak{b}\ell}(\mathcal{J}^{\ddot{\imath}}\hbar,\mathcal{J}^{\ddot{\imath}}\mathfrak{q},\vartheta) \leq \rho\mathfrak{C}_{\mathfrak{b}\ell}(\hbar,\mathfrak{q},\vartheta)$for some $\ddot{\imath} \in \mathbb{N}$ and all $\hbar,\mathfrak{q} \in \mathfrak{S}, \vartheta > 0, 0 < \rho < 1$. Then $\mathcal{J}$ has a unique fixed point $\mathfrak{d} \in \mathfrak{S}$ moreover$\mathfrak{A}_{\mathfrak{b}\ell}(\mathfrak{d},\mathfrak{d},\vartheta) = 1, \mathfrak{B}_{\mathfrak{b}\ell}(\mathfrak{d},\mathfrak{d},\vartheta) = 0$and $\mathfrak{C}_{\mathfrak{b}\ell}(\mathfrak{d},\mathfrak{d},\vartheta) = 0$for all $\vartheta > 0$.

Proof: Assume that $\mathfrak{d} \in \mathfrak{S}$ is the unique fixed point of $\mathcal{J}^{\ddot{\imath}}$ as determined by Theorem (3.4) and $\mathfrak{A}_{\mathfrak{b}\ell}(\mathfrak{d},\mathfrak{d},\vartheta) = 1, \mathfrak{B}_{\mathfrak{b}\ell}(\mathfrak{d},\mathfrak{d},\vartheta) = 0and\mathfrak{C}_{\mathfrak{b}\ell}(\mathfrak{d},\mathfrak{d},\vartheta) = 0$for all $\vartheta > 0$. So$\mathfrak{d}$ is another fixed point of $\mathcal{J}^{\ddot{\imath}}$ as $\mathcal{J}^{\ddot{\imath}}(\mathcal{J}\mathfrak{d}) = \mathcal{J}\mathfrak{d}$ and by Theorem(3.4), $\mathcal{J}\mathfrak{d} = \mathfrak{d}$ and so $\mathfrak{d}$ is the unique fixed point, since the unique fixed point of $\mathcal{J}$ is also the unique fixed point of $\mathcal{J}^{\ddot{\imath}}$.

Example 3.6. Let $\mathfrak{S} = [0,1]$ and the *CTN* and *CTCN* respectively defined by $\mathfrak{u} * \mathfrak{v} = \mathfrak{u}\mathfrak{v}$ and $\mathfrak{u} \diamond \mathfrak{v} = \max\{\mathfrak{u},\mathfrak{v}\}$. Consider $\mathfrak{A}_{\mathfrak{b}\ell},\mathfrak{B}_{\mathfrak{b}\ell}$ and $\mathfrak{C}_{\mathfrak{b}\ell}$by $\mathfrak{A}_{\mathfrak{b}\ell}(\hbar,\mathfrak{q},\vartheta) = e^{\frac{-(\max\{\hbar,\mathfrak{q}\})^2}{\vartheta}}$, $\mathfrak{B}_{\mathfrak{b}\ell}(\hbar,\mathfrak{q},\vartheta) = 1 - e^{\frac{-(\max\{\hbar,\mathfrak{q}\})^2}{\vartheta}}$and $\mathfrak{C}_{\mathfrak{b}\ell}(\hbar,\mathfrak{q},\vartheta) = e^{\frac{(\max\{\hbar,\mathfrak{q}\})^2}{\vartheta}} - 1$

for all $\hbar,\mathfrak{q} \in \mathfrak{S}$ and $\vartheta > 0$. Then $(\mathfrak{S},\mathfrak{A}_{\mathfrak{b}\ell},\mathfrak{B}_{\mathfrak{b}\ell},\mathfrak{C}_{\mathfrak{b}\ell},*,\diamond)$ is a complete *NbMLS*.

Define $\mathcal{J}:\mathfrak{S}\to\mathfrak{S}$ as$\mathcal{J}\hbar = \begin{cases} 0, \hbar = \frac{1}{2} \\ \frac{\hbar}{2}, \hbar \in [0,\frac{1}{2}) \\ \frac{\hbar}{4}, \hbar \in (\frac{1}{2},1] \end{cases}$

Then we have eight cases:

Case (i) If $\hbar = \mathfrak{q} = \frac{1}{2}$, then $\mathcal{J}\hbar = \mathcal{J}\mathfrak{q} = 0$.

Case (ii) If $\hbar = \frac{1}{2}$and $\mathfrak{q} \in \left[0,\frac{1}{2}\right)$ then $\mathcal{J}\hbar = 0$ and $\mathcal{J}\mathfrak{q} = \frac{\mathfrak{q}}{2}$.

Case (iii) If $\hbar = \frac{1}{2}$and $\mathfrak{q} \in (\frac{1}{2},1]$, then $\mathcal{J}\hbar = 0$ and $\mathcal{J}\mathfrak{q} = \frac{\mathfrak{q}}{4}$.

Case (iv) If $\hbar \in \left[0,\frac{1}{2}\right)$and $\mathfrak{q} \in (\frac{1}{2},1]$, then $\mathcal{J}\hbar = \frac{\hbar}{2}$ and $\mathcal{J}\mathfrak{q} = \frac{\mathfrak{q}}{4}$.

Case (v) If $\hbar \in [0,\frac{1}{2})$and $\mathfrak{q} \in [0,\frac{1}{2})$, then $\mathcal{J}\hbar = \frac{\hbar}{2}$ and $\mathcal{J}\mathfrak{q} = \frac{\mathfrak{q}}{2}$.

Case (vi) If $\hbar \in [0,\frac{1}{2})$and $\mathfrak{q} = \frac{1}{2}$, then $\mathcal{J}\hbar = \frac{\hbar}{2}$ and $\mathcal{J}\mathfrak{q} = 0$.

Case (vii) If $\hbar \in \left(\frac{1}{2},1\right]$and $\mathfrak{q} = \frac{1}{2}$, then $\mathcal{J}\hbar = \frac{\hbar}{4}$ and $\mathcal{J}\mathfrak{q} = 0$.

Case (viii) If $\hbar \in \left(\frac{1}{2},1\right]$and $\mathfrak{q} \in (\frac{1}{2},1]$, then $\mathcal{J}\hbar = \frac{\hbar}{4}$ and $\mathcal{J}\mathfrak{q} = \frac{o}{4}$.

NbML contraction is satisfied in all of the above cases:

$$\frac{1}{\mathfrak{A}_{\mathfrak{b}\ell}(\mathcal{J}\hbar,\mathcal{J}\mathfrak{q},\vartheta)} - 1 \leq \rho\left[\frac{1}{\mathfrak{A}_{\mathfrak{b}\ell}(\hbar,\mathfrak{q},\vartheta)} - 1\right],$$

$\mathfrak{B}_{\mathfrak{b}\ell}(\mathcal{J}\hbar,\mathcal{J}\mathfrak{q},\vartheta) \leq \rho\mathfrak{B}_{\mathfrak{b}\ell}(\hbar,\mathfrak{q},\vartheta)$and

$$\mathfrak{C}_{\mathfrak{b}\ell}(\mathcal{J}\hbar,\mathcal{J}\mathfrak{q},\vartheta) \leq \rho\mathfrak{C}_{\mathfrak{b}\ell}(\hbar,\mathfrak{q},\vartheta)$$

with the *NbML* contractive constant $\rho \in \left[\frac{1}{2},1\right)$.

Hence $\mathcal{J}$ is an *NbML* contractive mapping with $\rho \in \left[\frac{1}{2},1\right)$.

All the requirements of Theorem (3.1) have been met.

Moreover, 0 is the unique fixed point of T and $\mathfrak{A}_{b\ell}(0,0,\vartheta) = 1$, $\mathfrak{B}_{b\ell}(0,0,\vartheta) = 0$and $\mathfrak{C}_{b\ell}(0,0,\vartheta) = 0$for all $\vartheta > 0$.

4. Conclusion

In this paper, we develop *NbMLS* and demonstrate the fixed point theorem in order to demonstrate the unique fixed point in this space. This work is the extended form of fuzzy b-metric like space [6]. We hope that the result proved in this paper will form new connection for those who are working in the *NbMLS* space and this work opens a new path for researchers in the concerned field.

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Received: March 3, 2024. Accepted: July 28, 2024

Neutrosophic Sets and Systems, Vol. 70, 2024
University of New Mexico

A new approach of tree soft set with MCDM technique for leadership in international climate policy: measuring renewable energy sources and policies

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Abstract

Green technological progress can be achieved by mitigation efforts of leadership in climate policy. Renewable energies need to develop technological innovations for more competitive advantages. This study proposed a decision-making methodology for evaluating renewable energy sources under leadership in international climate policy. This study used the multi-criteria decision-making methodology to deal with conflicting criteria in the decision-making problem. The evaluation is based on the distance from the average solution (EDAS) method, which ranks the alternatives. The criteria weights are computed by using the average method. The neutrosophic set deals with uncertainty and vague information in the evaluation process. The neutrosophic set is integrated with the MCDM methodology. This study uses the concept of TreeSoft Set to deal with the problem as a tree and solve it easily. 15 criteria and 10 alternatives were used in this study. An empirical application was conducted to show the effectiveness of the proposed methodology.

Keywords: Treesoft Set; MCDM; Climate Policy; Leadership; Energy; Policy Making; Organization Ethically and Effectively.

1. Introduction

Countries and governments have the tools to solve the climate crisis in the world. The energy coming from the sun and the wind is not cheap but can be more favorable than conventional technologies[1], [2]. Renewable energy comes from the sky, so technological development is not needed for it[3]. However, using renewable energy at a competitive level requires much effort in developing technological innovations[4], [5].

The green technological program needs effective leadership to materialize and achieve mitigation efforts. Photovoltaics need effective leadership to reduce costs due to the results of policy measures taken by various countries[6], [7]. Hence, these green technologies need to prove the game changers of global climate policy and reduce costs[8], [9].

Multi-criteria decision-making (MCDM) concept is used to deal with various conflicting criteria[10]. MCDM methodology is used to deal with decision-making problems that have various criteria and alternatives. Experts and decision-makers evaluated the criteria and alternatives in decision-making problems[11], [12].

A neutrosophic set (NS) was built to overcome the uncertainty and inconsistency in real issues, and it is better than the fuzzy set (FS)[13]. NS has three aspects in decision-making issues like: truth, indeterminacy, and falsified membership functions. In classical FS, there is only a single membership function of FS[14][15]. NS considers three functions to overcome uncertainty in the evaluation decision-making problem. NS has the indeterminacy function, unlike the intuitionistic Fs[16], [17]. NS membership function helps experts and decision-makers to explain their opinions more accurately[18]. NS has various advantages over the intuitionistic FS, like

- The sum of membership functions may be at most three and can be assigned independently,
- The indeterminacy value cannot depend on the truth and falsify membership degrees.
- They extend decision makers and experts' disagreements.

NS can be applied in decision-making issues such as evaluating renewable sources under leadership and policy.

The Evaluation based on the Distance from Average Solution (EDAS) approach is an MCDM methodology. The EDAS method is efficient in conditions with contradictory criteria. EDAS computes the best alternative by computing the distance of every alternative from the best value. The EDAS method is applied to solve various decision-making problems. The EDAS method has various advantages, such as being comparable to compensatory approaches, allowing criteria to be independent of each other, and converting qualitative criteria to quantitative criteria[19], [20].

The main contributions of this study are:

- This study employed a decision-making methodology for evaluating renewable energy for leadership in international climate policy.
- This study used the neutrosophic set with the MCDM method to overcome uncertainty.
- This study used the treeSoft set with MCDM methodology.
- This study used the EDAS method to rank the alternatives and compute the criteria weights using the average method. It has 15 criteria and 10 alternatives.

The rest of this study is organized as follows: Section 2 defines SVNSs. Section 3 shows the treeSoft set. Section 4 shows the methodology of this study. Section 5 shows the results of this study. Section 6 shows the conclusions.

2. Single-Valued Neutrosophic Sets (SVNSs)

SVNSs are an extension of neutrosophic sets. This part introduces some definitions of SVNSs[21].

Definition 1.
SVNSs can be defined as:

$$0 \leq T_n(x) + I_n(x) + F_n(x) \leq 3 \quad (1)$$

Where $T_n(x), T_n(x), and\ T_n(x)$ refers to the truth, indeterminacy, and falsify functions.

Definition 2.

Let two SVNNs as $a_1 = \left(T_{a_1}, I_{a_1}, F_{a_1}\right) and\ a_2 = \left(T_{a_2}, I_{a_2}, F_{a_2}\right)$ (2)

Some of SVNSs operations can be defined as:

$$a_1 \oplus a_2 = \left(T_{a_1}(x) + T_{a_2}(x) - T_{a_1}(x)T_{a_2}(x), I_{a_1}(x)I_{a_2}(x), F_{a_1}(x)F_{a_2}(x)\right) \quad (3)$$

$$a_1 \otimes a_2 = \begin{pmatrix} T_{a_1}(x)T_{a_2}(x), I_{a_1}(x) + I_{a_2}(x) - I_{a_1}(x)I_{a_2}(x), \\ F_{a_1}(x) + F_{a_2}(x) - F_{a_1}(x)F_{a_2}(x) \end{pmatrix} \quad (4)$$

$$a_1 \cup a_2 = \left(\max\left(T_{a_1}(x), T_{a_2}(x)\right), \min\left(I_{a_1}(x), I_{a_2}(x)\right), \min\left(F_{a_1}(x), F_{a_2}(x)\right)\right) \quad (5)$$

$$a_1 \cap a_2 = \left(\min\left(T_{a_1}(x), T_{a_2}(x)\right), \max\left(I_{a_1}(x), I_{a_2}(x)\right), \max\left(F_{a_1}(x), F_{a_2}(x)\right)\right) \quad (6)$$

Definition 3.

The Euclidean distance and normalized Euclidean distance can be computed as:

$$Distance\ (a_1, a_2) = \sqrt{\sum_{i=1}^{m} \begin{Bmatrix} \left(T_{a_1}(x_i) - T_{a_{12}}(x_i)\right)^2 + \\ \left(I_{a_1}(x_i) - I_{a_2}(x_i)\right)^2 + \\ \left(F_{a_1}(x_i) - F_{a_2}(x_i)\right)^2 \end{Bmatrix}} \quad (7)$$

$$Normalized\ Distance\ (a_1, a_2) = \sqrt{\frac{1}{3m} \times \sum_{i=1}^{m} \begin{Bmatrix} \left(T_{a_1}(x_i) - T_{a_{12}}(x_i)\right)^2 + \\ \left(I_{a_1}(x_i) - I_{a_2}(x_i)\right)^2 + \\ \left(F_{a_1}(x_i) - F_{a_2}(x_i)\right)^2 \end{Bmatrix}} \quad (8)$$

3. TreeSoft Set (TSS)

Let X be a universe disclosure and Y a non-empty subset of X, with $P(Y)$ be a power of Y [22], [23]. Let TSS be a set of criteria for the study,

$$TSS = \{TSS_1, TSS_2, TSS_3, \dots, TSS_n\}, n \geq 1 \quad (9)$$

Where $TSS_1, TSS_2, \dots, TSS_n$ refer to the criteria of the first level of the tree.

Every criterion $TSS_1, 1 \leq i \leq n$ is built by sub-criteria as:

$$TSS_1 = \{TSS_{1,1}, TSS_{1,2}, TSS_{1,3}, TSS_{1,4}, \dots,\} \quad (10)$$

$$TSS_2 = \{TSS_{2,1}, TSS_{2,2}, TSS_{2,3}, TSS_{2,4}, \dots,\} \quad (11)$$

$$TSS_3 = \{TSS_{3,1}, TSS_{3,2}, TSS_{3,3}, TSS_{3,4}, \dots, \} \quad (12)$$

$$TSS_4 = \{TSS_{4,1}, TSS_{4,2}, TSS_{4,3}, TSS_{4,4}, \dots, \} \quad (13)$$

$$TSS_5 = \{TSS_{5,1}, TSS_{5,2}, TSS_{5,3}, TSS_{5,4}, \dots, \} \quad (14)$$

.

.

$$TSS_n = \{TSS_{n,1}, TSS_{n,2}, TSS_{n,3}, TSS_{n,4}, \dots, \} \quad (15)$$

Where $TSS_{i,j}$ refers to the sub-criteria.

The TSS can be built as:

$$F: P\big(Tree(TSS)\big) \to P(Y) \quad (16)$$

Where $Tree(TSS)$ refers to all nodes and leaves from level 1 to level m.

$P\big(Tree(TSS)\big)$ is the power set of the $Tree(TSS)$

$$Tree(TSS) = \begin{pmatrix} \{TSS_i |\ i_1 = 1,2, \dots\} \cup \{TSS_i | i_1, i_2, i_3 = 1,2, , \dots .\} \\ \cup \dots ., \cup (TSS_i | i_1, i_2, \dots, i_m = 1,2, \dots) \end{pmatrix} \quad (17)$$

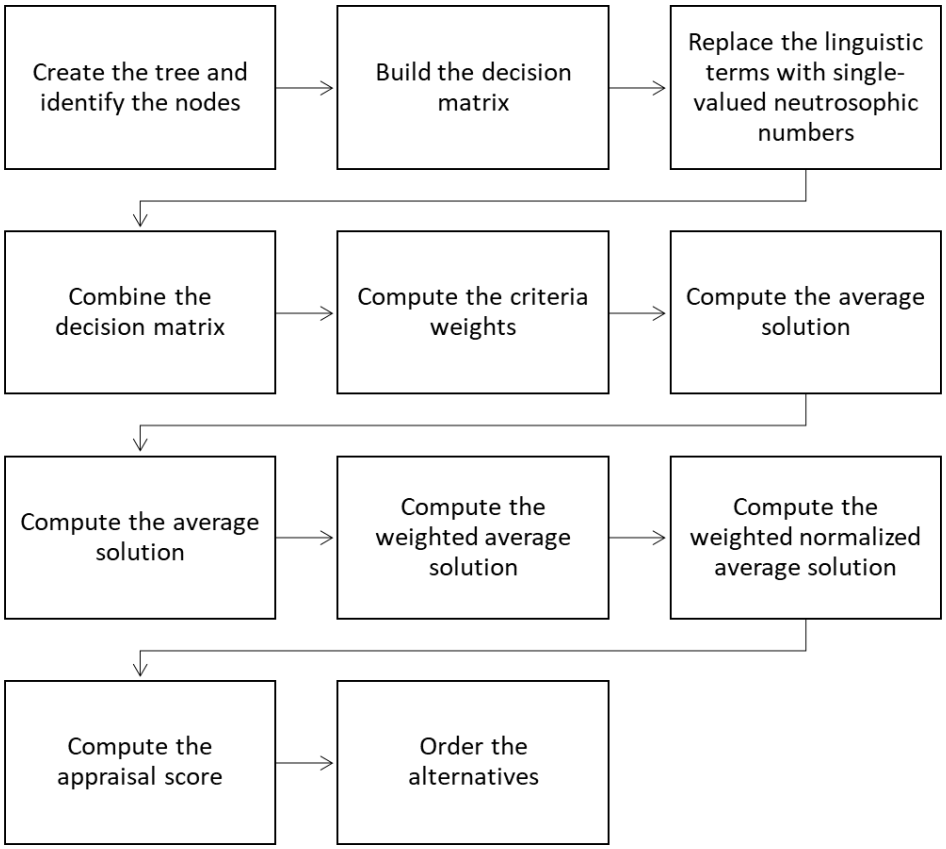
Create the tree and identify the nodes
Build the decision matrix
Replace the linguistic terms with single-valued neutrosophic numbers
Combine the decision matrix
Compute the criteria weights
Compute the average solution
Compute the average solution
Compute the weighted average solution
Compute the weighted normalized average solution
Compute the appraisal score
Order the alternatives

Figure 1. The research methodology

4. TreeSoft Set with MCDM Methodology

This part introduces the steps of TSS with EDAS methodology[24] as shown in Figure 1.

Phase 1: In this phase, Create the tree and identify the nodes. There are various levels in the tree and the criteria are presented in the first level. The second level shows the sub-factors.

Phase 2: In this phase, The decision matrix is built by using the opinions of experts. The experts used the linguistic terms of SVNSs to evaluate the criteria and alternatives.

Phase 3. In this phase, We replaced the linguistic terms with the single-valued neutrosophic numbers. Then we obtain the crisp values. We used the score function to obtain the crisp values[21].

Phase 4. Combine the decision matrix. We combined the opinions of experts into a single matrix as:

$$T = \begin{bmatrix} t_{11} & t_{12} & t_{13} & \cdots & t_{1n-2} & t_{1n-1} & t_{1n} \\ & \vdots & & \ddots & & \vdots & \\ t_{m1} & t_{m2} & t_{m3} & \cdots & t_{m-2n-2} & t_{m-1n-1} & t_{mn} \end{bmatrix}_{ij} \quad i = 1,2,\dots,m; j = 1,2,\dots,n \tag{18}$$

Phase 5. Compute the criteria weights. The criteria weights are computed based on the opinions of experts from the decision matrix by using the average approach. Then we ranked the criteria from highest to lowest.

Phase 6. This phase is used to compute the average solution of every criterion.

$$V_j = \frac{\sum_{i=1}^{m} t_{ij}}{m}; j = 1,2,\dots,n \tag{19}$$

Phase 7. Compute the positive distance from the average solution and the negative distance from the average solution.

$$R_{ij} = \frac{\max\left(0,(t_{ij} - V_j)\right)}{V_j} \tag{20}$$

$$R_{ij} = \frac{\max\left(0,(V_j - t_{ij})\right)}{V_j} \tag{21}$$

$$R_{ij} = \frac{\max\left(0,(V_j - t_{ij})\right)}{V_j} \tag{22}$$

$$R_{ij} = \frac{\max\left(0,(t_{ij} - v_j)\right)}{V_j} \tag{23}$$

Phase 8. In this phase, we compute the weighted average solution from the positive and negative distance as

$$Q_i = \sum_{j=1}^{n} R_{ij} W_j \tag{24}$$

Phase 9. This phase computes the weighted normalized average solution for positive and negative criteria.

$$S_i = \frac{Q_i}{\max\limits_i(Q_i)} \tag{25}$$

Phase 10. Compute the appraisal score

$$D_i = \frac{1}{2}\left(S_{i_{posotive}} + S_{i_{negative}}\right) \tag{26}$$

Phase 11. Order the alternatives. The alternatives are ordered based on the descending orders of the appraisal score.

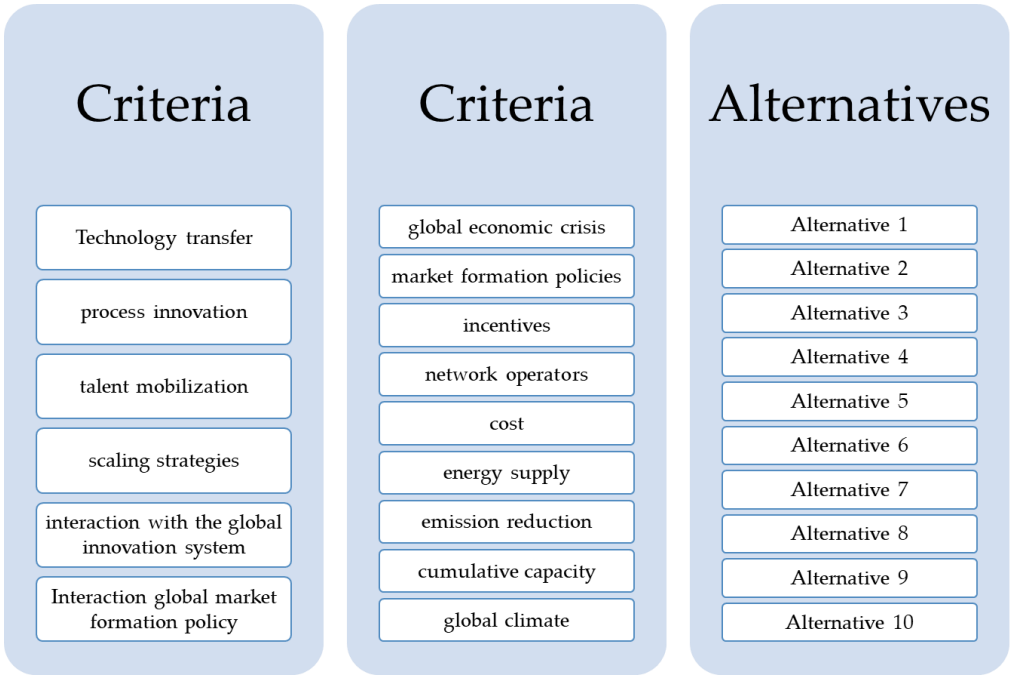

Figure 2. The criteria and alternatives.

5. Results and Discussions

This part shows the results of the SVNS-TSS and EDAS methods.

Phase 1: The tree is built and the nodes are defined. There are 15 criteria collected in this study and 10 alternatives as shown in Figure 2.

Phase 2: The decision matrix between criteria and alternatives is built as shown in Table 1.

Phase 3. Replace the linguistic terms with single-valued neutrosophic numbers.

We obtained the crisp values by using the score function.

Phase 4. The decision matrix is combined to obtain a single decision matrix.

Phase 5. The criteria weights are computed as shown in Figure 3.

We showed that criterion 2 has the highest weight equal to 0.076, followed by criterion 1 with a weight equal to 0.075, and criterion 5 with a weight equal to 0.071. We showed that criterion 3 has the lowest weight equal to 0.0564, followed by criterion 7 with a weight equal to 0.568, and criterion 8 with a weight equal to 0.61.

Table 1. The decision matrix.

	RWR_1	RWR_2	RWR_3	RWR_4	RWR_5	RWR_6	RWR_7	RWR_8	RWR_9	RWR_{10}
RWC_1	(0.9,0.1,0.05)	(0.65,0.35,0.30)	(0.8,0.2,0.15)	(0.9,0.1,0.05)	(0.2,0.75,0.80)	(0.35,0.65,0.60)	(0.5,0.5,0.45)	(0.65,0.35,0.30)	(0.8,0.2,0.15)	(0.8,0.2,0.15)
RWC_2	(0.8,0.2,0.15)	(0.5,0.5,0.45)	(0.2,0.75,0.80)	(0.5,0.5,0.45)	(0.8,0.2,0.15)	(0.8,0.2,0.15)	(0.9,0.1,0.05)	(0.9,0.1,0.05)	(0.5,0.5,0.45)	(0.9,0.1,0.05)
RWC_3	(0.65,0.35,0.30)	(0.35,0.65,0.60)	(0.2,0.75,0.80)	(0.5,0.5,0.45)	(0.2,0.75,0.80)	(0.35,0.65,0.60)	(0.35,0.65,0.60)	(0.5,0.5,0.45)	(0.65,0.35,0.30)	(0.2,0.75,0.80)
RWC_4	(0.5,0.5,0.45)	(0.2,0.75,0.80)	(0.9,0.1,0.05)	(0.9,0.1,0.05)	(0.8,0.2,0.15)	(0.9,0.1,0.05)	(0.5,0.5,0.45)	(0.65,0.35,0.30)	(0.8,0.2,0.15)	(0.2,0.75,0.80)
RWC_5	(0.35,0.65,0.60)	(0.9,0.1,0.05)	(0.8,0.2,0.15)	(0.8,0.2,0.15)	(0.9,0.1,0.05)	(0.8,0.2,0.15)	(0.65,0.35,0.30)	(0.8,0.2,0.15)	(0.9,0.1,0.05)	(0.35,0.65,0.60)
RWC_6	(0.2,0.75,0.80)	(0.8,0.2,0.15)	(0.65,0.35,0.30)	(0.65,0.35,0.30)	(0.9,0.1,0.05)	(0.65,0.35,0.30)	(0.8,0.2,0.15)	(0.9,0.1,0.05)	(0.2,0.75,0.80)	(0.5,0.5,0.45)
RWC_7	(0.2,0.75,0.80)	(0.5,0.5,0.45)	(0.5,0.5,0.45)	(0.5,0.5,0.45)	(0.2,0.75,0.80)	(0.5,0.5,0.45)	(0.9,0.1,0.05)	(0.35,0.65,0.60)	(0.35,0.65,0.60)	(0.65,0.35,0.30)
RWC_8	(0.9,0.1,0.05)	(0.35,0.65,0.60)	(0.35,0.65,0.60)	(0.35,0.65,0.60)	(0.2,0.75,0.80)	(0.35,0.65,0.60)	(0.2,0.75,0.80)	(0.2,0.75,0.80)	(0.5,0.5,0.45)	(0.8,0.2,0.15)
RWC_9	(0.9,0.1,0.05)	(0.2,0.75,0.80)	(0.2,0.75,0.80)	(0.9,0.1,0.05)	(0.35,0.65,0.60)	(0.2,0.75,0.80)	(0.35,0.65,0.60)	(0.5,0.5,0.45)	(0.65,0.35,0.30)	(0.9,0.1,0.05)
RWC_{10}	(0.9,0.1,0.05)	(0.65,0.35,0.30)	(0.2,0.75,0.80)	(0.8,0.2,0.15)	(0.5,0.5,0.45)	(0.2,0.75,0.80)	(0.5,0.5,0.45)	(0.8,0.2,0.15)	(0.8,0.2,0.15)	(0.2,0.75,0.80)
RWC_{11}	(0.8,0.2,0.15)	(0.65,0.35,0.30)	(0.5,0.5,0.45)	(0.65,0.35,0.30)	(0.65,0.35,0.30)	(0.8,0.2,0.15)	(0.65,0.35,0.30)	(0.9,0.1,0.05)	(0.9,0.1,0.05)	(0.2,0.75,0.80)
RWC_{12}	(0.8,0.2,0.15)	(0.5,0.5,0.45)	(0.5,0.5,0.45)	(0.2,0.75,0.80)	(0.8,0.2,0.15)	(0.9,0.1,0.05)	(0.8,0.2,0.15)	(0.5,0.5,0.45)	(0.35,0.65,0.60)	(0.35,0.65,0.60)
RWC_{13}	(0.65,0.35,0.30)	(0.35,0.65,0.60)	(0.2,0.75,0.80)	(0.9,0.1,0.05)	(0.9,0.1,0.05)	(0.8,0.2,0.15)	(0.8,0.2,0.15)	(0.65,0.35,0.30)	(0.5,0.5,0.45)	(0.5,0.5,0.45)
RWC_{14}	(0.5,0.5,0.45)	(0.2,0.75,0.80)	(0.5,0.5,0.45)	(0.5,0.5,0.45)	(0.35,0.65,0.60)	(0.2,0.75,0.80)	(0.9,0.1,0.05)	(0.8,0.2,0.15)	(0.65,0.35,0.30)	(0.65,0.35,0.30)
RWC_{15}	(0.5,0.5,0.45)	(0.9,0.1,0.05)	(0.8,0.2,0.15)	(0.65,0.35,0.30)	(0.5,0.5,0.45)	(0.35,0.65,0.60)	(0.2,0.75,0.80)	(0.2,0.75,0.80)	(0.9,0.1,0.05)	(0.8,0.2,0.15)
	RWR_1	RWR_2	RWR_3	RWR_4	RWR_5	RWR_6	RWR_7	RWR_8	RWR_9	RWR_{10}
RWC_1	(0.9,0.1,0.05)	(0.65,0.35,0.30)	(0.5,0.5,0.45)	(0.5,0.5,0.45)	(0.2,0.75,0.80)	(0.2,0.75,0.80)	(0.8,0.2,0.15)	(0.35,0.65,0.60)	(0.9,0.1,0.05)	(0.8,0.2,0.15)
RWC_2	(0.8,0.2,0.15)	(0.5,0.5,0.45)	(0.2,0.75,0.80)	(0.2,0.75,0.80)	(0.8,0.2,0.15)	(0.35,0.65,0.60)	(0.9,0.1,0.05)	(0.9,0.1,0.05)	(0.5,0.5,0.45)	(0.9,0.1,0.05)
RWC_3	(0.65,0.35,0.30)	(0.35,0.65,0.60)	(0.5,0.5,0.45)	(0.2,0.75,0.80)	(0.2,0.75,0.80)	(0.8,0.2,0.15)	(0.35,0.65,0.60)	(0.9,0.1,0.05)	(0.65,0.35,0.30)	(0.2,0.75,0.80)
RWC_4	(0.5,0.5,0.45)	(0.2,0.75,0.80)	(0.2,0.75,0.80)	(0.8,0.2,0.15)	(0.35,0.65,0.60)	(0.9,0.1,0.05)	(0.9,0.1,0.05)	(0.65,0.35,0.30)	(0.8,0.2,0.15)	(0.2,0.75,0.80)
RWC_5	(0.35,0.65,0.60)	(0.5,0.5,0.45)	(0.5,0.5,0.45)	(0.2,0.75,0.80)	(0.2,0.75,0.80)	(0.8,0.2,0.15)	(0.35,0.65,0.60)	(0.9,0.1,0.05)	(0.9,0.1,0.05)	(0.35,0.65,0.60)
RWC_6	(0.2,0.75,0.80)	(0.5,0.5,0.45)	(0.2,0.75,0.80)	(0.2,0.75,0.80)	(0.5,0.5,0.45)	(0.2,0.75,0.80)	(0.2,0.75,0.80)	(0.8,0.2,0.15)	(0.35,0.65,0.60)	(0.9,0.1,0.05)
RWC_7	(0.2,0.75,0.80)	(0.5,0.5,0.45)	(0.5,0.5,0.45)	(0.2,0.75,0.80)	(0.2,0.75,0.80)	(0.8,0.2,0.15)	(0.35,0.65,0.60)	(0.9,0.1,0.05)	(0.35,0.65,0.60)	(0.65,0.35,0.30)
RWC_8	(0.9,0.1,0.05)	(0.5,0.5,0.45)	(0.2,0.75,0.80)	(0.5,0.5,0.45)	(0.2,0.75,0.80)	(0.2,0.75,0.80)	(0.8,0.2,0.15)	(0.35,0.65,0.60)	(0.9,0.1,0.05)	(0.8,0.2,0.15)
RWC_9	(0.9,0.1,0.05)	(0.5,0.5,0.45)	(0.2,0.75,0.80)	(0.2,0.75,0.80)	(0.5,0.5,0.45)	(0.2,0.75,0.80)	(0.2,0.75,0.80)	(0.8,0.2,0.15)	(0.35,0.65,0.60)	(0.9,0.1,0.05)
RWC_{10}	(0.5,0.5,0.45)	(0.2,0.75,0.80)	(0.5,0.5,0.45)	(0.2,0.75,0.80)	(0.2,0.75,0.80)	(0.8,0.2,0.15)	(0.35,0.65,0.60)	(0.9,0.1,0.05)	(0.8,0.2,0.15)	(0.2,0.75,0.80)
RWC_{11}	(0.8,0.2,0.15)	(0.65,0.35,0.30)	(0.5,0.5,0.45)	(0.5,0.5,0.45)	(0.2,0.75,0.80)	(0.2,0.75,0.80)	(0.8,0.2,0.15)	(0.35,0.65,0.60)	(0.9,0.1,0.05)	(0.2,0.75,0.80)
RWC_{12}	(0.8,0.2,0.15)	(0.5,0.5,0.45)	(0.2,0.75,0.80)	(0.2,0.75,0.80)	(0.8,0.2,0.15)	(0.35,0.65,0.60)	(0.9,0.1,0.05)	(0.5,0.5,0.45)	(0.35,0.65,0.60)	(0.35,0.65,0.60)
RWC_{13}	(0.5,0.5,0.45)	(0.2,0.75,0.80)	(0.2,0.75,0.80)	(0.8,0.2,0.15)	(0.35,0.65,0.60)	(0.9,0.1,0.05)	(0.9,0.1,0.05)	(0.65,0.35,0.30)	(0.5,0.5,0.45)	(0.5,0.5,0.45)
RWC_{14}	(0.5,0.5,0.45)	(0.5,0.5,0.45)	(0.2,0.75,0.80)	(0.2,0.75,0.80)	(0.8,0.2,0.15)	(0.35,0.65,0.60)	(0.9,0.1,0.05)	(0.35,0.65,0.60)	(0.9,0.1,0.05)	(0.65,0.35,0.30)
RWC_{15}	(0.5,0.5,0.45)	(0.2,0.75,0.80)	(0.2,0.75,0.80)	(0.8,0.2,0.15)	(0.5,0.5,0.45)	(0.2,0.75,0.80)	(0.2,0.75,0.80)	(0.8,0.2,0.15)	(0.35,0.65,0.60)	(0.9,0.1,0.05)
	RWR_1	RWR_2	RWR_3	RWR_4	RWR_5	RWR_6	RWR_7	RWR_8	RWR_9	RWR_{10}
RWC_1	(0.9,0.1,0.05)	(0.65,0.35,0.30)	(0.65,0.35,0.30)	(0.65,0.35,0.30)	(0.5,0.5,0.45)	(0.65,0.35,0.30)	(0.65,0.35,0.30)	(0.5,0.5,0.45)	(0.35,0.65,0.60)	(0.2,0.75,0.80)
RWC_2	(0.8,0.2,0.15)	(0.5,0.5,0.45)	(0.65,0.35,0.30)	(0.65,0.35,0.30)	(0.5,0.5,0.45)	(0.65,0.35,0.30)	(0.65,0.35,0.30)	(0.5,0.5,0.45)	(0.35,0.65,0.60)	(0.2,0.75,0.80)
RWC_3	(0.65,0.35,0.30)	(0.65,0.35,0.30)	(0.65,0.35,0.30)	(0.5,0.5,0.45)	(0.35,0.65,0.60)	(0.2,0.75,0.80)	(0.65,0.35,0.30)	(0.5,0.5,0.45)	(0.35,0.65,0.60)	(0.2,0.75,0.80)
RWC_4	(0.5,0.5,0.45)	(0.2,0.75,0.80)	(0.65,0.35,0.30)	(0.65,0.35,0.30)	(0.5,0.5,0.45)	(0.35,0.65,0.60)	(0.2,0.75,0.80)	(0.65,0.35,0.30)	(0.8,0.2,0.15)	(0.2,0.75,0.80)
RWC_5	(0.35,0.65,0.60)	(0.65,0.35,0.30)	(0.65,0.35,0.30)	(0.5,0.5,0.45)	(0.35,0.65,0.60)	(0.65,0.35,0.30)	(0.65,0.35,0.30)	(0.5,0.5,0.45)	(0.35,0.65,0.60)	(0.2,0.75,0.80)
RWC_6	(0.2,0.75,0.80)	(0.8,0.2,0.15)	(0.65,0.35,0.30)	(0.65,0.35,0.30)	(0.5,0.5,0.45)	(0.35,0.65,0.60)	(0.2,0.75,0.80)	(0.9,0.1,0.05)	(0.2,0.75,0.80)	(0.5,0.5,0.45)
RWC_7	(0.2,0.75,0.80)	(0.5,0.5,0.45)	(0.5,0.5,0.45)	(0.5,0.5,0.45)	(0.2,0.75,0.80)	(0.65,0.35,0.30)	(0.65,0.35,0.30)	(0.5,0.5,0.45)	(0.35,0.65,0.60)	(0.2,0.75,0.80)
RWC_8	(0.9,0.1,0.05)	(0.35,0.65,0.60)	(0.35,0.65,0.60)	(0.65,0.35,0.30)	(0.65,0.35,0.30)	(0.65,0.35,0.30)	(0.65,0.35,0.30)	(0.5,0.5,0.45)	(0.35,0.65,0.60)	(0.2,0.75,0.80)
RWC_9	(0.65,0.35,0.30)	(0.65,0.35,0.30)	(0.5,0.5,0.45)	(0.65,0.35,0.30)	(0.65,0.35,0.30)	(0.5,0.5,0.45)	(0.35,0.65,0.60)	(0.2,0.75,0.80)	(0.65,0.35,0.30)	(0.9,0.1,0.05)
RWC_{10}	(0.9,0.1,0.05)	(0.65,0.35,0.30)	(0.65,0.35,0.30)	(0.65,0.35,0.30)	(0.5,0.5,0.45)	(0.35,0.65,0.60)	(0.2,0.75,0.80)	(0.8,0.2,0.15)	(0.8,0.2,0.15)	(0.2,0.75,0.80)
RWC_{11}	(0.8,0.2,0.15)	(0.65,0.35,0.30)	(0.65,0.35,0.30)	(0.5,0.5,0.45)	(0.65,0.35,0.30)	(0.65,0.35,0.30)	(0.5,0.5,0.45)	(0.35,0.65,0.60)	(0.2,0.75,0.80)	(0.2,0.75,0.80)
RWC_{12}	(0.8,0.2,0.15)	(0.5,0.5,0.45)	(0.65,0.35,0.30)	(0.65,0.35,0.30)	(0.5,0.5,0.45)	(0.35,0.65,0.60)	(0.2,0.75,0.80)	(0.5,0.5,0.45)	(0.35,0.65,0.60)	(0.35,0.65,0.60)
RWC_{13}	(0.65,0.35,0.30)	(0.35,0.65,0.60)	(0.2,0.75,0.80)	(0.65,0.35,0.30)	(0.65,0.35,0.30)	(0.5,0.5,0.45)	(0.35,0.65,0.60)	(0.2,0.75,0.80)	(0.5,0.5,0.45)	(0.5,0.5,0.45)
RWC_{14}	(0.65,0.35,0.30)	(0.65,0.35,0.30)	(0.5,0.5,0.45)	(0.65,0.35,0.30)	(0.65,0.35,0.30)	(0.5,0.5,0.45)	(0.35,0.65,0.60)	(0.2,0.75,0.80)	(0.65,0.35,0.30)	(0.65,0.35,0.30)
RWC_{15}	(0.65,0.35,0.30)	(0.65,0.35,0.30)	(0.5,0.5,0.45)	(0.65,0.35,0.30)	(0.65,0.35,0.30)	(0.5,0.5,0.45)	(0.35,0.65,0.60)	(0.2,0.75,0.80)	(0.9,0.1,0.05)	(0.8,0.2,0.15)

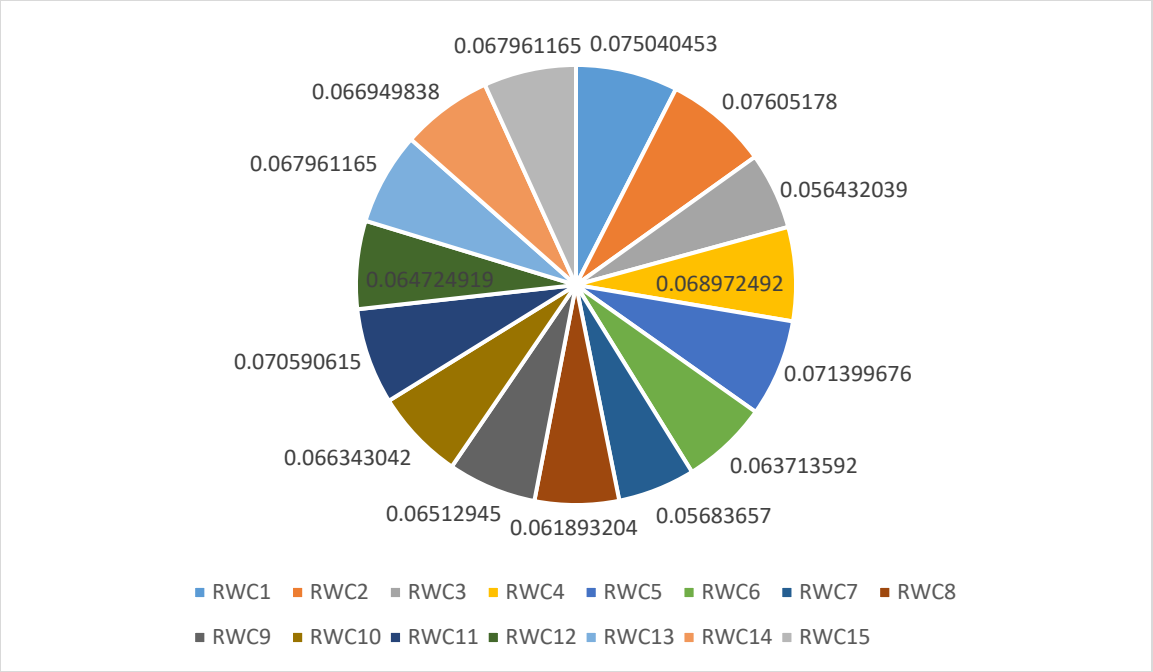

Figure 3. The criteria weighs.

Phase 6. Eq. (19) is used to compute the average solution.

Phase 7. Eqs. (20-23) are used to compute the positive and negative distance from the average solution as shown in Table 2.

Phase 8. Eq. (24) is used to compute the weighted average solution as shown in Table 3.

Phase 9. Eq. (25) is used to compute the weighted normalized average solution as shown in Table 4.

Table 2. The distance from the positive and negative average solution.

	RWR_1	RWR_2	RWR_3	RWR_4	RWR_5	RWR_6	RWR_7	RWR_8	RWR_9	RWR_{10}
RWC_1	0.48248	0.078167	0.078167	0.132075	0	0	0.078167	0	0.132075	0
RWC_2	0.303191	0	0	0	0.143617	0	0.329787	0.25	0	0.090426
RWC_3	0.433692	0.003584	0.003584	0	0	0.003584	0.003584	0.397849	0.218638	0
RWC_4	0	0	0.055718	0.407625	0	0.290323	0	0.173021	0.43695	0
RWC_5	0	0.189802	0.133144	0	0	0.303116	0	0.274788	0.246459	0
RWC_6	0	0.365079	0	0	0.238095	0	0	0.68254	0	0.238095
RWC_7	0	0.103203	0.103203	0	0	0.423488	0.3879	0.281139	0	0.103203
RWC_8	0.797386	0	0	0.013072	0	0	0.111111	0	0.176471	0.20915
RWC_9	0.552795	0	0	0.118012	0	0	0	0	0.055901	0.708075
RWC_{10}	0.432927	0	0	0.036585	0	0	0	0.554878	0.493902	0
RWC_{111}	0.404011	0.146132	0	0	0	0	0.146132	0	0.174785	0
RWC_{12}	0.53125	0	0	0	0.34375	0.03125	0.21875	0	0	0
RWC_{13}	0.10119	0	0	0.428571	0.160714	0.339286	0.25	0	0	0
RWC_{14}	0.02719	0	0	0	0.117825	0	0.329305	0	0.359517	0.208459
RWC_{15}	0	0	0.077381	0	0	0.345238	0.52381	0.255952	0	0

Table 3. The weighted average solution.

	RWR_1	RWR_2	RWR_3	RWR_4	RWR_5	RWR_6	RWR_7	RWR_8	RWR_9	RWR_{10}
RWC_1	0.036206	0.005866	0.005866	0.009911	0	0	0.005866	0	0.009911	0
RWC_2	0.023058	0	0	0	0.010922	0	0.025081	0.019013	0	0.006877
RWC_3	0.024474	0.000202	0.000202	0	0	0.000202	0.000202	0.022451	0.012338	0
RWC_4	0	0	0.003843	0.028115	0	0.020024	0	0.011934	0.030138	0
RWC_5	0	0.013552	0.009506	0	0	0.021642	0	0.01962	0.017597	0
RWC_6	0	0.023261	0	0	0.01517	0	0	0.043487	0	0.01517
RWC_7	0	0.005866	0.005866	0	0	0.02407	0.022047	0.015979	0	0.005866
RWC_8	0.049353	0	0	0.000809	0	0	0.006877	0	0.010922	0.012945
RWC_9	0.036003	0	0	0.007686	0	0	0	0	0.003641	0.046117
RWC_{10}	0.028722	0	0	0.002427	0	0	0	0.036812	0.032767	0
RWC_{111}	0.028519	0.010316	0	0	0	0	0.010316	0	0.012338	0
RWC_{12}	0.034385	0	0	0	0.022249	0.002023	0.014159	0	0	0
RWC_{13}	0.006877	0	0	0.029126	0.010922	0.023058	0.01699	0	0	0
RWC_{14}	0.00182	0	0	0	0.007888	0	0.022047	0	0.02407	0.013956
RWC_{15}	0	0	0.005259	0	0	0.023463	0.035599	0.017395	0	0

Table 4. The weighted normalized average solution.

	RWR_1	RWR_2	RWR_3	RWR_4	RWR_5	RWR_6	RWR_7	RWR_8	RWR_9	RWR_{10}
RWC_1	1	0.162011	0.162011	0.273743	0	0	0.162011	0	0.273743	0
RWC_2	0.919355	0	0	0	0.435484	0	1	0.758065	0	0.274194
RWC_3	1	0.008264	0.008264	0	0	0.008264	0.008264	0.917355	0.504132	0
RWC_4	0	0	0.127517	0.932886	0	0.66443	0	0.395973	1	0
RWC_5	0	0.626168	0.439252	0	0	1	0	0.906542	0.813084	0
RWC_6	0	0.534884	0	0	0.348837	0	0	1	0	0.348837
RWC_7	0	0.243697	0.243697	0	0	1	0.915966	0.663866	0	0.243697
RWC_8	1	0	0	0.016393	0	0	0.139344	0	0.221311	0.262295
RWC_9	0.780702	0	0	0.166667	0	0	0	0	0.078947	1
RWC_{10}	0.78022	0	0	0.065934	0	0	0	1	0.89011	0
RWC_{111}	1	0.361702	0	0	0	0	0.361702	0	0.432624	0
RWC_{12}	1	0	0	0	0.647059	0.058824	0.411765	0	0	0
RWC_{13}	0.236111	0	0	1	0.375	0.791667	0.583333	0	0	0
RWC_{14}	0.07563	0	0	0	0.327731	0	0.915966	0	1	0.579832
RWC_{15}	0	0	0.147727	0	0	0.659091	1	0.488636	0	0

Phase 10. Eq. (26) is used to compute the appraisal score value as shown in Figure 4.

Phase 11. Order the alternatives

Figure 4 shows the alternative 1 has the highest score, followed by alternative 8 and alternative 7. Alternative 3 has the lowest rank followed by alternative 2 and alternative 5.

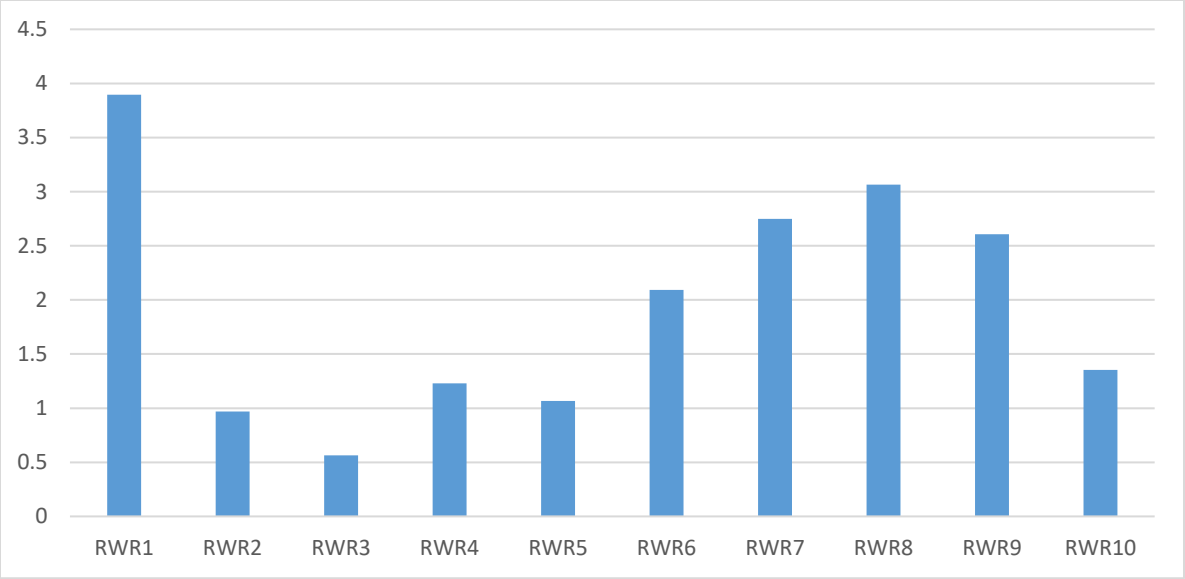

Figure 4. The appraisal score for each alternative.

6. Conclusions

This study proposed a decision-making methodology to evaluate the renewable energy sources under leadership in international climate policy. This study applied the EDAS method to rank the alternatives. A single-valued neutrosophic set is used to deal with vague and uncertain information. Three experts and decision-makers with experience in leadership, policy, and renewable energy are invited to evaluate the criteria and alternatives. They used the linguistic terms of single-valued neutrosophic sets. Then, we replace these terms with single-valued neutrosophic numbers. Then, we obtained the crisp values. Then, we combined the decision matrix into a single decision matrix. This study used the TreeSoft set to deal with the problem as a tree. Fifteen criteria and 10 alternatives were used in this study. The criteria weights are computed by using the average method. The results show that Criterion 2 has the highest importance and Criterion 3 has the lowest. EDAS method is used to rank the alternatives. The average method is computed between criteria and alternatives. The positive and negative distance from the positive and negative criteria are calculated. The rank results show that alternative 1 is the best and alternative 3 is the worst.

Various MCDM methods, such as AHP, BWM, and DEMATEL, can be applied to compute the criteria weights. The relationship between criteria can be calculated. There are various rank methods can be used, such as TOPSIS, VIKOR, etc.

Conflict of interest

The authors declare that there is no conflict of interest in the research.

Funding

This research has no funding source.

Ethical approval

This article does not contain any studies with human participants or animals performed by any of the authors.

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Received: February 26, 2024. **Accepted**: July 29, 2024.

Neutrosophic Sets and Systems, Vol. 70, 2024
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Generalized Symmetric Fermatean Neutrosophic Fuzzy Matrices

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Abstract – This study explores a new type of matrix called a range-symmetric Fermatean neutrosophic fuzzy matrix (FNFM), inspired by the concept of range-Hermitian matrices. We demonstrate that all FNFMs inherently possess a specific property we term "Pythagorean neutrosophic fuzzy," (PNFM) but the reverse is not always true. Furthermore, we delve into graphical representations of FNFMs with specific symmetry properties (kernel-symmetric (KS), column symmetric, and range-symmetric (RS)) and show that these properties hold for all isomorphic graphs. The study goes on to establish equivalent characterizations for range-symmetric FNFMs and identify conditions for KS FNFMs. We introduce a novel concept: k-KS and RS FNFMs. Examples illustrate that KS FNFMs inherently possess k-KS, but not necessarily the other way around. This research contributes to a deeper understanding of symmetric FNFM and their potential applications, highlighting their importance in mathematical and computational fields.

Keywords: PNFM, FNFM, NFM

1. Introduction

This paper delves into Generalized Symmetric FNFM, a recent development in representing uncertainty. We begin with the fundamental concept of FS, introduced by Zadeh [1], which use membership degrees to handle vagueness. Recognizing limitations in assigning non-membership values, Atanassov introduced intuitionistic fuzzy sets [2]. Smarandache further extended this framework with neutrosophic sets (NSs) to encompass indeterminacy [3]. Building on these ideas, Wang et al. [4, 5] proposed single-valued and interval-valued neutrosophic sets, expanding their applicability.

The concept of symmetric fuzzy matrices with properties based on range and kernel was explored by Kim and Roush [6]. They showed that range symmetry implies kernel symmetry, but not vice versa. Meenakshi [7] introduced FM with a fixed product, leading to further research on k-real and k-Hermitian matrices [8]. Baskett and Katz [9] and Schwerdtfeger [10] also contributed significantly to the field.

Recent studies by Meenakshi and colleagues [11, 12, 13, 14] explored various aspects of symmetric fuzzy matrices, including k-kernel symmetric and k-range symmetric properties. Sumathi and Arockiarani [15] proposed new operations on FNSM, while Meenakshi and Krishnamoorthy [16] introduced k-EP matrices. Jaya Shree [17] studied secondary κ-RSFM. Anandhkumar et al. [18] characterized Generalized Symmetric NFMs, and Broumi et al. [19] discussed Fermatean neutrosophic matrices. Silambarasan [20] further explored Fermatean fuzzy matrices.

Neutrosophic theory, introduced by Smarandache [3], embraces indeterminacy, acknowledging that truth, falsity, and indeterminacy can coexist. Neutrosophic sets address this by using membership degrees for truth, falsity, and indeterminacy. The study of matrices has evolved significantly to accommodate uncertainty, leading to the development of GSFNFM. This paper aims to explore the development of GSFNFM, discuss their theoretical foundations, mathematical properties, and potential applications. Anandhkumar [21] et.al, have studied Interval Valued Secondary k-Range Symmetric NFMs,

GSFNFM represent a novel approach to modeling uncertainty by combining Fermatean algebra, neutrosophic theory, and fuzzy logic. Fermatean algebra extends classical algebra to include three logical states: true, false, and indeterminate. This framework allows for a structured representation of uncertainty. We will explore the theoretical underpinnings of GSFNFM, their mathematical properties, and potential applications in various domains. Anandhkumar [22] et.al, have studied Secondary K- CSNFM.

1.1 Research Gap

In our research, we aim to introduce two innovative categories of NFM the RS-FNFM and the KS-FNFM. These matrices draw inspiration from EP-matrices within the complex domain and offer fresh perspectives on representing uncertainty and indeterminacy.

Our study establishes that while every FNFM qualifies as a PNFM, the reverse relationship does not always hold. Additionally, we depict visual representations of KS, CS, and RS adjacency FNFM to demonstrate their versatility in various scenarios, particularly in depicting relationships within isomorphic graphs.

Furthermore, we introduce RS-FNFM and derive environments for KS-FNFM, shedding light on their characteristics and interrelations. This exploration aids in comprehending the basic structures and constraints foremost these matrices.

Moreover, we present the idea of k-KS and RS-FNFM, providing illustrations that illustrate their relations. Exactly, we demonstrate that KS implies k-KS, thereby deepening our understanding of the interactions between different forms of SNFM.

Our research expands the comprehension of symmetric NFM and their practical applications, mostly in mathematical and computational sciences. By presenting novel matrix types and investigating their properties, we pay to forward both theoretic frameworks and applied uses of NFM. This underscores the worth of our findings in proceeding the understanding and utilization of SFM.

Prior research has laid a foundation for understanding symmetric fuzzy matrices, including explorations of k-kernel symmetric and k-range symmetric properties [11, 12, 13, 14]. However, the specific application of these symmetry concepts to Fermatean neutrosophic fuzzy matrices (FNFMs) remains underexplored. This gap in knowledge motivates our current investigation.

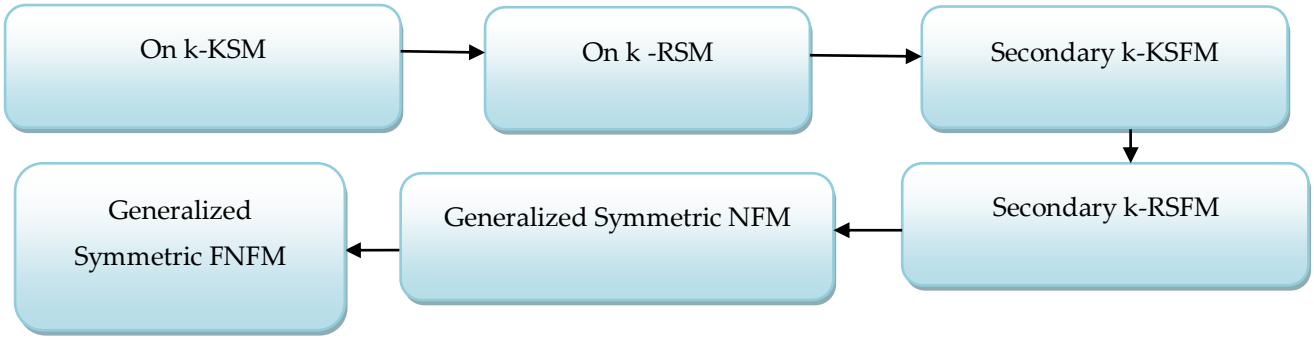
Building on the work of Anandhkumar et al. [18] who presented range and KS to NFM, we extend these principles to FNFMs. While their work represents a significant advancement, a critical research gap persists regarding generalized symmetric properties in FNFMs. In particular, no prior research has investigated how range and KS principles apply to FNFMs, nor have the properties of such matrices been well-characterized.

Our study addresses this gap by:

We propose two new types of FNFMs: RS-FNFM) and KS-FNFM. These matrices draw inspiration from EP-matrices in the complex domain and offer a fresh perspective for representing uncertainty and indeterminacy, especially when neutrosophic logic is relevant. We go beyond simply introducing new matrix types. Our research explores the connections between these novel FNFMs and existing concepts. We will establish that all FNFMs possess a property we call "Pythagorean NFM," but the converse is not always true. This distinction provides valuable insights into the characteristics of these matrices. To enhance understanding and highlight their applicability, we will explore graphical representations of KS, CS, and RS adjacency FNFMs. This visualization is particularly useful in representing relationships within isomorphic graphs. We will establish equivalent characterizations for RS-FNFM and identify conditions for KS-FNFM. These findings will provide a deeper understanding of the fundamental constructions and relationships leading these matrices. We will present the concept of k-KS and RS-FNFMs and illustrate their connections through examples. We will show that a kernel-symmetric FNFM inherently possesses k-KS, though the converse does not always hold. This analysis sheds light on the interplay between different types of symmetric NFM.

By addressing these research objectives, our work aims to significantly advance the understanding of symmetric NFM and their applications, particularly in mathematical and computational domains.

We introduce novel matrix types, explore their properties and relationships with existing concepts, and showcase their potential for representing uncertainty in various contexts.



1.2 Novelty

This study introduces several key elements that contribute to its originality and significance:

We propose two new types of matrices, RS-FNFM and KS-FNFM. Inspired by EP-matrices, these matrices offer a fresh perspective for representing uncertainty and indeterminacy, particularly when dealing with neutrosophic logic. Our research goes beyond simply introducing new matrices. We extend the well-established principles of symmetric matrices to the domain of FNFMs. This expansion leads to the development of equivalent characterizations for RS-FNFM and the identification of conditions for KS-FNFM. These findings provide deeper insights into the properties and underlying structures governing these novel matrices. We delve into the connections between different types of symmetric NFMs. For instance, we demonstrate that a kernel-symmetric FNFM inherently possesses k-KS, although the converse is not always true. This analysis sheds light on the intricate interplay between various symmetry properties within the framework of NFMs. To enhance understanding and showcase the applicability of these matrices, we explore graphical representations of KS, CS, and RS adjacency FNFMs.

This visualization is particularly valuable in representing relationships within isomorphic graphs. Our research identifies and addresses a critical gap in the existing literature. While prior studies have explored various types of symmetric fuzzy matrices, the specific application of these concepts to FNFM has remained largely unexamined. This work fills this void by introducing novel matrix types, exploring their properties, and highlighting their potential applications. By introducing these novel matrices and exploring their properties, we contribute to a more comprehensive understanding of symmetric NFM and their potential applications in mathematical and computational domains. This work advances the theoretical foundations of NFM and paves the way for their broader practical use. These elements collectively demonstrate the originality and significance of our research, offering valuable insights and expanding the existing framework for representing uncertainty in various fields.

1.3 Notations:

For FNFM of $P=[P_T,P_I,P_F]\in(FNFM)_n$

$[P_T,P_I,P_F]^T$: Transpose of $[P_T,P_I,P_F]$,

R($[P_T,P_I,P_F]$) : Row space of $[P_T,P_I,P_F]$,

C($[P_T,P_I,P_F]$) : Column space of $[P_T,P_I,P_F]$

N($[P_T,P_I,P_F]$) : Null Space of $[P_T,P_I,P_F]$,

$[P_T,P_I,P_F]^+$: Moore-Penrose inverse of $[P_T,P_I,P_F]$,

GSFNFM: Generalized Symmetric Fermatean Neutrosophic Fuzzy Matrices.

1.4. PRELIMINARIES: The permutation matrix K is satisfied using the subsequent

(P_1) $K = K^T$, $K^2 = I$ for all $P=[P_T,P_I,P_F]\in(FNFM)_n$

(P_2) $N([P_T,P_I,P_F]) = N([P_T,P_I,P_F]K) = N(K[P_T,P_I,P_F])$

(P_3) $([P_T,P_I,P_F]K)^+ = K[P_T,P_I,P_F]^+$ and $([P_T,P_I,P_F])^+ = [P_T,P_I,P_F]^+$ K exists, if $[P_T,P_I,P_F]^+$ exists.

(P_4) $[P_T,P_I,P_F]^T$ is a Generalized inverse of $[P_T,P_I,P_F]$ iff $[P_T,P_I,P_F]^+$ occur.

2. DEFINITIONS AND THEOREMS

Definition:2.1 (NFM): A NS P on the set X is well-defined as $P=\{<x,T,I,F>,x\in X\}$, where

$T,I,F:X\to\,]^-0,1^+[$ and $0\le T+I+F\le 3.$

Example2.1: Consider a NFM $P=\begin{bmatrix}(0.7,0.8,0.5) & (0.2,0.4,0.6) & (0.3,0.7,0.4)\\ (0.4,0.5,0.6) & (0.3,0.2,0.1) & (0.3,0.2,0.1)\\ (0.1,0.2,0.3) & (0.7,0.2,0.1) & (0.2,0.2,0.2)\end{bmatrix}$

Definition 2.2 PNFM: PNFM P with m × n matrix is given by $P=[X_{ij},<T,I,F>]_{mxn}$ where $T,I,F\in[0,1]$ are referred to as the degrees of the truth, the indeterminacy, and the falsity of in P, which preserving the form $0\le T^2+I^2+F^2\le 2$ where $0\le T^2+F^2\le 2$ and $0\le I^2\le 1$.

Example2.2: Consider a PNFM $P=\begin{bmatrix}(1,1,0) & (0.5,0.3,0.4) & (0.3,0.4,0.1)\\ (0.6,1,0.2) & (0.4,0.1,0.6) & (1,1,0)\\ (0.5,0.5,1) & (1,1,0) & (0.4,0.4,0.5)\end{bmatrix}$

Definition 2.3 FNFM: FNFM with dimensions m × n is given by $P=[X,<T,I,F>]_{mxn}$ where $T,I,F\in[0,1]$ are referred to as the degrees of the truth, the indeterminacy, and the falsity of in P, which preserving the state $0\le T^3+I^3+F^3\le 2$ where $0\le T^3+F^3\le 1$ and $0\le I^3\le 1.$

Example2.3: Consider a NFM $P=\begin{bmatrix}<0.7,0.7,0.8> & <1,0,1> & <0.2,0.3,0.4>\\ <1,0,1> & <1,0,1> & <1,0,1>\\ <0.2,0.3,0.4> & <1,0,1> & <1,0,1>\end{bmatrix}$ is not PNFM but it P is a FNFM.

$(0.7)^2+(0.7)^2+(0.8)^2=1.62\le 2$, $(0.7)^2+(0.8)^2=1.13>1$ not PNFM

$(0.7)^3+(0.7)^3+(0.8)^3=1.198\le 2$, $(0.7)^3+(0.8)^3=0.855<1$ is FNFM

Therefore every FNFM is PNFM but converse need not be true.

Definition: 2.4 Let $P=[P_T,P_I,P_F]\in(FNFM)_n$ be a FNFM, if R $[[P_T,P_I,P_F]]$ = R $[[P_T,P_I,P_F]^T]$ and R $[[P_T,P_I,P_F]]$ = R $[[P_T,P_I,P_F]^T]$ then $P=[P_T,P_I,P_F]$ is called as RS.

Example: 2.4 Consider a FNFM

$$P=[P_T,P_I,P_F]=\begin{bmatrix}<0.4,0,1> & <1,0,1> & <0.2,0.3,0.4>\\ <1,0,1> & <1,0,1> & <1,0,1>\\ <0.2,0.3,0.4> & <1,0,1> & <1,0,1>\end{bmatrix}$$

Here, R $[[(0.2,0.4,0.4)\ (0.1,0.2,0.3)\ (0.1,0.2,0.3)]\notin R([P_T,P_I,P_F]^T)]$ = R $[[P_T,P_I,P_F]^T]$

The following matrix does not meet the RS-FNFM condition.

$$[P_T,P_I,P_F]=\begin{bmatrix}<0.3,0,0.2> & <0.1,0.2,0.3> & <0.5,0.6,0.4>\\ <0.1,0.2,0.3> & <0.1,0.2,0.3> & <0.1,0.2,0.3>\\ <0.2,0.4,0.4> & <0.1,0.2,0.3> & <0.1,0.2,0.3>\end{bmatrix},$$

$$[P_T,P_I,P_F]=\begin{bmatrix}<0.3,0,0.2> & <0.1,0.2,0.3> & <0.2,0.4,0.4>\\ <0.1,0.2,0.3> & <0.1,0.2,0.3> & <0.1,0.2,0.3>\\ <0.5,0.6,0.4> & <0.1,0.2,0.3> & <0.1,0.2,0.3>\end{bmatrix},$$

$[(0.3,0,0.2)\ (0.1,0.2,0.3)\ (0.5,0.6,0.4)]\in R([P_T,P_I,P_F])$,

$\left[(0.3,0,0.2)\ (0.1,0.2,0.3)\ (0.5,0.6,0.4)\right] \notin R([P_T,P_I,P_F]^T)$

$\left[(0.1,0.2,0.3)\ (0.1,0.2,0.3)\ (0.1,0.2,0.3)\right] \in R([P_T,P_I,P_F])$,

$\left[(0.1,0.2,0.3)\ (0.1,0.2,0.3)\ (0.1,0.2,0.3)\right] \in R([P_T,P_I,P_F]^T)$

$\left[(0.2,0.4,0.4)\ (0.1,0.2,0.3)\ (0.1,0.2,0.3)\right] \in R([P_T,P_I,P_F])$,

$\left[(0.2,0.4,0.4)\ (0.1,0.2,0.3)\ (0.1,0.2,0.3)\right] \notin R([P_T,P_I,P_F]^T)$

$R([P_T,P_I,P_F]) \notin R([P_T,P_I,P_F]^T)$

Note:2.1 For FNFM P with det $[P_T,P_I,P_F]$ > <0,1,1> has non- null rows columns, hereafter N($[P_T,P_I,P_F]$) = <0,1,1> = N($[P_T,P_I,P_F]^T$). Additionally, a SM $[P_T,P_I,P_F]$ = $[P_T,P_I,P_F]^T$ that is N($[P_T,P_I,P_F]$)= N($[P_T,P_I,P_F]^T$).

Definition : 2.5 Let $P=[P_T,P_I,P_F]\in\left(FNFM\right)_n$ if N($[P_T,P_I,P_F]$) = N($[P_T,P_I,P_F]^T$) and P is said to be KS- FNFM where N($[P_T,P_I,P_F]$) = {y/y$[P_T,P_I,P_F]$ = (0,1,1) and y∈ $F_{1\times n}$}.

Example: 2.5 Consider a FNFM

$$[A_T,A_I,A_F]=\begin{bmatrix} <0.4,0.4,0.6> & <0.6,0.7,0.7> & <0.5,0.6,0.7> \\ <0.6,0.8,0.7> & <0.7,0.9,0.2> & <0.3,0.7,0.2> \\ <0.7,0.6,0.7> & <0.5,0.6,0.6> & <0.5,0.5,0.6> \end{bmatrix}$$

$N([P_T,P_I,P_F])=N([P_T,P_I,P_F]^T)=(0,1,1).$

Definition 2.6. Symmetric FNFM. If $P=[P_T,P_I,P_F]\in\left(FNFM\right)_n$ is called SFNFM if $p_{ij} = p_{ji}$.

Example: 2.6 Consider a FNFM

$$[P_T,P_I,P_F]=\begin{bmatrix} <0.5,0,1> & <1,0,1> & <0.5,0.6,0.7> \\ <1,0,1> & <1,0,1> & <1,0,1> \\ <0.5,0.6,0.7> & <1,0,1> & <1,0,1> \end{bmatrix}$$

Here, $[P_T,P_I,P_F]$ = $[P_T,P_I,P_F]^T$

Definition 2.7. Permutation NFM

A NFPM is a square matrix where every row and every column contain exactly one <1,1,0> entry, with all other entries being <0,0,1>.

Example: 2.7 Consider a NFPM,

$$K = \begin{bmatrix} (0,0,1) & (0,1,1) & (0,0,1) \\ (0,0,1) & (0,0,1) & (1,1,0) \\ (1,1,0) & (1,0,0) & (0,0,1) \end{bmatrix}$$

3**. Graphical Representation of Range symmetric, CS and KS Adjacency NFM.**

Definition 3.1. Adjacency FNFM

An adjacency FNFM is a square matrix used to represent a finite graph. The elements of this matrix indicate whether pairs of vertices in the graph are connected. For a finite simple graph, this matrix can be defined as a binary matrix, often referred to as a (1,1,0) and (0,0,1) matrix, where all diagonal elements are consistently set to (0,0,1). Let G(V, E) represent a simple graph with n vertices. The adjacency matrix P = [Pij] is a SM defined P = [Pij] = (1,1,0) vi is adjacent to vj and (0,0,1) otherwise denoted by P(G).

Example: 3.1 Consider a FNFM

$$A_G = \begin{bmatrix} (0,0,1) & (0,0,1) & (1,1,0) \\ (0,0,1) & (0,0,1) & (1,1,0) \\ (1,1,0) & (1,1,0) & (0,0,1) \end{bmatrix}, A_H = \begin{bmatrix} (0,0,1) & (1,1,0) & (1,1,0) \\ (1,1,0) & (0,0,1) & (0,0,1) \\ (1,1,0) & (0,0,1) & (0,0,1) \end{bmatrix}$$

Equivalent adjacency graph.

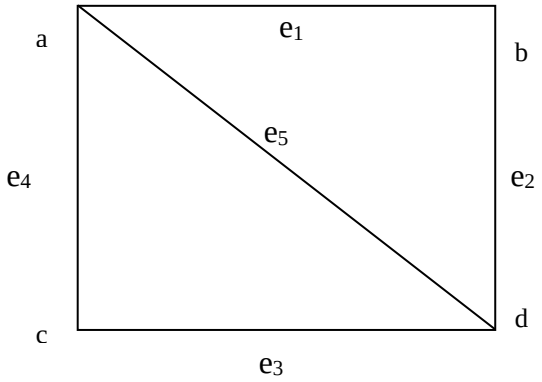
Definition 3.2. Incidence FNFM

The incidence NFM I = [m_{ij}] is a $n \times m$ matrix defined by I = [mij] = (1,1,0), vi is incident to vj and (0,0,1) otherwise denoted by P(G).

Example:3.2 Consider a FNFM and its equivalent graph.

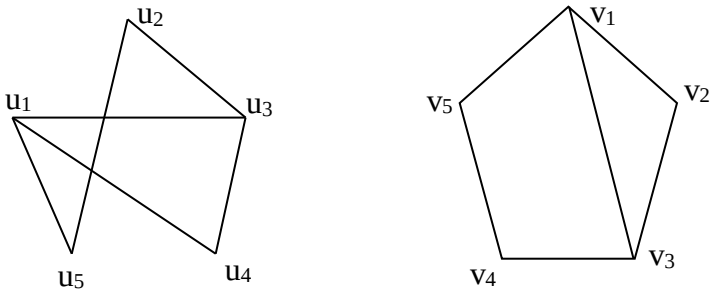
$$I = \begin{matrix} a \\ b \\ c \\ d \end{matrix} \begin{bmatrix} (1,1,0) & (0,0,1) & (0,0,1) & (1,1,0) & (1,1,0) \\ (1,1,0) & (1,1,0) & (0,0,1) & (0,0,1) & (0,0,1) \\ (0,0,1) & (0,0,1) & (1,1,0) & (1,1,0) & (0,0,1) \\ (0,0,1) & (1,1,0) & (1,1,0) & (0,0,1) & (1,1,0) \end{bmatrix}$$

Corresponding graph



3.1 Relation between isomorphism , non-isomorphism and KS, RS and CS- FNFM

Graph: I



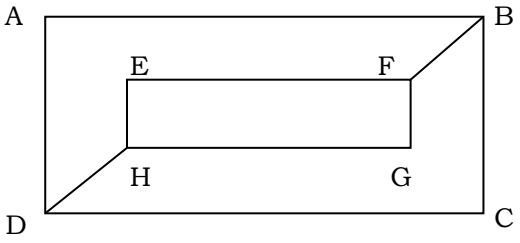
$$A_U = \begin{bmatrix} & u_1 & u_2 & u_3 & u_4 & u_5 \\ u_1 & (0,0,1) & (1,1,0) & (1,1,0) & (0,0,1) & (1,1,0) \\ u_2 & (1,1,0) & (0,0,1) & (1,1,0) & (0,0,1) & (0,0,1) \\ u_3 & (1,1,0) & (1,1,0) & (0,0,1) & (1,1,0) & (0,0,1) \\ u_4 & (0,0,1) & (0,0,1) & (1,1,0) & (0,0,1) & (1,1,0) \\ u_5 & (1,1,0) & (0,0,1) & (0,0,1) & (1,1,0) & (0,0,1) \end{bmatrix}$$

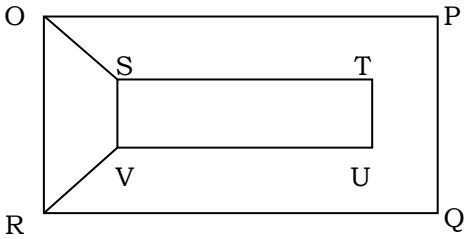
$$A_V = \begin{bmatrix} & v_1 & v_2 & v_3 & v_4 & v_5 \\ v_1 & (0,0,1) & (1,1,0) & (1,1,0) & (0,0,1) & (1,1,0) \\ v_2 & (1,1,0) & (0,0,1) & (1,1,0) & (0,0,1) & (0,0,1) \\ v_3 & (1,1,0) & (1,1,0) & (0,0,1) & (1,1,0) & (0,0,1) \\ v_4 & (0,0,1) & (0,0,1) & (1,1,0) & (0,0,1) & (1,1,0) \\ v_5 & (1,1,0) & (0,0,1) & (0,0,1) & (1,1,0) & (0,0,1) \end{bmatrix}$$

Graph: I Two graphs have the equal number of vertices, the equal number of edges, the equal degree sequence, and the FNFM are equal. Consequently, the given graphs are isomorphic

and also KS, CS, RS adjacency FNFM.

Graph: II





$$A_U = \begin{bmatrix} & O & R & P & Q & S & V & T & U \\ O & (0,0,1) & (1,1,0) & (1,1,0) & (0,0,1) & (0,0,1) & (0,0,1) & (0,0,1) & (0,0,1) \\ R & (1,1,0) & (0,0,1) & (0,0,1) & (1,1,0) & (0,0,1) & (1,1,0) & (0,0,1) & (0,0,1) \\ P & (1,1,0) & (0,0,1) & (0,0,1) & (1,1,0) & (0,0,1) & (0,0,1) & (0,0,1) & (0,0,1) \\ Q & (0,0,1) & (1,1,0) & (1,1,0) & (0,0,1) & (0,0,1) & (0,0,1) & (0,0,1) & (0,0,1) \\ S & (1,1,0) & (0,0,1) & (0,0,1) & (0,0,1) & (0,0,1) & (1,1,0) & (1,1,0) & (0,0,1) \\ V & (0,0,1) & (1,1,0) & (0,0,1) & (0,0,1) & (1,1,0) & (0,0,1) & (0,0,1) & (1,1,0) \\ T & (0,0,1) & (0,0,1) & (0,0,1) & (0,0,1) & (1,1,0) & (0,0,1) & (0,0,1) & (1,1,0) \\ U & (0,0,1) & (0,0,1) & (0,0,1) & (0,0,1) & (0,0,1) & (1,1,0) & (1,1,0) & (0,0,1) \end{bmatrix}$$

$$A_V = \begin{bmatrix} & A & B & D & C & E & F & H & G \\ A & (0,0,1) & (1,1,0) & (1,1,0) & (0,0,1) & (0,0,1) & (0,0,1) & (0,0,1) & (0,0,1) \\ B & (1,1,0) & (0,0,1) & (0,0,1) & (1,1,0) & (0,0,1) & (1,1,0) & (0,0,1) & (0,0,1) \\ D & (1,1,0) & (0,0,1) & (0,0,1) & (1,1,0) & (0,0,1) & (0,0,1) & (1,1,0) & (0,0,1) \\ C & (0,0,1) & (1,1,0) & (1,1,0) & (0,0,1) & (0,0,1) & (0,0,1) & (0,0,1) & (0,0,1) \\ E & (0,0,1) & (0,0,1) & (0,0,1) & (0,0,1) & (0,0,1) & (1,1,0) & (1,1,0) & (0,0,1) \\ F & (0,0,1) & (1,1,0) & (0,0,1) & (0,0,1) & (1,1,0) & (0,0,1) & (0,0,1) & (1,1,0) \\ H & (0,0,1) & (0,0,1) & (1,1,0) & (0,0,1) & (1,1,0) & (0,0,1) & (0,0,1) & (1,1,0) \\ G & (0,0,1) & (0,0,1) & (0,0,1) & (0,0,1) & (0,0,1) & (1,1,0) & (1,1,0) & (0,0,1) \end{bmatrix}$$

Two graphs have the equal number of vertices, the equal number of edges, the degree sequence are not equal.

Consequently, the graphs II, G and H are not isomorphic.

4.Theorems and Results

Theorem:4.1 For a FNFM $P=[P_T,P_I,P_F]$, $Q=[Q_T,Q_I,Q_F]$ and K be a NFPM if

$$N([P_T,P_I,P_F])=N([Q_T,Q_I,Q_F])\Leftrightarrow N(K[P_T,P_I,P_F]\,K^T)=N(K[Q_T,Q_I,Q_F]K^T).$$

Proof: Let $z\in N(K[P_T,P_I,P_F]K^T)$

$$\Rightarrow z(K[P_T,P_I,P_F]K^T)=(0,1,1)$$

$\Rightarrow xK^T = (0,1,1),$

everywhere $x = zK([P_T, P_I, P_F])$

$\Rightarrow x \in N(K^T)$

$\det K = \det K^T > (0,1,1)$

Consequently,

$N(K^T) = (0,1,1)$

Hereafter

$x = (0,0,1)$

$\Rightarrow zK([P_T, P_I, P_F]) = (0,1,1)$

$\Rightarrow zK \in N([P_T, P_I, P_F]) = N([Q_T, Q_I, Q_F])$

$\Rightarrow zK([Q_T, Q_I, Q_F])K^T = (0,0,1)$

$\Rightarrow z \in N(K([Q_T, Q_I, Q_F])K^T)$

$N(K[\mathrm{P}_\mu, \mathrm{P}_\lambda, \mathrm{P}_\nu]_L K^T) \subseteq N(K[Q_T, Q_I, Q_F]K^T)$

Also, $N(K[Q_T, Q_I, Q_F]K^T) \subseteq N(K[P_T, P_I, P_F]K^T)$

Consequently,

$N([P_T, P_I, P_F]) = N([Q_T, Q_I, Q_F]) \Leftrightarrow N(K[P_T, P_I, P_F]K^T) = N(K[Q_T, Q_I, Q_F]K^T)$

Conversely, if $N(K[P_T, P_I, P_F]K^T) = N(K[Q_T, Q_I, Q_F]K^T)$,

N $([P_T, P_I, P_F])$ = N (K^T (K$[P_T, P_I, P_F]$ K^T) K)

= N (K^T (K$([Q_T, Q_I, Q_F])$ K^T) K)

$N([P_T, P_I, P_F]) = N([Q_T, Q_I, Q_F])$

Example: 4.1 Consider a FNFM

$$[P_T, P_I, P_F] = \begin{bmatrix} <0.3,0.3,0.4> & <0.5,0.3,0.6> & <0.2,0.3,0.4> \\ <0.4,0.7,0.6> & <0.4,0.8,0.1> & <0.2,0.5,0.1> \\ <0.2,0.4,0.4> & <0.4,0.4,0.5> & <0.4,0.5,0.6> \end{bmatrix}$$

$$K = \begin{bmatrix} (1,1,0) & (0,0,1) & (0,0,1) \\ (0,0,1) & (1,1,0) & (0,0,1) \\ (0,0,1) & (0,0,1) & (1,1,0) \end{bmatrix}$$

$$[Q_T, Q_I, Q_F] = \begin{bmatrix} <0.5,0.3,0.7> & <0.4,0.3,0.6> & <0.3,0.4,0.4> \\ <0.5,0.7,0.5> & <0.4,0.8,0.2> & <0.2,0.5,0.2> \\ <0.2,0.4,0.4> & <0.4,0.4,0.5> & <0.4,0.5,0.6> \end{bmatrix}$$

$$K[P_T, P_I, P_F]K^T = \begin{bmatrix} (1,1,0) & (0,0,1) & (0,0,1) \\ (0,0,1) & (1,1,0) & (0,0,1) \\ (0,0,1) & (0,0,1) & (1,1,0) \end{bmatrix}$$

$$\begin{bmatrix} <0.3,0.3,0.4> & <0.5,0.3,0.6> & <0.2,0.3,0.4> \\ <0.4,0.7,0.6> & <0.4,0.8,0.1> & <0.2,0.5,0.1> \\ <0.2,0.4,0.4> & <0.4,0.4,0.5> & <0.4,0.5,0.6> \end{bmatrix} \begin{bmatrix} (1,1,0) & (0,0,1) & (0,0,1) \\ (0,0,1) & (1,1,0) & (0,0,1) \\ (0,0,1) & (0,0,1) & (1,1,0) \end{bmatrix}$$

$$= \begin{bmatrix} <0.3,0.3,0.4> & <0.5,0.3,0.6> & <0.2,0.3,0.4> \\ <0.4,0.7,0.6> & <0.4,0.8,0.1> & <0.2,0.5,0.1> \\ <0.2,0.4,0.4> & <0.4,0.4,0.5> & <0.4,0.5,0.6> \end{bmatrix}$$

Let $w \in N(K[P_T, P_I, P_F]K^T)$

$w = [(0,0,1) \quad (0,0,1) \quad (0,0,1)]$

By definition 2.7

$\Rightarrow w(K[P_T, P_I, P_F]K^T)$

$$= [(0,0,1) \quad (0,0,1) \quad (0,0,1)] \begin{bmatrix} <0.3,0.3,0.4> & <0.5,0.3,0.6> & <0.2,0.3,0.4> \\ <0.4,0.7,0.6> & <0.4,0.8,0.1> & <0.2,0.5,0.1> \\ <0.2,0.4,0.4> & <0.4,0.4,0.5> & <0.4,0.5,0.6> \end{bmatrix} = (0,0,1)$$

$\Rightarrow xK^T = (0,1,1)$ where $x = zK([P_T, P_I, P_F])$

Hence

$x = (0,1,1)$

$$K[P_T, P_I, P_F] = \begin{bmatrix} <0.3,0.3,0.4> & <0.5,0.3,0.6> & <0.2,0.3,0.4> \\ <0.4,0.7,0.6> & <0.4,0.8,0.1> & <0.2,0.5,0.1> \\ <0.2,0.4,0.4> & <0.4,0.4,0.5> & <0.4,0.5,0.6> \end{bmatrix}$$

$$\Rightarrow zK([P_T, P_I, P_F]) = (0,1,1)$$

$$\Rightarrow zK \in N([P_T, P_I, P_F]) = N([Q_T, Q_I, Q_F])$$

$$\Rightarrow zK([Q_T, Q_I, Q_F])K^T = (0,0,1)$$

$$\Rightarrow z \in N(K([Q_T, Q_I, Q_F])K^T)$$

$$N(K[P_\mu, P_\lambda, P_\nu]_L K^T) \subseteq N(K[Q_T, Q_I, Q_F]K^T)$$

Also, $N(K[Q_T, Q_I, Q_F]K^T) \subseteq N(K[P_T, P_I, P_F]K^T)$

Consequently,

$$N([P_T, P_I, P_F]) = N([Q_T, Q_I, Q_F]) \Leftrightarrow N(K[P_T, P_I, P_F]K^T) = N(K[Q_T, Q_I, Q_F]K^T)$$

Conversely, if $N(K[P_T, P_I, P_F]K^T) = N(K[Q_T, Q_I, Q_F]K^T)$,

N $([P_T, P_I, P_F])$ = N (K^T (K$[P_T, P_I, P_F]$ K^T) K)

= N (K^T (K$([Q_T, Q_I, Q_F])$ K^T) K)

$$N([P_T, P_I, P_F]) = N([Q_T, Q_I, Q_F])$$

Theorem:4.2 For a FNFM $P = [P_T, P_I, P_F] \in (FNFM)_n$ and K be a NFPM if

$N([P_T, P_I, P_F]) = N([P_T, P_I, P_F]^T) \Leftrightarrow$N(K$[P_T, P_I, P_F]$ K^T) = N(K $[P_T, P_I, P_F]^T$ K^T) .

Proof: The proof is like to theorem 4.1

Theorem: 4.3 For $P = [P_T, P_I, P_F] \in (FNFM)_n$ is KS- FNFM, then N($[P_T, P_I, P_F]$ $[P_T, P_I, P_F]^T$) =

N($[P_T, P_I, P_F]$) = N($[P_T, P_I, P_F]^T[P_T, P_I, P_F]$).

Example: 4.2 Consider a FNFM

$$[P_T, P_I, P_F] = \begin{bmatrix} <0.1,0.3,0.5> & <0.4,0.3,0.6> & <0.2,0.3,0.4> \\ <0.4,0.7,0.6> & <0.4,0.7,0.1> & <0.2,0.5,0.1> \\ <0.2,0.3,0.5> & <0.4,0.3,0.5> & <0.4,0.2,0.1> \end{bmatrix}$$

Theorem:4.4 For a FNFM $P = [P_T, P_I, P_F], Q = [Q_T, Q_I, Q_F] \in (FNFM)_n$ and K NFPM,

R ($[P_T, P_I, P_F]$) = R ($[Q_T, Q_I, Q_F]$) ⇔ R (K$[P_T, P_I, P_F]$ K^T) = R (K$[Q_T, Q_I, Q_F]$ K^T) .

Proof: Let R ($[P_T, P_I, P_F]$) = R ($[Q_T, Q_I, Q_F]$)

Then, R ($[P_T, P_I, P_F]$ K^T) = R ($[P_\mu, P_\lambda, P_\nu]_L$) K^T

= R ($[P_T, P_I, P_F]$) K^T

= R ($[P_T, P_I, P_F]$ K^T)

Let $z \in \{R(K[P_T, P_I, P_F]K^T)\}$

$z = w(K[P_T, P_I, P_F]K^T)$ for some $w \in V^n$

$z = r[P_T, P_I, P_F]K^T, \ r = wK$

$z \in R([P_T, P_I, P_F]K^T) = R([Q_T, Q_I, Q_F](K^T))$

$z = u[Q_T, Q_I, Q_F]K^T$ for some $u \in V^n$

$z = (uK^T)K[Q_T, Q_I, Q_F]K^T$

$z = vK[Q_T, Q_I, Q_F]K^T$ for some $v \in V^n$

$z \in R(K[Q_T, Q_I, Q_F]K^T)$

Therefore, $R(K[P_T, P_I, P_F]K^T) \subseteq R(K[Q_T, Q_I, Q_F]K^T)$

Similarly, $R(K[Q_T, Q_I, Q_F]K^T) \subseteq R(K[P_T, P_I, P_F]K^T)$

Therefore, R(K$[P_T, P_I, P_F]$ K^T) = R(K$[Q_T, Q_I, Q_F]$ K^T)

Conversely, Let R(K$[P_T, P_I, P_F]$ K^T) = R(K$[Q_T, Q_I, Q_F]$ K^T)

$$R([P_T, P_I, P_F]) = R[K^T(K[P_T, P_I, P_F]K^T)K]$$

$$= R[K^T(K[Q_T, Q_I, Q_F]K^T)K]$$

$$= R([Q_T, Q_I, Q_F])$$

$$R([P_T, P_I, P_F]) = R([Q_T, Q_I, Q_F])$$

Example: 4.3 Consider a FNFM

$$[P_T, P_I, P_F] = \begin{bmatrix} <0.5,0.6,0.4> & <0.3,0.1,0.2> & <0.2,0.4,0.5> \\ <0.3,0.1,0.2> & <0.4,0.5,0.1> & <0.2,0.5,0.5> \\ <0.2,0.4,0.5> & <0.2,0.5,0.5> & <0.4,0.2,0.1> \end{bmatrix}$$

$$[Q_T, Q_I, Q_F] = \begin{bmatrix} <0.2,0.4,0.5> & <0.2,0.5,0.5> & <0.4,0.2,0.1> \\ <0.3,0.1,0.2> & <0.4,0.5,0.1> & <0.2,0.5,0.5> \\ <0.5,0.6,0.4> & <0.3,0.1,0.2> & <0.2,0.4,0.5> \end{bmatrix}$$

$$K = \begin{bmatrix} (0,0,1) & (1,1,0) & (0,0,1) \\ (0,0,1) & (0,0,1) & (1,1,0) \\ (1,1,0) & (0,0,1) & (0,0,1) \end{bmatrix}$$

Theorem:4.5 For $P = [P_T, P_I, P_F] \in (FNFM)_n$ be the FNFM and K be a NFPM, R($[P_T, P_I, P_F]$)

= R($[P_T, P_I, P_F]^T$) ⇔ R(K$[P_T, P_I, P_F]$ K^T) = R(K $[P_T, P_I, P_F]^T$ K^T) .

Proof: The proof is comparable to theorem 4.4

Example: 4.4 Consider a FNFM

$$[P_T, P_I, P_F] = \begin{bmatrix} <0.2,0.1,0.3> & <0.3,0.3,0.2> & <0.4,0.5,0.6> \\ <0.3,0.3,0.2> & <0.4,0.7,0.1> & <0.4,0.5,0.2> \\ <0.4,0.5,0.6> & <0.4,0.5,0.2> & <0.3,1,1> \end{bmatrix}$$

Theorem:4.6 For a FNFM $P = [P_T, P_I, P_F], Q = [Q_T, Q_I, Q_F] \in (FNFM)_n$ and K NFPM

C($[P_T, P_I, P_F]$) = C($[Q_T, Q_I, Q_F]$) ⇔ C(K$[P_T, P_I, P_F]$ K^T) = C(K $[Q_T, Q_I, Q_F]$ K^T).

Proof: The proof is like to theorem 4.4

Example: 4.5 Consider a FNFM

$$[P_T, P_I, P_F] = \begin{bmatrix} <0.1,0.2,0.3> & <0.4,0.5,0.1> & <0.3,0.2,0.1> \\ <0.1,0.4,0.5> & <0.3,0.1,0.8> & <0.6,0.5,0.4> \\ <0.2,0.3,0.1> & <0.8,0.2,0.1> & <0.5,0.6,0.2> \end{bmatrix}$$

$$[Q_T, Q_I, Q_F] = \begin{bmatrix} <0.3,0.2,0.1> & <0.4,0.5,0.1> & <0.1,0.2,0.3> \\ <0.6,0.5,0.4> & <0.3,0.1,0.8> & <0.1,0.4,0.5> \\ <0.5,0.6,0.2> & <0.8,0.2,0.1> & <0.2,0.3,0.1> \end{bmatrix}$$

$$[P_T, P_I, P_F] = \begin{bmatrix} <0.1,0.2,0.3> & <0.4,0.5,0.1> & <0.3,0.2,0.1> \\ <0.1,0.4,0.5> & <0.3,0.1,0.8> & <0.6,0.5,0.4> \\ <0.2,0.3,0.1> & <0.8,0.2,0.1> & <0.5,0.6,0.2> \end{bmatrix}$$

$$[Q_T, Q_I, Q_F] = \begin{bmatrix} <0.3,0.2,0.1> & <0.4,0.5,0.1> & <0.1,0.2,0.3> \\ <0.6,0.5,0.4> & <0.3,0.1,0.8> & <0.1,0.4,0.5> \\ <0.5,0.6,0.2> & <0.8,0.2,0.1> & <0.2,0.3,0.1> \end{bmatrix}$$

5.**k-KERNEL SYMMETRIC IVNFM**

Definition: 5.1 Let $P = [P_T, P_I, P_F] \in (FNFM)_n$ is said to be k-KS- FNFM if N($[P_T, P_I, P_F]$) = N(K$[P_T, P_I, P_F]^T$K) .

Theorem: 5.1 The subsequent conditions are equivalent for $P = [P_T, P_I, P_F] \in (FNFM)_n$

(i) N($[P_T, P_I, P_F]$) = N(K$[P_T, P_I, P_F]^T$K),

(ii) N(K$[P_T, P_I, P_F]$) = N((K$[P_T, P_I, P_F]$)^T),

(iii) N($[P_T, P_I, P_F]$K) = N(($[P_T, P_I, P_F]$K)^T),

(iv) N($[P_T, P_I, P_F]^T$) = N(K$[P_T, P_I, P_F]$),

(v) N($[P_T, P_I, P_F]$) = N(($[P_T, P_I, P_F]$K)^T),

(vi) $[P_T, P_I, P_F]^+$ is k-KSIVNFM,

(vii) N($[P_T, P_I, P_F]$) = N($[P_T, P_I, P_F]^+$K),

(viii) K $[P_T, P_I, P_F]^+[P_T, P_I, P_F]$ = $[P_T, P_I, P_F]$ $[P_T, P_I, P_F]^+$K,

(ix) $[P_T, P_I, P_F]^+[P_T, P_I, P_F]\,\mathrm{K} = \mathrm{K}[P_T, P_I, P_F]\,[P_T, P_I, P_F]^+$

Proof: The proof is like to Ref [12]

6. Comparison Study:

Fuzzy Matrices	**Neutrosophic Fuzzy Matrices (NFM)**	**GSFNFM**
Representation of Uncertainty		
This research leverages fuzzy sets, a mathematical framework that utilizes membership degrees between 0 and 1. These degrees quantify the extent to which an element belongs to a particular set, enabling the representation of uncertainty with varying levels of inclusion.	In NFM, the authors extend the capabilities of traditional fuzzy matrices by incorporating neutrosophic logic. Neutrosophic sets introduce membership degrees for truth (T), indeterminacy (I), and falsity (F), allowing us to capture not only vagueness but also ambiguity and inconsistency within a single framework.	This study introduces GSFNFM, a novel extension of neutrosophic fuzzy matrices. GSFNFM incorporates symmetric properties and Fermatean neutrosophic elements, offering a more comprehensive and nuanced approach to representing complex forms of uncertainty encountered in real-world scenarios.
Symmetry Properties		
Prior research on fuzzy or neutrosophic matrices may not have explicitly addressed symmetry properties as a core concept.	While some studies might have introduced the idea of symmetry, they likely did not delve into it as deeply as the current work.	This research emphasizes the significance of various symmetry properties, such as kernel symmetry, range symmetry, and column symmetry. By extensively exploring these properties, the study enhances our understanding and broadens the potential applications of the matrices.
Computational Complexity		
Many fundamental matrix operations, such as addition, multiplication, and inversion, have	Introducing neutrosophic elements into the matrix framework can potentially increase the computational complexity compared to	Incorporating symmetric properties like kernel or range symmetry can further add to the computational complexity. Specialized algorithms may be

well-established and efficient algorithms. These algorithms allow for fast and reliable computations involving traditional matrices.	traditional matrices. This is because neutrosophic elements involve additional membership degrees (truth, indeterminacy, falsity) compared to the single membership value used in classical matrices.	necessary to handle these properties efficiently when performing matrix operations involving GSFNFM. Developing such algorithms will be crucial for practical applications of GSFNFM.
Interpretability		
Traditional fuzzy sets offer a high degree of interpretability due to their use of single membership values between 0 and 1. These values directly correspond to the likelihood of an element belonging to a set, making the results easy to understand.	The introduction of neutrosophic elements, including indeterminacy and falsity memberships, can introduce some complexity into the interpretation process. Researchers and users need to be aware of the nuances of these additional degrees to avoid misinterpretations.	The complexity of symmetric properties, like kernel or range symmetry, can affect interpretability. While these properties offer valuable insights, understanding their impact on the overall meaning of the matrix might require advanced visualization techniques. These techniques can help to represent the data visually and enhance clarity, especially when dealing with intricate symmetric relationships.
Real-world Applications		
Traditional fuzzy set theory has proven valuable in numerous real-world applications, including decision-making, pattern recognition, and control systems. These applications leverage the ability of fuzzy sets to represent uncertainty with varying degrees of membership.	Neutrosophic sets extend the capabilities of fuzzy sets by incorporating indeterminacy and falsity memberships. This additional information makes them particularly well-suited for domains with inherent uncertainty and ambiguity, such as medical diagnosis and financial forecasting.	Generalized Symmetric Fermatean Neutrosophic Fuzzy Matrices (GSFNFM) represent a novel approach with the potential to address even more complex real-world scenarios. The combination of symmetric properties and Fermatean neutrosophic elements opens doors for applications in network analysis, image processing, and expert systems, where these features can play a significant role in modeling and reasoning with intricate forms

		of uncertainty.
Research Focus		
Much of the existing research in fuzzy logic and related fields has primarily focused on improving the efficiency of computational methods and expanding the applicability of these techniques to various real-world domains. This focus ensures that these tools can be used effectively in practical applications.	This research takes a broader perspective, shifting the focus towards enhancing the representational capabilities of these frameworks. By introducing concepts like neutrosophic elements, we aim to address complex uncertainty scenarios that may not be adequately captured by traditional methods.	Our study delves deeper by exploring the intricate interplay between symmetric properties and neutrosophic elements within the context of GSFNFM. This exploration opens doors for the development of novel algorithms specifically tailored for efficient manipulation of these matrices. Furthermore, by investigating these properties and algorithms, we aim to extend the applicability of GSFNFM to a wider range of diverse domains, enabling researchers and practitioners to leverage its strengths in tackling intricate problems.

Generalized Symmetric FFNFM emerge as a powerful new tool for representing and reasoning with uncertainty. This framework builds upon traditional fuzzy sets and neutrosophic fuzzy matrices by incorporating symmetric properties and FNFM elements. This innovation offers a more nuanced and comprehensive approach to capturing the complexities of uncertainty encountered in real-world scenarios.

The introduction of symmetric properties within GSFNFM unlocks new possibilities for analyzing relationships and structures. Furthermore, Fermatean neutrosophic elements provide a richer framework for representing situations involving not only vagueness but also indeterminacy and inconsistency.

While GSFNFM holds significant promise, future research is crucial to unlocking its full potential. In particular, further exploration is needed to validate its effectiveness through empirical studies and practical applications across diverse domains. Additionally, research efforts focused on developing efficient algorithms for GSFNFM operations will be essential for ensuring its widespread adoption.

In conclusion, GSFNFM represents a significant leap forward in the field of uncertainty modeling. By harnessing the strengths of prior frameworks and introducing novel features, GSFNFM paves

the way for more robust and insightful decision-making processes in various fields grappling with intricate uncertainties.

7. CONCLUSION

Our investigation into Generalized Symmetric FNFM sheds light on a powerful tool for representing and reasoning with uncertain information. This novel framework holds significant promise for various fields, offering a more nuanced approach to decision-making under uncertainty.

Key Contributions:

We have delved into the theoretical foundations of GSFNFM, unveiling their core properties and relationships between different types, such as Range-Symmetric (RS-FNFM) and Kernel-Symmetric (KS-FNFM) matrices. Our exploration has introduced the concept of k-kernel-symmetric FNFMs (k-KS-FNFM) and illustrated their connections to KS-FNFM through examples. The graphical representation of adjacency and incidence FNFMs with specific symmetry properties provides valuable insights into their applicability, particularly in the context of isomorphic graphs.

Future works:

This research lays a solid foundation for further exploration of GSFNFM's potential. Future endeavors can extend our findings in several key directions: Investigating properties related to generalized inverses of k-Kernel Symmetric FNFMs can offer deeper understanding and potential applications. Devising efficient algorithms for performing matrix operations involving GSFNFM is crucial for practical applications. Integration with existing computational frameworks for uncertainty management holds significant promise. While theoretical advancements are important, further research should focus on validating the effectiveness of GSFNFM in real-world scenarios. Empirical studies demonstrating its practical utility across various domains are essential for broader adoption. The efficacy of GSFNFM relies on the quality and availability of data inputs. Future research should explore strategies for obtaining accurate and comprehensive data that can be effectively represented using GSFNFM, particularly in domains with inherent uncertainty and variability. By addressing these future directions, the potential of GSFNFM can be fully realized, leading to more robust and reliable decision-making processes in complex real-world problems. As we navigate the ever-present uncertainties in various fields, GSFNFM emerges as a powerful tool, paving the way for a more informed future.

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Received: March 20, 2024. Accepted: July 27, 2024

Neutrosophic Sets and Systems, Vol. 70, 2024
University of New Mexico

Operators Based On Multiple Generalized Set-Valued Neutrosophic Quadruple Sets

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Abstract: In this study, as a generalized form of multiple valued neutrosophic numbers and neutrosophic quadruple numbers, multiple generalized set-valued neutrosophic quadruple numbers and sets are defined and their basic properties are given. Thus, a new structure is obtained in which multi-valued neutrosophic sets and neutrosophic quadruple sets are used together. Also, optimistic union, pessimistic union, average union, pessimistic intersection, optimistic intersection, average intersection operators are defined for multiple generalized set-valued neutrosophic quadruples. Thus, multiple generalized set-valued neutrosophic quadruple sets were made usable in decision-making applications. Researchers can obtain more objective results in decision making applications by using these operators.

Keywords: Multi-valued Neutrosophic Number, Multiple Generalized Set-Valued Neutrosophic Quadruple Numbers, Multiple Generalized Set-Valued Neutrosophic Quadruple Sets, Generalized Set-Valued Neutrosophic Quadruple Number

1. Introduction

Smarandache defined neutrosophic sets (NS) [1] in 1998. A neutrosophic structure is generally represented as (T, I, F). In this notation, T is defined as the degree of membership, F as the degree of non-membership and I as the degree of indeterminacy, and these three components are given independently. This facilitates the mathematical description of uncertainties. It has found its place in the application areas in many different disciplines. Therefore, researchers have been conducting studies on decision-making structures in engineering, health services, data mining, cyber security, social sciences and many other fields. Şahin et al. defined metric spaces and normed spaces for neutrosophic numbers [2] in 2023. Miranda et al. measured the effectiveness of an indigenous knowledge-inspired training program for forest management applied to forest engineering students at the National University of Central Peru using neutrosophic 2-tuples [3]. Alqazzaz and Sallam studied the evaluation of sustainable waste valuation using treesoft set with neutrosophic nets [4]. Jdid and Smarandache worked on neutrosophic vision for decision making under the expected opportunity loss criterion (NEOL) risk [5]. Bedirhanoğlu and Atlas studied production planning with fuzzy neutrosophic multi-objective optimization technique [6]. Chen et al. studied logarithmic similarity measurement of neutrosophic Z-number sets for undergraduate teaching quality assessment [7]. El-Massry et al. obtained on augmenting artificial intelligence techniques with soft computation of neutrosophic theory in mysterious conditions for plant diseases [8]. Vergara et al.

studied the use of logistic regression, ambiguous likert scales and similarity measures for insightful analysis of child labor, informality and poverty in Ecuador [9]. Surya et al. studied a neutrosophic inventory system [10] for perishable goods with price-dependent demand. Obbineni et al. studied neutrosophic cognitive maps for clinical decision making in mental health care, a unified learning approach [11]. Farag et al. worked on the integration between bioinformatics algorithms and neutrosophic theory [12]. Salama et al. performed a comparative study for medical image quality enhancement, leukemia detection and classification using neutrosophic fuzzy domain and multilevel enhancement transforms [13]. El-Henawy et al. worked on modeling the affected criteria in the unbalanced challenges of classifiers based on tree soft sets (TrSS) [14], supported by the uncertain nature of neutrosophic theory. Ram et al. introduced the concepts of categorical aspects [15] related to neutrosophic automata and inverse neutrosophic automata by studying various properties and relationships between neutrosophic automata, inverse neutrosophic automata and double neutrosophic automata. Salama et al. introduced a new and efficient approach for encoding and decoding of digital data by neutrosophic encoding and decoding algorithm for ASCII code system [16]. More recently, Sahin et al. have made research and definitions on quadruple neutrosophic theory and its applications [17] in 2023.

Chatterjee et al. defined a multivalued neutrosophic set (MVNS) [18] in 2015. Peng, J. J. et al. introduced MVNS operations, developed a comparison method and defined an approach for multi-criteria decision making problems using aggregation operators [19] in 2015. Peng, H. G. et al. Defined MVNS [20] in 2018, which allows accuracy-membership, uncertainty-membership and inaccuracy-membership to have a range of net values between zero and one, respectively.

Smarandache defined the neutrosophic quadruple set (NQS) [21] in 2015. In general, a neutrosophic quadruple number is of the form (k, lT, mI, nF) (k, l, m, n ∈ R or C). The components T, I and F in this notation are components in neutrosophic logic. However, unlike the NS, NQS has a known part (k) and an unknown part ((lT, mI, nF)). Therefore, NQS is a new type of set that also includes NS. In addition, Şahin and Kargın defined set-valued neutrosophic quadruple numbers (SVNQN) [22] in 2019. Unlike neutrosophic quadruple numbers, in set-valued neutrosophic quadruple numbers, k, l, m, n are used as sets instead of real or complex numbers. That is, a set-valued neutrosophic quadruple number (SVNQN) over a set A is of the form (K, LT, MI, NF) (K, L, M, N ∈ P(A), where P(A) is the power set of A). Şahin et al. defined generalized set-valued neutrosophic quadruple numbers (GSVNQN) [23] in 2020. Thanks to this new structure, the NQS theory has become available in the field of applications. Kargın et al. In 2021, Kargın et al. studied the generalized hamming similarity measure based on neutrosophic quadruple numbers and its applications to legal sciences [24]. Şahin and Kargın defined interval generalized set-valued neutrosophic quadruple sets [25] in 2022. A new structure was obtained by using interval neutrosophic sets and generalized set-valued neutrosophic quadruple sets together. Sahin et al. defined Pythagorean neutrosophic quadruple numbers and set-valued Pythagorean neutrosophic quadruple numbers [26] in 2022. Kargın and Şahin defined neutrosophic triplet normed spaces [27] in 2023, based on set-valued neutrosophic quadruple numbers. Sahin et al. defined interval generalized set-valued neutrosophic quadruple graphs [28] in 2023. Şahin and Kargın defined interval generalized set-valued Pythagorean neutrosophic quadruple numbers [29] in 2023.

In this study, In The Preliminaries Section, the basic definitions and properties to be used in this study are given by mentioning NQS, MGNS, SVNQN and SVNQS, known-unknown part, membership degrees, indeterminacy and non-membership degrees. In the research and findings section, we define the MGSVNQN and MGSVNQS and the main features of these new concepts are given. Thus, we have obtained a new structure for NQS, MVS. Optimistic union, pessimistic union, average union, pessimistic intersection, optimistic intersection, average intersection operators were

defined for MGSVNQS. Sample sets from daily life were created and operations were performed on the sets for all the aforementioned properties and definitions. In The Conclusion Section, the results obtained in the study and suggestions for future studies are presented.

2. Preliminaries

Definition 2.1:[5] Let $\mathbf{X}$ be a universal set. For $\forall \widetilde{\boldsymbol{d}}_{\boldsymbol{\beta}} \in \mathbf{X}$,

$$\mathbf{0} \leq \boldsymbol{T}^{\widetilde{K}}_{(\widetilde{d}_{\beta})} + \boldsymbol{I}^{\widetilde{K}}_{(\widetilde{d}_{\beta})} + \boldsymbol{F}^{\widetilde{K}}_{(\widetilde{d}_{\beta})} \leq \mathbf{3}$$

$$\mathbf{T}^{\widetilde{K}}: \mathbf{X} \to [\mathbf{0}, \mathbf{1}] \ , \ \mathbf{I}^{\widetilde{K}}: \mathbf{X} \to [\mathbf{0}, \mathbf{1}] \quad \text{and} \quad \mathbf{F}^{\widetilde{K}}: \mathbf{X} \to [\mathbf{0}, \mathbf{1}]$$

With functions, a $\widetilde{\boldsymbol{K}}$ single-valued NS on X; It is defined as

$$\widetilde{\boldsymbol{K}} = \left\{ \langle \widetilde{\boldsymbol{d}}_{\boldsymbol{\beta}}, \boldsymbol{T}^{\widetilde{K}}_{(\widetilde{d}_{\beta})}, \boldsymbol{I}^{\widetilde{K}}_{(\widetilde{d}_{\beta})}, \boldsymbol{F}^{\widetilde{K}}_{(\widetilde{d}_{\beta})} \rangle : \widetilde{\boldsymbol{d}}_{\boldsymbol{\beta}} \in \mathbf{X} \right\}.$$

$\boldsymbol{T}^{\widetilde{K}}_{(\widetilde{d}_{\beta})}, \boldsymbol{I}^{\widetilde{K}}_{(\widetilde{d}_{\beta})}, \boldsymbol{F}^{\widetilde{K}}_{(\widetilde{d}_{\beta})}$ are the degree of truth, indeterminancy and falsity of $\widetilde{\boldsymbol{d}}_{\boldsymbol{\beta}} \in \mathbf{X}$, respectively.

Definition 2.2: [6] Let $\mathbf{E}$ be an universal set, a MVNS $\widetilde{\boldsymbol{K}}$ over the set $\mathbf{E}$ can be defined as follows.

For $\forall \mathbf{x_j} \in \mathbf{E}$; i=1,...,p and $\boldsymbol{\beta}$ =1,...,n;

$$\widetilde{\boldsymbol{K}} = \{ \langle \widetilde{\boldsymbol{d}}_{\boldsymbol{\beta}}, \left(\boldsymbol{T}^{\widetilde{K}}_{(\widetilde{d}_{\beta})}{}^{\mathbf{1}}, \boldsymbol{T}^{\widetilde{K}}_{(\widetilde{d}_{\beta})}{}^{\mathbf{2}}, \dots, \boldsymbol{T}^{\widetilde{K}}_{(\widetilde{d}_{\beta})}{}^{\boldsymbol{P}} \right), \left(\boldsymbol{I}^{\widetilde{K}}_{(\widetilde{d}_{\beta})}{}^{\mathbf{1}}, \boldsymbol{I}^{\widetilde{K}}_{(\widetilde{d}_{\beta})}{}^{\mathbf{2}}, \dots, \boldsymbol{I}^{\widetilde{K}}_{(\widetilde{d}_{\beta})}{}^{\boldsymbol{P}} \right), \left(\boldsymbol{F}^{\widetilde{K}}_{(\widetilde{d}_{\beta})}{}^{\mathbf{1}}, \boldsymbol{F}^{\widetilde{K}}_{(\widetilde{d}_{\beta})}{}^{\mathbf{2}}, \dots, \boldsymbol{F}^{\widetilde{K}}_{(\widetilde{d}_{\beta})}{}^{\boldsymbol{P}} \right) \rangle : \widetilde{\boldsymbol{d}}_{\boldsymbol{\beta}} \in \mathbf{E} \}$$

$$\boldsymbol{T}^{\widetilde{K}}_{(\widetilde{d}_{\beta})}{}^{\mathbf{1}}, \boldsymbol{T}^{\widetilde{K}}_{(\widetilde{d}_{\beta})}{}^{\mathbf{2}}, \dots, \boldsymbol{T}^{\widetilde{K}}_{(\widetilde{d}_{\beta})}{}^{\boldsymbol{P}} : \mathbf{E} \to [\mathbf{0}, \mathbf{1}] \ ,$$

$$\boldsymbol{I}^{\widetilde{K}}_{(\widetilde{d}_{\beta})}{}^{\mathbf{1}}, \boldsymbol{I}^{\widetilde{K}}_{(\widetilde{d}_{\beta})}{}^{\mathbf{2}}, \dots, \boldsymbol{I}^{\widetilde{K}}_{(\widetilde{d}_{\beta})}{}^{\boldsymbol{P}} : \mathbf{E} \to [\mathbf{0}, \mathbf{1}],$$

$$\boldsymbol{F}^{\widetilde{K}}_{(\widetilde{d}_{\beta})}{}^{\mathbf{1}}, \boldsymbol{F}^{\widetilde{K}}_{(\widetilde{d}_{\beta})}{}^{\mathbf{2}}, \dots, \boldsymbol{F}^{\widetilde{K}}_{(\widetilde{d}_{\beta})}{}^{\boldsymbol{P}} : \mathbf{E} \to [\mathbf{0}, \mathbf{1}]$$

It is also,

$$\mathbf{0} \leq \boldsymbol{T}^{\widetilde{K}}_{(\widetilde{d}_{\beta})}{}^{\boldsymbol{i}} + \boldsymbol{I}^{\widetilde{K}}_{(\widetilde{d}_{\beta})}{}^{\boldsymbol{i}} + \boldsymbol{F}^{\widetilde{K}}_{(\widetilde{d}_{\beta})}{}^{\dot{\mathbf{I}}} \leq \mathbf{3}$$

$$\left(\boldsymbol{T}^{\widetilde{K}}_{(\widetilde{d}_{\beta})}{}^{\mathbf{1}}, \boldsymbol{T}^{\widetilde{K}}_{(\widetilde{d}_{\beta})}{}^{\mathbf{2}}, \dots, \boldsymbol{T}^{\widetilde{K}}_{(\widetilde{d}_{\beta})}{}^{\boldsymbol{P}} \right), \left(\boldsymbol{I}^{\widetilde{K}}_{(\widetilde{d}_{\beta})}{}^{\mathbf{1}}, \boldsymbol{I}^{\widetilde{K}}_{(\widetilde{d}_{\beta})}{}^{\mathbf{2}}, \dots, \boldsymbol{I}^{\widetilde{K}}_{(\widetilde{d}_{\beta})}{}^{\boldsymbol{P}} \right) \text{ and } \left(\boldsymbol{F}^{\widetilde{K}}_{(\widetilde{d}_{\beta})}{}^{\mathbf{1}}, \boldsymbol{F}^{\widetilde{K}}_{(\widetilde{d}_{\beta})}{}^{\mathbf{2}}, \dots, \boldsymbol{F}^{\widetilde{K}}_{(\widetilde{d}_{\beta})}{}^{\boldsymbol{P}} \right)$$

The degree of truth, the degree of indeterminacy, the degree of falsity, respectively.

Definition 2.3:[7] Let N be a set and let P(N) be the power set of N. $\widetilde{\boldsymbol{K}}^{\boldsymbol{N}}$ SVNQN is of the form

$$\widetilde{\boldsymbol{K}}^{\boldsymbol{N}} = (\widetilde{\boldsymbol{K}}_{\boldsymbol{\beta}_1}, \widetilde{\boldsymbol{K}}_{\boldsymbol{\beta}_2} \mathbf{T}_{\widetilde{K}_{\beta_2}}, \widetilde{\boldsymbol{K}}_{\boldsymbol{\beta}_3} \mathbf{I}_{\widetilde{K}_{\beta_3}}, \widetilde{\boldsymbol{K}}_{\boldsymbol{\beta}_4} \mathbf{F}_{\widetilde{K}_{\beta_4}}).$$

Here T, I and F are the degree of membership, degree of uncertainty and degree of non-membership as in the neutrosophic theory, respectively and $\widetilde{\boldsymbol{K}}_{\boldsymbol{\beta}_1}, \widetilde{\boldsymbol{K}}_{\boldsymbol{\beta}_2}, \widetilde{\boldsymbol{K}}_{\boldsymbol{\beta}_3}, \widetilde{\boldsymbol{K}}_{\boldsymbol{\beta}_4} \in \mathbf{P}(\mathbf{N})$. Also, a SVNQS is defined such that

$$\widetilde{\boldsymbol{K}} = \{ (\widetilde{\boldsymbol{K}}_{\boldsymbol{\beta}_1}, \widetilde{\boldsymbol{K}}_{\boldsymbol{\beta}_2} \boldsymbol{T}^{\widetilde{K}}_{\boldsymbol{\beta}_2}, \widetilde{\boldsymbol{K}}_{\boldsymbol{\beta}_3} \boldsymbol{I}^{\widetilde{K}}_{\boldsymbol{\beta}_3}, \widetilde{\boldsymbol{K}}_{\boldsymbol{\beta}_4} \boldsymbol{F}^{\widetilde{K}}_{\boldsymbol{\beta}_4}) : \ \widetilde{\boldsymbol{K}}_{\boldsymbol{\beta}_1}, \widetilde{\boldsymbol{K}}_{\boldsymbol{\beta}_2}, \widetilde{\boldsymbol{K}}_{\boldsymbol{\beta}_3}, \widetilde{\boldsymbol{K}}_{\boldsymbol{\beta}_4} \in \mathbf{P}(\mathbf{N}) \}$$

Where, similar to neutrosophic quadruple sets,

$$\widetilde{\boldsymbol{K}}_{\boldsymbol{\beta_1}}$$

is called the known part and

$$(\widetilde{\boldsymbol{K}}_{\boldsymbol{\beta_2}}\boldsymbol{T}^{\widetilde{K}}_{\boldsymbol{\beta_2}}, \widetilde{\boldsymbol{K}}_{\boldsymbol{\beta_3}}\boldsymbol{I}^{\widetilde{K}}_{\boldsymbol{\beta_3}}, \widetilde{\boldsymbol{K}}_{\boldsymbol{\beta_4}}\boldsymbol{F}^{\widetilde{K}}_{\boldsymbol{\beta_4}})$$

is called the unknown part.

Definition 2.4: [9] Let X be a set and let P(X) be the power set of X. $\widetilde{\boldsymbol{K}}_{\boldsymbol{i}}$ set of a GSVNQN form

$$\widetilde{\boldsymbol{K}} = \left\{(\widetilde{\boldsymbol{K}}_{\boldsymbol{\beta_{1_i}}}, \widetilde{\boldsymbol{K}}_{\boldsymbol{\beta_{2_i}}}\boldsymbol{T}^{\widetilde{K}}_{\boldsymbol{\beta_{2_i}}}, \widetilde{\boldsymbol{K}}_{\boldsymbol{\beta_{3_i}}}\boldsymbol{I}^{\widetilde{K}}_{\boldsymbol{\beta_{3_i}}}, \widetilde{\boldsymbol{K}}_{\boldsymbol{\beta_{4_i}}}\boldsymbol{F}^{\widetilde{K}}_{\boldsymbol{\beta_{4_i}}}) : \widetilde{\boldsymbol{K}}_{\boldsymbol{\beta_{1_i}}}, \widetilde{\boldsymbol{K}}_{\boldsymbol{\beta_{2_i}}}, \widetilde{\boldsymbol{K}}_{\boldsymbol{\beta_{3_i}}}, \widetilde{\boldsymbol{K}}_{\boldsymbol{\beta_{4_i}}} \in \mathbf{P(X)}; \mathbf{i = 1, 2, 3, \dots, n}\right\}$$

$\boldsymbol{T}^{\widetilde{K}}_{\boldsymbol{\beta_{2_i}}}$, $\boldsymbol{I}^{\widetilde{K}}_{\boldsymbol{\beta_{3_i}}}$ and $\boldsymbol{F}^{\widetilde{K}}_{\boldsymbol{\beta_{4_i}}}$ are the usual neutrosophic logic tools. Also, a GSVNQS is defined such that

$$\widetilde{\boldsymbol{K}}^{\boldsymbol{N}}_{\boldsymbol{i}} = \left(\widetilde{\boldsymbol{K}}_{\boldsymbol{\beta_{1_i}}}, \widetilde{\boldsymbol{K}}_{\boldsymbol{\beta_{2_i}}}\boldsymbol{T}^{\widetilde{K}}_{\boldsymbol{\beta_{2_i}}}, \widetilde{\boldsymbol{K}}_{\boldsymbol{\beta_{3_i}}}\boldsymbol{I}^{\widetilde{K}}_{\boldsymbol{\beta_{3_i}}}, \widetilde{\boldsymbol{K}}_{\boldsymbol{\beta_{4_i}}}\boldsymbol{F}^{\widetilde{K}}_{\boldsymbol{\beta_{4_i}}}\right).$$

Where, a GSVNQN representing an entity that can be a number, an idea, an object, etc. For a GSVNQN $\left(\widetilde{\boldsymbol{K}}_{\boldsymbol{\beta_{1_i}}}, \widetilde{\boldsymbol{K}}_{\boldsymbol{\beta_{2_i}}}\boldsymbol{T}^{\widetilde{K}}_{\boldsymbol{\beta_{2_i}}}, \widetilde{\boldsymbol{K}}_{\boldsymbol{\beta_{3_i}}}\boldsymbol{I}^{\widetilde{K}}_{\boldsymbol{\beta_{3_i}}}, \widetilde{\boldsymbol{K}}_{\boldsymbol{\beta_{4_i}}}\boldsymbol{F}^{\widetilde{K}}_{\boldsymbol{\beta_{4_i}}}\right)$

$$\widetilde{\boldsymbol{K}}_{\boldsymbol{\beta_{1_i}}}$$

is called the known part and

$$\left(\widetilde{\boldsymbol{K}}_{\boldsymbol{\beta_{2_i}}}\boldsymbol{T}^{\widetilde{K}}_{\boldsymbol{\beta_{2_i}}}, \widetilde{\boldsymbol{K}}_{\boldsymbol{\beta_{3_i}}}\boldsymbol{I}^{\widetilde{K}}_{\boldsymbol{\beta_{3_i}}}, \widetilde{\boldsymbol{K}}_{\boldsymbol{\beta_{4_i}}}\boldsymbol{F}^{\widetilde{K}}_{\boldsymbol{\beta_{4_i}}}\right)$$

is called the unknown part.

We can also show that the GSVNQN consisting of GSVNQS

$$\widetilde{\boldsymbol{K}} = \left\{\widetilde{\boldsymbol{K}}^{\boldsymbol{N}}_{\boldsymbol{i}}; \mathbf{i = 1, 2, \dots, n}\right\}$$

3. Research and Findings

Definition 3.1: Let E be universal set and P(E) be power set of E. $\ddot{C}$ MGSVNS over the set E is defined as

$$\ddot{C} = \{\langle\left(M^1_{1(x_i)}, M^2_{1(x_i)}, M^3_{1(x_i)}, \dots, M^n_{1(x_i)}\right), \left(M^1_{2(x_i)}, M^2_{2(x_i)}, M^3_{2(x_i)}, \dots, M^n_{2(x_i)}\right)\left(T^1_{M_{2(x_i)}}, T^2_{M_{2(x_i)}}, T^3_{M_{2(x_i)}}, \dots, T^n_{M_{2(x_i)}}\right),$$

$$\left(M^1_{3(x_i)}, M^2_{3(x_i)}, M^3_{3(x_i)}, \dots, M^n_{3(x_i)}\right)\left(I^1_{M_{3(x_i)}}, I^2_{M_{3(x_i)}}, I^3_{M_{3(x_i)}}, \dots, I^n_{M_{3(x_i)}}\right), \left(M^1_{4(x_i)}, M^2_{4(x_i)}, M^3_{4(x_i)}, \dots, M^n_{4(x_i)}\right)$$

$$\left(F^1_{M_{4(x_i)}}, F^2_{M_{4(x_i)}}, F^3_{M_{4(x_i)}}, \dots, F^n_{M_{4(x_i)}}\right)\rangle : M^n_{1(x_i)}, M^n_{2(x_i)}, M^n_{3(x_i)}, M^n_{4(x_i)} \quad \in P(E)\}.$$

Here,

$$\text{i=1,…,j; n=1,…,p}$$

$$T^1_{M_2}, T^2_{M_2}, T^3_{M_2}, \dots, T^n_{M_2} : E \to [0,1],$$

$$I_{M_3}^1, I_{M_3}^2, I_{M_3}^3, \dots, I_{M_3}^n : E \to [0,1],$$

$$F_{M_4}^1, F_{M_4}^2, F_{M_4}^3, \dots, F_{M_4}^n : E \to [0,1]$$

$$0 \leq \dots, T_{M_{2(x_i)}}^n + I_{M_{3(x_i)}}^n + F_{M_{4(x_i)}}^n \leq 3$$

and

$$T_{M_2}^1(x_j), T_{M_2}^2(x_j), T_{M_2}^3(x_j), \dots, T_{M_2}^n(x_j),$$

$$I_{M_3}^1(x_j), I_{M_3}^2(x_j), I_{M_3}^3(x_j), \dots, I_{M_3}^n(x_j),$$

$$F_{M_4}^1(x_j), F_{M_4}^2(x_j), F_{M_4}^3(x_j) \dots, F_{M_4}^n(x_j)$$

degrees of truth, degrees of indeterminacy, degrees of falsity, respectively.

Note 3.2: From Definition 2.2 and Definition 2.4, MGSVNSs satisfy the conditions both GSVNS and MVNSs. Also, in the MGSVNS, p is the number of elements of the set and n is the number of components of each element.

Definition 3.3: Let

$$\ddot{C} = \{\langle (M_{1(x_i)}^1, M_{1(x_i)}^2, M_{1(x_i)}^3, \dots, M_{1(x_i)}^n), (M_{2(x_i)}^1, M_{2(x_i)}^2, M_{2(x_i)}^3, \dots, M_{2(x_i)}^n)\left(T_{M_{2(x_i)}}^1, T_{M_{2(x_i)}}^2, T_{M_{2(x_i)}}^3, \dots, T_{M_{2(x_i)}}^n\right),$$

$$(M_{3(x_i)}^1, M_{3(x_i)}^2, M_{3(x_i)}^3, \dots, M_{3(x_i)}^n)\left(I_{M_{3(x_i)}}^1, I_{M_{3(x_i)}}^2, I_{M_{3(x_i)}}^3, \dots, I_{M_{3(x_i)}}^n\right), (M_{4(x_i)}^1, M_{4(x_i)}^2, M_{4(x_i)}^3, \dots, M_{4(x_i)}^n)$$

$$\left(F_{M_{4(x_i)}}^1, F_{M_{4(x_i)}}^2, F_{M_{4(x_i)}}^3, \dots, F_{M_{4(x_i)}}^n\right)\rangle : M_{1(x_i)}^n, M_{2(x_i)}^n, M_{3(x_i)}^n, M_{4(x_i)}^n \in P(E)\}.$$

be a MGSVNS and let P(E) be the power set of E. A MGSVNQN $\ddot{C}$ is defined as

$$\ddot{C}_1 = \{(M_{1_{S_{x_1}}1}, M_{1_{S_{x_1}}2}, \dots, M_{1_{S_{x_1}}n}), (M_{2_{S_{x_1}}1}, M_{2_{S_{x_1}}2}, \dots, M_{2_{S_{x_1}}n})(T_{S_{x_1}1}(x_1), T_{S_{x_1}2}(x_1), \dots, T_{S_{x_1}n}(x_1)),$$

$$(M_{3_{S_{x_1}}1}, M_{3_{S_{x_1}}2}, \dots, M_{3_{S_{x_1}}n})(I_{S_{x_1}1}(x_1), I_{S_{x_1}2}(x_1), \dots, I_{S_{x_1}n}(x_1)), (M_{4_{S_{x_1}}1}, M_{4_{S_{x_1}}2}, \dots, M_{4_{S_{x_1}}n})$$

$$(F_{S_{x_1}1}(x_1), F_{S_{x_1}2}(x_1), \dots, F_{S_{x_1}n}(x_1))\}.$$

Where, i=1; n=1,…,p.

As in NQN, an MVNQN representing an entity that can be a number, an idea, an object, etc. For

$$\left(M_{1_{S_{x_1}}}, M_{2_{S_{x_1}}}, M_{3_{S_{x_1}}}, M_{4_{S_{x_1}}}\right)$$

$\left(M_{1_{S_{x_1}}}\right)$ is called the known part and

$$\left(M_{2_{S_{x_1}}}, M_{3_{S_{x_1}}}, M_{4_{S_{x_1}}}\right)$$

is called the unknown part.

MGSVNQS can also be represented in the form of

$$\ddot{C} = \{\ddot{C}_i \ ; \ i = 1,2, \dots, j\}.$$

Example 3.4: $\ddot{C}$ = {Biology, Geography, Religious Culture and Moral Knowledge, Literature, Geometry, English, Chemistry, Mathematics}

Let the set represent different textbooks studied in a month. For four consecutive weeks of a student in a month, we represent the lessons solved by H_1, H_2, H_3, H_4 GSVNQS. In addition, let $\ddot{C}_{n_1}, \ddot{C}_{n_2}, \ddot{C}_{n_3}, \ddot{C}_{n_4}, \ddot{C}_{n_5}$ be the MGSVNQN representation of the courses with questions solved on Monday, Tuesday, Wednesday, Thursday and Friday, respectively. (Biology 'B', Geography 'Ge', Religious Culture and Moral Knowledge 'D', Literature 'E', Geometry 'G', English 'I', Chemistry 'C', Mathematics 'M' will be expressed as abbreviations in the sets $\ddot{C}_{n_1}, \ddot{C}_{n_2}, \ddot{C}_{n_3}, \ddot{C}_{n_4}, \ddot{C}_{n_5}$)

$$H_1 = \left\{ \begin{array}{c} Monday_1 = \{\{Mathematics, Geometry\}, \{Religion, Literature\}(0.3), \\ \{English, Geography, \}(0.5), \{Biology\}(0.1)\} \\ Tuesday_1 = \{\{English\}, \{Chemistry, Literature\}(0.2), \\ \{Mathematics, Geometry\}(0.6), \{D.k, Biology\}(0.3)\} \\ Wednesday_1 = \{\{Biology, Geography, English\}, \{Literature\}(0.3), \\ \{Geometry, Religious\ \ Studies\}(0.5), \{Mathematics\}(0.2)\} \\ Thursday_1 = \{\{Mathematics\}, \{Literature, Biology\}(0.3), \\ \{Religion, English\}(0.4), \{Geography\}(0.3)\} \\ Friday_1 = \{\{Geometry, English\}, \{Geography, Maths\}(0.6), \\ \{Chemistry, Religious\ \ Studies\}(0.2), \{Literature, Biology\}(0.3)\} \end{array} \right\}$$

$$H_2 = \left\{ \begin{array}{c} Monday_2 = \{\{Religion, Literature\}, \{Chemistry\}(0.2), \\ \{Mathematics, Geometry\}(0.5), \{Geography, Biology\}(0.3)\} \\ Tuesday_2 = \{\{English, Maths\}, \{Geography, Geometry\}(0.4), \\ \{Religious\ \ Studies\}(0.3), \{Literature, Chemistry\}(0.3)\} \\ Wednesday_2 = \{\{Geography, English\}, \{Literature, Maths\}(0.5), \\ \{Geometry, Religious\ \ Studies\}(0.3), \{Biology\}(0.2)\} \\ Thursday_2 = \{\{Geometry, Maths\}, \{Geography, Chemistry, Biology\}(0.7), \\ \{Religious\ \ Studies\}(0.2), \{English\}(0.1)\} \\ Friday_2 = \{\{Chemistry\}, \{Mathematics\}(0.3), \\ \{Biology, Literature\}(0.4), \{Religion, English\}(0.3)\} \end{array} \right\}$$

$$H_3 = \left\{ \begin{array}{c} Monday_3 = \{\{Mathematics\}, \{Geometry\}(0.1), \\ \{Chemistry, Religious\ \ Studies\}(0.3), \{Literature, Geography\}(0.3)\} \\ Tuesday_3 = \{\{Religion, Geometry\}, \{English\}(0.3), \\ \{Chemistry\}(0.1), \{Literature, Geography, Mathematics\}(0.6)\} \\ Wednesday_3 = \{\{Literature, Chemistry, Biology\}, \{Mathematics, Geometry\}(0.5), \\ \{Religious\ \ Studies\}(0.2), \{English\}(0.3)\} \\ Thursday_3 = \{\{Chemistry\}, \{Mathematics, Geometry\}(0.5), \\ \{Religion\ \ K.\}(0.3), \{Biology\}(0.3)\} \\ Friday_3 = \{\{English\}, \{Literature, Geometry\}(0.3), \\ \{Chemistry, Religious\ \ Studies\}(0.1), \{Geography, Mathematics, Biology\}(0.6)\} \end{array} \right\}$$

$$H_4 = \left\{ \begin{array}{c} Monday_4 = \{\{English, Geometry\}, \{Chemistry, Literature\}(0.4), \\ \{Biology\}(0.4), \{Mathematics, Geography\}(0.3)\} \\ Tuesday_4 = \{\{Biology, Maths, Geometry\}, \{Literature\}(0.1), \\ \{Chemistry\}(0.7), \{Geography, English, Chemistry\}(0.2)\} \\ Wednesday_4 = \{\{Geography\}, \{Geometry, Chemistry, Biology\}(0.8), \\ \{English, Maths\}(0.1), \{Religious\ \ Studies\}(0.1)\} \\ Thursday_4 = \{\{Mathematics, Chemistry, Geometry\}, \{Literature\}(0.2), \\ \{Geography, Religious\ \ Studies\}(0.5), \{Biology\}(0.3)\} \\ Friday_4 = \{\{Geometry\}, \{English, Maths\}(0.5), \\ \{Chemistry, Religious\ \ Studies\}(0.3), \{Literature, Biology\}(0.2)\} \end{array} \right\}$$

$$\ddot{C}_{n_1} = \left\{ \begin{array}{c} (\{M, G\}, \{R, L\}, \{M\}, \{E, Ge\}), (\{R, L\}, \{C\}, \{G\}\{C, L\})(0.3,0.2,0.1,0.4), \\ (\{E, Ge\}, \{M, G\}, \{C, R\}, \{B\})(0.5,0.5,0.3,0.4), (\{B\}, \{Ge, B\}, \{L, Ge\}, \{M, Ge\})(0.1,0.3,0.3,0.3) \end{array} \right\}$$

$$\ddot{C}_{n_2} = \left\{ \begin{array}{c} (\{E\}, \{E, M\}, \{R, G\}, \{B, M, G\}), (\{C, L\}, \{Ge, G\}, \{E\}, \{L\}(0.2,0.4,0.3,0.1), \\ (\{M, G\}, \{R\}, \{C\}, \{C\})(0.6,0.3,0.1,0.7), (\{R, B\}, \{L, C\}, \{L, Ge, M\}, \{Ge, E, C\})(0.3,0.3,0.6,0.2) \end{array} \right\}$$

$$\ddot{C}_{n_3} = \left\{ \begin{array}{c} (\{B, Ge, E\}, \{Ge, E\}, \{L, C, B\}, \{Ge\}), (\{L\}, \{L, M\}, \{M, G\}, \{G, C, B\})(0.3,0.5,0.5,0.8), \\ (\{G, R\}, \{G, R\}, \{R\}, \{E, M\})(0.5,0.3,0.2,0.1), (\{M\}, \{B\}, \{E\}, \{R\})(0.2,0.2,0.3,0.1) \end{array} \right\}$$

$$\ddot{C}_{n_4} = \left\{ \begin{array}{c} (\{G, M\}, \{C\}, \{Ge, E, C\}, \{M, C, G\}), (\{L, B\}, \{Ge, C, B\}, \{M, G\}, \{L\})(0.3,0.7,0.5,0.2), \\ (\{R, E\}, \{R\}, \{R\}, \{Ge, R\})(0.4,0.2,0.3,0.5), (\{Ge\}, \{E\}, \{B\}, \{B\})(0.3,0.1,0.3,0.3) \end{array} \right\}$$

$$\ddot{C}_{n_5} = \left\{ \begin{array}{c} (\{G, E\}, \{C\}, \{E\}, \{G\}), (\{Ge, M\}, \{M\}, \{L, G\}, \{E, M\})(0.6,0.3,0.3,0.5), \\ (\{C, R\}, \{B, L\}, \{C, R\}, \{C, R\})(0.2,0.4,0.1,0.3), (\{L, B\}, \{R, E\}, \{Ge, M, B\}, \{L, B\})(0.3,0.3,0.6,0.2) \end{array} \right\}$$

Now, if we represent the month with these four weeks as a single set, we obtain $\ddot{C}_n$ MGSVNQS such that

$$\ddot{C}_n = \left\{ \begin{array}{c} \{(\{M, G\}, \{R, L\}, \{M\}, \{E, Ge\}), (\{R, L\}, \{C\}, \{G\}\{C, L\})(0.3,0.2,0.1,0.4), \\ (\{E, Ge\}, \{M, G\}, \{C, R\}, \{B\})(0.5,0.5,0.3,0.4), (\{B\}, \{Ge, B\}, \{L, Ge\}, \{M, Ge\})(0.1,0.3,0.3,0.3)\}, \\ \{(\{E\}, \{E, M\}, \{R, G\}, \{B, M, G\}), (\{C, L\}, \{Ge, G\}, \{E\}, \{L\}(0.2,0.4,0.3,0.1), \\ (\{M, G\}, \{R\}, \{C\}, \{C\})(0.6,0.3,0.1,0.7), (\{R, B\}, \{L, C\}, \{L, Ge, M\}, \{Ge, E, C\})(0.3,0.3,0.6,0.2)\}, \\ \{(\{B, Ge, E\}, \{Ge, E\}, \{L, C, B\}, \{Ge\}), (\{L\}, \{L, M\}, \{M, G\}, \{G, C, B\})(0.3,0.5,0.5,0.8), \\ (\{G, R\}, \{G, R\}, \{R\}, \{E, M\})(0.5,0.3,0.2,0.1), (\{M\}, \{B\}, \{E\}, \{R\})(0.2,0.2,0.3,0.1)\}, \\ \{(\{G, M\}, \{C\}, \{Ge, E, C\}, \{M, C, G\}), (\{L, B\}, \{Ge, C, B\}, \{M, G\}, \{L\})(0.3,0.7,0.5,0.2), \\ (\{R, E\}, \{R\}, \{R\}, \{Ge, R\})(0.4,0.2,0.3,0.5), (\{Ge\}, \{E\}, \{B\}, \{B\})(0.3,0.1,0.3,0.3)\}, \\ \{(\{G, E\}, \{C\}, \{E\}, \{G\}), (\{Ge, M\}, \{M\}, \{L, G\}, \{E, M\})(0.6,0.3,0.3,0.5), \\ (\{C, R\}, \{B, L\}, \{C, R\}, \{C, R\})(0.2,0.4,0.1,0.3), (\{L, B\}, \{R, E\}, \{Ge, M, B\}, \{L, B\})(0.3,0.3,0.6,0.2)\} \end{array} \right\}$$

Where, thanks to $\ddot{C}_n$ MGSVNQS, the sets $\ddot{C}_{n_1}, \ddot{C}_{n_2}, \ddot{C}_{n_3}, \ddot{C}_{n_4}, \ddot{C}_{n_5}$ were expressed as a single set.

Definition 3.5: Let

$$M^{\ddot{C}} = \{\langle (M^1_{1(x_i)}, M^2_{1(x_i)}, M^3_{1(x_i)}, \dots, M^n_{1(x_i)}), (M^1_{2(x_i)}, M^2_{2(x_i)}, M^3_{2(x_i)}, \dots, M^n_{2(x_i)})$$

$$\left(T^1_{M_{2(x_i)}}, T^2_{M_{2(x_i)}}, T^3_{M_{2(x_i)}}, \dots, T^n_{M_{2(x_i)}}\right), (M^1_{3(x_i)}, M^2_{3(x_i)}, M^3_{3(x_i)}, \dots, M^n_{3(x_i)}) \left(I^1_{M_{3(x_i)}}, I^2_{M_{3(x_i)}}, I^3_{M_{3(x_i)}}, \dots, I^n_{M_{3(x_i)}}\right)$$

$$(M^1_{4(x_i)}, M^2_{4(x_i)}, M^3_{4(x_i)}, \dots, M^n_{4(x_i)}) \left(F^1_{M_{4(x_i)}}, F^2_{M_{4(x_i)}}, F^3_{M_{4(x_i)}}, \dots, F^n_{M_{4(x_i)}}\right)\rangle : M^n_{1(x_i)}, M^n_{2(x_i)}, M^n_{3(x_i)}, M^n_{4(x_i)} \in P(E)\}$$

and

$$N^{\ddot{C}} = \{\langle (N^1_{1(x_i)}, N^2_{1(x_i)}, N^3_{1(x_i)}, \dots, N^n_{1(x_i)}), (N^1_{2(x_i)}, N^2_{2(x_i)}, N^3_{2(x_i)}, \dots, N^n_{2(x_i)})$$

$$\left(T^1_{N_{2(x_i)}}, T^2_{N_{2(x_i)}}, T^3_{N_{2(x_i)}}, \dots, T^n_{N_{2(x_i)}}\right), \left(N^1_{3(x_i)}, N^2_{3(x_i)}, N^3_{3(x_i)}, \dots, N^n_{3(x_i)}\right) \left(I^1_{N_{3(x_i)}}, I^2_{N_{3(x_i)}}, I^3_{N_{3(x_i)}}, \dots, I^n_{N_{3(x_i)}}\right)$$

$$\left(N^1_{4(x_i)}, N^2_{4(x_i)}, N^3_{4(x_i)}, \dots, N^n_{4(x_i)}\right) \left(F^1_{N_{4(x_i)}}, F^2_{N_{4(x_i)}}, F^3_{N_{4(x_i)}}, \dots, F^n_{N_{4(x_i)}}\right)\Big\rangle : N^n_{1(x_i)}, N^n_{2(x_i)}, N^n_{3(x_i)}, N^n_{4(x_i)} \in P(E)\}$$

be two MGSVNQSs.

i. If the following conditions are satisfied, we say that $M^{\ddot{C}}$ is a subset of $N^{\ddot{C}}$ and denote $M^{\ddot{C}} \subset N^{\ddot{C}}$.

$$M^n_1(x_i) \subset N^n_1(x_i)$$

$$M^n_2(x_i) \subset N^n_2(x_i)$$

$$M^n_3(x_i) \subset N^n_3(x_i)$$

$$M^n_4(x_i) \subset N^n_4(x_i)$$

and

$$T^n_{M_{2(x_i)}} \leq T^n_{N_{2(x_i)}}$$

$$I^n_{M_{3(x_i)}} \geq I^n_{N_{3(x_i)}}$$

$$F^n_{M_{4(x_i)}} \geq F^n_{M_{4(x_i)}}$$

i=1,…,j; n=1,…,p.

ii. If the following conditions are satisfied, $M^{\ddot{C}}$ is equal to $N^{\ddot{C}}$ and is denoted as $M^{\ddot{C}} = N^{\ddot{C}}$.

$$M^n_1(x_i) = N^n_1(x_i)$$

$$M^n_2(x_i) = N^n_2(x_i)$$

$$M^n_3(x_i) = N^n_3(x_i)$$

$$M^n_4(x_i) = N^n_4(x_i)$$

and

$$T^n_{M_{2(x_i)}} = T^n_{N_{2(x_i)}}$$

$$I^n_{M_{3(x_i)}} = I^n_{N_{3(x_i)}}$$

$$F^n_{M_{4(x_i)}} = F^n_{M_{4(x_i)}}$$

i=1,…,j; n=1,…,p.

Definition 3.6: Let

$$M^{\ddot{C}} = \{\langle (M^1_{1(x_i)}, M^2_{1(x_i)}, M^3_{1(x_i)}, \dots, M^n_{1(x_i)}), (M^1_{2(x_i)}, M^2_{2(x_i)}, M^3_{2(x_i)}, \dots, M^n_{2(x_i)})$$

$$\left(T^1_{M_{2(x_i)}}, T^2_{M_{2(x_i)}}, T^3_{M_{2(x_i)}}, \dots, T^n_{M_{2(x_i)}}\right), \left(M^1_{3(x_i)}, M^2_{3(x_i)}, M^3_{3(x_i)}, \dots, M^n_{3(x_i)}\right)\left(I^1_{M_{3(x_i)}}, I^2_{M_{3(x_i)}}, I^3_{M_{3(x_i)}}, \dots, I^n_{M_{3(x_i)}}\right)$$

$$\left(M^1_{4(x_i)}, M^2_{4(x_i)}, M^3_{4(x_i)}, \dots, M^n_{4(x_i)}\right)\left(F^1_{M_{4(x_i)}}, F^2_{M_{4(x_i)}}, F^3_{M_{4(x_i)}}, \dots, F^n_{M_{4(x_i)}}\right)\Big\rangle : M^n_{1(x_i)}, M^n_{2(x_i)}, M^n_{3(x_i)}, M^n_{4(x_i)} \in P(E)\}$$

and

$$N^{\ddot{C}} = \{\langle\left(N^1_{1(x_i)}, N^2_{1(x_i)}, N^3_{1(x_i)}, \dots, N^n_{1(x_i)}\right), \left(N^1_{2(x_i)}, N^2_{2(x_i)}, N^3_{2(x_i)}, \dots, N^n_{2(x_i)}\right)$$

$$\left(T^1_{N_{2(x_i)}}, T^2_{N_{2(x_i)}}, T^3_{N_{2(x_i)}}, \dots, T^n_{N_{2(x_i)}}\right), \left(N^1_{3(x_i)}, N^2_{3(x_i)}, N^3_{3(x_i)}, \dots, N^n_{3(x_i)}\right)\left(I^1_{N_{3(x_i)}}, I^2_{N_{3(x_i)}}, I^3_{N_{3(x_i)}}, \dots, I^n_{N_{3(x_i)}}\right)$$

$$\left(N^1_{4(x_i)}, N^2_{4(x_i)}, N^3_{4(x_i)}, \dots, N^n_{4(x_i)}\right)\left(F^1_{N_{4(x_i)}}, F^2_{N_{4(x_i)}}, F^3_{N_{4(x_i)}}, \dots, F^n_{N_{4(x_i)}}\right)\Big\rangle : N^n_{1(x_i)}, N^n_{2(x_i)}, N^n_{3(x_i)}, N^n_{4(x_i)} \in P(E)\}$$

be two MGSVNQSs.

i. Average ∪ operation for $M^{\ddot{C}}$ *and* $N^{\ddot{C}}$ is defined as

$$M^{\ddot{C}} \ \breve{\cup}_A \ N^{\ddot{C}} = \{\langle((M,N)^1{}_1(x_i), (M,N)^2{}_1(x_i), \dots, (M,N)^n{}_1(x_i)),$$

$$((M,N)^1{}_2(x_i), (M,N)^2{}_2(x_i), \dots, (M,N)^n{}_2(x_i))\ (T^1_{(M,N)_2(x_i)}, T^2_{(M,N)_2(x_i)}, \dots,\ T^n_{(M,N)_2(x_i)}),$$

$$((M,N)^1{}_3(x_i), (M,N)^2{}_3(x_i), \dots, (M,N)^n{}_3(x_i))\ \ (I^1_{(M,N)_2(x_i)}, I^2_{(M,N)_2(x_i)}, \dots,\ I^n_{(M,N)_2(x_i)}),$$

$$((M,N)^1{}_4(x_i), (M,N)^2{}_4(x_i), \dots, (M,N)^n{}_4(x_i))\ \ (F^1_{(M,N)_2(x_i)}, F^2_{(M,N)_2(x_i)}, \dots, F^n_{(M,N)_2(x_i)})\rangle,$$

$$(M,N)^n{}_1(x_i), (M,N)^n{}_2(x_i), (M,N)^n{}_3(x_i), (M,N)^n{}_4(x_i) \in P(E)\}.$$

Where,

$$(M,N)^n{}_1(x_i) = M^n_1(x_i) \cup N^n_1(x_i)$$

$$(M,N)^n{}_2(x_i) = M^n_2(x_i) \cup N^n_2(x_i)$$

$$(M,N)^n{}_3(x_i) = M^n_3(x_i) \cup N^n_3(x_i)$$

$$(M,N)^n{}_4(x_i) = M^n_4(x_i) \cup N^n_4(x_i)$$

$$T^n_{(M,N)_2(x_i)} = \frac{T^n_{(M)_2(x_i)} + T^n_{(N)_2(x_i)}}{2}$$

$$I^n_{(M,N)_3(x_i)} = \frac{I^n_{(M)_3(x_i)} + I^n_{(N)_3(x_i)}}{2}$$

$$F^n_{(M,N)_4(x_i)} = \frac{F^n_{(M)_4(x_i)} + F^n_{(N)_4(x_i)}}{2}$$

i=1,…,j; n=1,…,p.

ii. Average ∩ operation for $M^{\ddot{C}}$ *and* $N^{\ddot{C}}$ is defined as

$$M^{\ddot{C}} \ \breve{\cap}_A \ N^{\ddot{C}} = \{\langle((M,N)^1{}_1(x_i), (M,N)^2{}_1(x_i), \dots, (M,N)^n{}_1(x_i)),$$

$$((M,N)^1{}_2(x_i),(M,N)^2{}_2(x_i),\dots,(M,N)^n{}_2(x_i))\ (T^1_{(M,N)_2(x_i)},T^2_{(M,N)_2(x_i)},\dots,\ T^n_{(M,N)_2(x_i)}),$$

$$((M,N)^1{}_3(x_i),(M,N)^2{}_3(x_i),\dots,(M,N)^n{}_3(x_i))\ \ (I^1_{(M,N)_2(x_i)},I^2_{(M,N)_2(x_i)},\dots,\ I^n_{(M,N)_2(x_i)}),$$

$$((M,N)^1{}_4(x_i),(M,N)^2{}_4(x_i),\dots,(M,N)^n{}_4(x_i))\ \ (F^1_{(M,N)_2(x_i)},F^2_{(M,N)_2(x_i)},\dots,F^n_{(M,N)_2(x_i)})\rangle,$$

$$(M,N)^n{}_1(x_i),(M,N)^n{}_2(x_i),(M,N)^n{}_3(x_i),(M,N)^n{}_4(x_i)\in P(E)\}.$$

Where,

$(M,N)^n{}_1(x_i)=M^n_1(x_i)\cap N^n_1(x_i)$

$(M,N)^n{}_2(x_i)=M^n_2(x_i)\cap N^n_2(x_i)$

$(M,N)^n{}_3(x_i)=M^n_3(x_i)\cap N^n_3(x_i)$

$(M,N)^n{}_4(x_i)=M^n_4(x_i)\cap N^n_4(x_i)$

$T^n_{(M,N)_2(x_i)}=\frac{T^n_{(M)_2(x_i)}+T^n_{(N)_2(x_i)}}{2}$

$I^n_{(M,N)_3(x_i)}=\frac{I^n_{(M)_3(x_i)}+I^n_{(N)_3(x_i)}}{2}$

$F^n_{(M,N)_4(x_i)}=\frac{F^n_{(M)_4(x_i)}+F^n_{(N)_4(x_i)}}{2}$

i=1,…,j; n=1,…,p.

iii. Optimistic ∪ operation for $M^{\ddot{C}}$ *and* $N^{\ddot{C}}$ is defined as

$$M^{\ddot{C}}\ \ \breve{\cup}_O\ N^{\ddot{C}}=\{\langle((M,N)^1{}_1(x_i),(M,N)^2{}_1(x_i),\dots,(M,N)^n{}_1(x_i)),$$

$$((M,N)^1{}_2(x_i),(M,N)^2{}_2(x_i),\dots,(M,N)^n{}_2(x_i))\ (T^1_{(M,N)_2(x_i)},T^2_{(M,N)_2(x_i)},\dots,\ T^n_{(M,N)_2(x_i)}),$$

$$((M,N)^1{}_3(x_i),(M,N)^2{}_3(x_i),\dots,(M,N)^n{}_3(x_i))\ \ (I^1_{(M,N)_2(x_i)},I^2_{(M,N)_2(x_i)},\dots,\ I^n_{(M,N)_2(x_i)}),$$

$$((M,N)^1{}_4(x_i),(M,N)^2{}_4(x_i),\dots,(M,N)^n{}_4(x_i))\ \ (F^1_{(M,N)_2(x_i)},F^2_{(M,N)_2(x_i)},\dots,F^n_{(M,N)_2(x_i)})\rangle,$$

$$(M,N)^n{}_1(x_i),(M,N)^n{}_2(x_i),(M,N)^n{}_3(x_i),(M,N)^n{}_4(x_i)\in P(E)\}.$$

Where,

$(M,N)^n{}_1(x_i)=M^n_1(x_i)\cup N^n_1(x_i)$

$(M,N)^n{}_2(x_i)=M^n_2(x_i)\cup N^n_2(x_i)$

$(M,N)^n{}_3(x_i)=M^n_3(x_i)\cup N^n_3(x_i)$

$(M,N)^n{}_4(x_i)=M^n_4(x_i)\cup N^n_4(x_i)$

$T^n_{(M,N)_2(x_i)=max}\{T^n_{(M)_2(x_i)},T^n_{(N)_2(x_i)}\}$

$I^n_{(M,N)_3(x_i)=min}\{I^n_{(M)_3(x_i)}, I^n_{(N)_3(x_i)}\}$

$F^n_{(M,N)_4(x_i)=min}\{F^n_{(M)_4(x_i)}, F^n_{(N)_4(x_i)}\}$

i=1,…,j; n=1,…,p.

iv. Optimistic ∩ operation for $M^{\ddot{C}}$ ve $N^{\ddot{C}}$ is defined as

$$M^{\ddot{C}} \ \breve{\cap}_O \ N^{\ddot{C}} = \{\langle((M,N)^1{}_1(x_i), (M,N)^2{}_1(x_i), \dots, (M,N)^n{}_1(x_i)),$$
$$((M,N)^1{}_2(x_i), (M,N)^2{}_2(x_i), \dots, (M,N)^n{}_2(x_i))\ (T^1_{(M,N)_2(x_i)}, T^2_{(M,N)_2(x_i)}, \dots, T^n_{(M,N)_2(x_i)}),$$
$$((M,N)^1{}_3(x_i), (M,N)^2{}_3(x_i), \dots, (M,N)^n{}_3(x_i))\ (I^1_{(M,N)_2(x_i)}, I^2_{(M,N)_2(x_i)}, \dots, I^n_{(M,N)_2(x_i)}),$$
$$((M,N)^1{}_4(x_i), (M,N)^2{}_4(x_i), \dots, (M,N)^n{}_4(x_i))\ (F^1_{(M,N)_2(x_i)}, F^2_{(M,N)_2(x_i)}, \dots, F^n_{(M,N)_2(x_i)})\rangle,$$
$$(M,N)^n{}_1(x_i), (M,N)^n{}_2(x_i), (M,N)^n{}_3(x_i), (M,N)^n{}_4(x_i) \in P(E)\}.$$

Where,

$(M,N)^n{}_1(x_i) = M^n_1(x_i) \cap N^n_1(x_i)$

$(M,N)^n{}_2(x_i) = M^n_2(x_i) \cap N^n_2(x_i)$

$(M,N)^n{}_3(x_i) = M^n_3(x_i) \cap N^n_3(x_i)$

$(M,N)^n{}_4(x_i) = M^n_4(x_i) \cap N^n_4(x_i)$

$T^n_{(M,N)_2(x_i)=min}\{T^n_{(M)_2(x_i)}, T^n_{(N)_2(x_i)}\}$

$I^n_{(M,N)_3(x_i)=max}\{I^n_{(M)_3(x_i)}, I^n_{(N)_3(x_i)}\}$

$F^n_{(M,N)_4(x_i)=max}\{F^n_{(M)_4(x_i)}, F^n_{(N)_4(x_i)}\}$

i=1,…,j; n=1,…,p.

v. Pessimistic ∪ operation for $M^{\ddot{C}}$ and $N^{\ddot{C}}$ is defined as

$$M^{\ddot{C}} \ \breve{\cup}_P \ N^{\ddot{C}} = \{\langle((M,N)^1{}_1(x_i), (M,N)^2{}_1(x_i), \dots, (M,N)^n{}_1(x_i)),$$
$$((M,N)^1{}_2(x_i), (M,N)^2{}_2(x_i), \dots, (M,N)^n{}_2(x_i))\ (T^1_{(M,N)_2(x_i)}, T^2_{(M,N)_2(x_i)}, \dots, T^n_{(M,N)_2(x_i)}),$$
$$((M,N)^1{}_3(x_i), (M,N)^2{}_3(x_i), \dots, (M,N)^n{}_3(x_i))\ (I^1_{(M,N)_2(x_i)}, I^2_{(M,N)_2(x_i)}, \dots, I^n_{(M,N)_2(x_i)}),$$
$$((M,N)^1{}_4(x_i), (M,N)^2{}_4(x_i), \dots, (M,N)^n{}_4(x_i))\ (F^1_{(M,N)_2(x_i)}, F^2_{(M,N)_2(x_i)}, \dots, F^n_{(M,N)_2(x_i)})\rangle,$$
$$(M,N)^n{}_1(x_i), (M,N)^n{}_2(x_i), (M,N)^n{}_3(x_i), (M,N)^n{}_4(x_i) \in P(E)\}.$$

Where,

$(M,N)^n{}_1(x_i) = M^n_1(x_i) \cup N^n_1(x_i)$

$(M,N)^n{}_2(x_i) = M_2^n(x_i) \cup N_2^n(x_i)$

$(M,N)^n{}_3(x_i) = M_3^n(x_i) \cup N_3^n(x_i)$

$(M,N)^n{}_4(x_i) = M_4^n(x_i) \cup N_4^n(x_i)$

$T^n_{(M,N)_2(x_i)=min}\{T^n_{(M)_2(x_i)}, T^n_{(N)_2(x_i)}\}$

$I^n_{(M,N)_3(x_i)=max}\{I^n_{(M)_3(x_i)}, I^n_{(N)_3(x_i)}\}$

$F^n_{(M,N)_4(x_i)=max}\{F^n_{(M)_4(x_i)}, F^n_{(N)_4(x_i)}\}$

i=1,…,j; n=1,…,p.

vi. Pessimistic ∩ operation for $M^{\ddot{C}}$ ve $N^{\ddot{C}}$ is defined as

$$M^{\ddot{C}} \ \breve{\cap}_P \ N^{\ddot{C}} = \{\langle((M,N)^1{}_1(x_i), (M,N)^2{}_1(x_i), \dots, (M,N)^n{}_1(x_i)),$$
$$((M,N)^1{}_2(x_i), (M,N)^2{}_2(x_i), \dots, (M,N)^n{}_2(x_i))\ (T^1_{(M,N)_2(x_i)}, T^2_{(M,N)_2(x_i)}, \dots, T^n_{(M,N)_2(x_i)}),$$
$$((M,N)^1{}_3(x_i), (M,N)^2{}_3(x_i), \dots, (M,N)^n{}_3(x_i))\ \ (I^1_{(M,N)_2(x_i)}, I^2_{(M,N)_2(x_i)}, \dots, I^n_{(M,N)_2(x_i)}),$$
$$((M,N)^1{}_4(x_i), (M,N)^2{}_4(x_i), \dots, (M,N)^n{}_4(x_i))\ \ (F^1_{(M,N)_2(x_i)}, F^2_{(M,N)_2(x_i)}, \dots, F^n_{(M,N)_2(x_i)})\rangle,$$
$$(M,N)^n{}_1(x_i), (M,N)^n{}_2(x_i), (M,N)^n{}_3(x_i), (M,N)^n{}_4(x_i) \in P(E)\}.$$

Where,

$(M,N)^n{}_1(x_i) = M_1^n(x_i) \cap N_1^n(x_i)$

$(M,N)^n{}_2(x_i) = M_2^n(x_i) \cap N_2^n(x_i)$

$(M,N)^n{}_3(x_i) = M_3^n(x_i) \cap N_3^n(x_i)$

$(M,N)^n{}_4(x_i) = M_4^n(x_i) \cap N_4^n(x_i)$

$T^n_{(M,N)_2(x_i)=min}\{T^n_{(M)_2(x_i)}, T^n_{(N)_2(x_i)}\}$

$I^n_{(M,N)_3(x_i)=max}\{I^n_{(M)_3(x_i)}, I^n_{(N)_3(x_i)}\}$

$F^n_{(M,N)_4(x_i)=max}\{F^n_{(M)_4(x_i)}, F^n_{(N)_4(x_i)}\}$

i=1,…,j; n=1,…,p.

Properties 3.7: Let

$$M^{\ddot{C}} = \{\langle(M^1_{1(x_i)}, M^2_{1(x_i)}, M^3_{1(x_i)}, \dots, M^n_{1(x_i)}), (M^1_{2(x_i)}, M^2_{2(x_i)}, M^3_{2(x_i)}, \dots, M^n_{2(x_i)})$$
$$\left(T^1_{M_{2(x_i)}}, T^2_{M_{2(x_i)}}, T^3_{M_{2(x_i)}}, \dots, T^n_{M_{2(x_i)}}\right), (M^1_{3(x_i)}, M^2_{3(x_i)}, M^3_{3(x_i)}, \dots, M^n_{3(x_i)}) \left(I^1_{M_{3(x_i)}}, I^2_{M_{3(x_i)}}, I^3_{M_{3(x_i)}}, \dots, I^n_{M_{3(x_i)}}\right)$$
$$(M^1_{4(x_i)}, M^2_{4(x_i)}, M^3_{4(x_i)}, \dots, M^n_{4(x_i)}) \left(F^1_{M_{4(x_i)}}, F^2_{M_{4(x_i)}}, F^3_{M_{4(x_i)}}, \dots, F^n_{M_{4(x_i)}}\right)\rangle : M^n_{1(x_i)}, M^n_{2(x_i)}, M^n_{3(x_i)}, M^n_{4(x_i)} \in P(E)\},$$

$$N^{\ddot{C}} = \{\langle\left(N^1_{1(x_i)}, N^2_{1(x_i)}, N^3_{1(x_i)}, \dots, N^n_{1(x_i)}\right), \left(N^1_{2(x_i)}, N^2_{2(x_i)}, N^3_{2(x_i)}, \dots, N^n_{2(x_i)}\right)$$
$$\left(T^1_{N_{2(x_i)}}, T^2_{N_{2(x_i)}}, T^3_{N_{2(x_i)}}, \dots, T^n_{N_{2(x_i)}}\right), \left(N^1_{3(x_i)}, N^2_{3(x_i)}, N^3_{3(x_i)}, \dots, N^n_{3(x_i)}\right)\left(I^1_{N_{3(x_i)}}, I^2_{N_{3(x_i)}}, I^3_{N_{3(x_i)}}, \dots, I^n_{N_{3(x_i)}}\right)$$
$$\left(N^1_{4(x_i)}, N^2_{4(x_i)}, N^3_{4(x_i)}, \dots, N^n_{4(x_i)}\right)\left(F^1_{N_{4(x_i)}}, F^2_{N_{4(x_i)}}, F^3_{N_{4(x_i)}}, \dots, F^n_{N_{4(x_i)}}\right)\rangle : N^n_{1(x_i)}, N^n_{2(x_i)}, N^n_{3(x_i)}, N^n_{4(x_i)} \in P(E)\}$$

and

$$P^{\ddot{C}} = \{\langle\left(P^1_{1(x_i)}, P^2_{1(x_i)}, P^3_{1(x_i)}, \dots, P^n_{1(x_i)}\right), \left(P^1_{2(x_i)}, P^2_{2(x_i)}, P^3_{2(x_i)}, \dots, P^n_{2(x_i)}\right)$$
$$\left(T^1_{P_{2(x_i)}}, T^2_{P_{2(x_i)}}, T^3_{P_{2(x_i)}}, \dots, T^n_{P_{2(x_i)}}\right), \left(P^1_{3(x_i)}, P^2_{3(x_i)}, P^3_{3(x_i)}, \dots, P^n_{3(x_i)}\right)\left(I^1_{P_{3(x_i)}}, I^2_{P_{3(x_i)}}, I^3_{P_{3(x_i)}}, \dots, I^n_{P_{3(x_i)}}\right)$$
$$\left(P^1_{4(x_i)}, P^2_{4(x_i)}, P^3_{4(x_i)}, \dots, P^n_{4(x_i)}\right)\left(F^1_{P_{4(x_i)}}, F^2_{P_{4(x_i)}}, F^3_{P_{4(x_i)}}, \dots, F^n_{P_{4(x_i)}}\right)\rangle : P^n_{1(x_i)}, P^n_{2(x_i)}, P^n_{3(x_i)}, P^n_{4(x_i)} \in P(E)\}.$$

be three MGSVNQSs. Then, the following properties are provided.

i) $M^{\ddot{C}} \breve{\cup}_A N^{\ddot{C}} = N^{\ddot{C}} \breve{\cup}_A M^{\ddot{C}}$

ii) $M^{\ddot{C}} \breve{\cup}_O N^{\ddot{C}} = N^{\ddot{C}} \breve{\cup}_O M^{\ddot{C}}$

iii) $M^{\ddot{C}} \cup'_P N^{\ddot{C}} = N^{\ddot{C}} \cup'_P M^{\ddot{C}}$

iv) $M^{\ddot{C}} \breve{\cap}_A N^{\ddot{C}} = N^{\ddot{C}} \breve{\cap}_A M^{\ddot{C}}$

v) $M^{\ddot{C}} \cap'_O N^{\ddot{C}} = N^{\ddot{C}} \cap'_O M^{\ddot{C}}$

vi) $M^{\ddot{C}} \cap'_P N^{\ddot{C}} = N^{\ddot{C}} \cap'_P M^{\ddot{C}}$

vii) $M^{\ddot{C}} \breve{\cup}_A (N^{\ddot{C}} \breve{\cup}_A P^{\ddot{C}}) = (M^{\ddot{C}} \breve{\cup}_A N^{\ddot{C}}) \breve{\cup}_A N^{\ddot{C}}$

viii) $M^{\ddot{C}} \breve{\cup}_O (N^{\ddot{C}} \breve{\cup}_O P^{\ddot{C}}) = (M^{\ddot{C}} \breve{\cup}_O N^{\ddot{C}}) \breve{\cup}_O P^{\ddot{C}}$

ix) $M^{\ddot{C}} \breve{\cup}_P (N^{\ddot{C}} \breve{\cup}_P P^{\ddot{C}}) = (M^{\ddot{C}} \breve{\cup}_P N^{\ddot{C}}) \breve{\cup}_P P^{\ddot{C}}$

x) $M^{\ddot{C}} \breve{\cap}_A (N^{\ddot{C}} \breve{\cap}_A P^{\ddot{C}}) = (M^{\ddot{C}} \breve{\cap}_A N^{\ddot{C}}) \breve{\cap}_A P^{\ddot{C}}$

xi) $M^{\ddot{C}} \cap'_O (N^{\ddot{C}} \cap'_O P^{\ddot{C}}) = (M^{\ddot{C}} \cap'_O N^{\ddot{C}}) \cap'_O P^{\ddot{C}}$

xii) $M^{\ddot{C}} \cap'_P (N^{\ddot{C}} \cap'_P P^{\ddot{C}}) = (M^{\ddot{C}} \cap'_P N^{\ddot{C}}) \cap'_P P^{\ddot{C}}$

xiii) $M^{\ddot{C}} \breve{\cap}_A (N^{\ddot{C}} \breve{\cup}_A P^{\ddot{C}}) = (M^{\ddot{C}} \breve{\cap}_A N^{\ddot{C}}) \breve{\cup}_A (M^{\ddot{C}} \breve{\cap}_A P^{\ddot{C}})$

xiv) $M^{\ddot{C}} \cap'_O (N^{\ddot{C}} \breve{\cup}_A P^{\ddot{C}}) = (M^{\ddot{C}} \cap'_O N^{\ddot{C}}) \breve{\cup}_A (M^{\ddot{C}} \cap'_O P^{\ddot{C}})$

xv) $M^{\ddot{C}} \cap'_P (N^{\ddot{C}} \breve{\cup}_P P^{\ddot{C}}) = (M^{\ddot{C}} \cap'_P N^{\ddot{C}}) \breve{\cup}_P (M^{\ddot{C}} \cap'_P P^{\ddot{C}})$

xvi) $M^{\ddot{C}} \breve{\cup}_A (N^{\ddot{C}} \breve{\cap}_A P^{\ddot{C}}) = (M^{\ddot{C}} \breve{\cup}_A N^{\ddot{C}}) \cap'_O (M^{\ddot{C}} \breve{\cup}_A P^{\ddot{C}})$

xvii) $M^{\ddot{C}} \breve{\cup}_A (N^{\ddot{C}} \cap'_O P^{\ddot{C}}) = (M^{\ddot{C}} \breve{\cup}_A N^{\ddot{C}}) \cap'_O (M^{\ddot{C}} \breve{\cup}_A P^{\ddot{C}})$

xviii) $M^{\ddot{C}} \breve{\cup}_P (M^{\ddot{C}} \cap'_P P^{\ddot{C}}) = (M^{\ddot{C}} \breve{\cup}_P N^{\ddot{C}}) \cap'_P (M^{\ddot{C}} \breve{\cup}_P P^{\ddot{C}})$

xix) İf $M^{\ddot{C}} = N^{\ddot{C}}$ then $M^{\ddot{C}} \breve{\cup}_A N^{\ddot{C}} = M^{\ddot{C}} \breve{\cup}_O N^{\ddot{C}} = M^{\ddot{C}} \cup'_P N^{\ddot{C}}$

xx) İf $M^{\ddot{C}} = N^{\ddot{C}}$ then $M^{\ddot{C}} \breve{\cap}_A N^{\ddot{C}} = M^{\ddot{C}} \cap'_O N^{\ddot{C}} = M^{\ddot{C}} \cap'_P N^{\ddot{C}}$

Proof:

i) From Definition 3.6 for $(M^{\ddot{C}} \breve{\cup}_A N^{\ddot{C}})$, we get

$(M,N)^n{}_1(x_i) = (M_1^n(x_i) \cup N_1^n(x_i))$, $(M,N)^n{}_2(x_i) = (M_2^n(x_i) \cup N_2^n(x_i))$,

$(M,N)^n{}_3(x_i) = (M_3^n(x_i) \cup N_3^n(x_i))$, $(M,N)^n{}_4(x_i) = (M_4^n(x_i) \cup N_4^n(x_i))$.

$T^n_{(M,N)_2(x_i)} = \frac{T^n_{(M)_2(x_i)} + T^n_{(N)_2(x_i)}}{2}$, $I^n_{(M,N)_3(x_i)} = \frac{I^n_{(M)_3(x_i)} + I^n_{(N)_3(x_i)}}{2}$, $F^n_{(M,N)_4(x_i)} = \frac{F^n_{(M)_4(x_i)} + F^n_{(N)_4(x_i)}}{2}$

i=1,…,j; n=1,…,p. (1)

From Definition 3.6 for $(N^{\ddot{C}} \breve{\cup}_A M^{\ddot{C}})$, we get

$(N,M)^n{}_1(x_i) = (N_1^n(x_i) \cup M_1^n(x_i))$, $(N,M)^n{}_2(x_i) = (N_2^n(x_i) \cup M_2^n(x_i))$,

$(N,M)^n{}_3(x_i) = (N_3^n(x_i) \cup M_3^n(x_i))$, $(N,M)^n{}_4(x_i) = (N_4^n(x_i) \cup M_4^n(x_i))$.

$T^n_{(N,M)_2(x_i)} = \frac{T^n_{(N)_2(x_i)} + T^n_{(M)_2(x_i)}}{2}$, $I^n_{(N,M)_3(x_i)} = \frac{I^n_{(N)_3(x_i)} + I^n_{(M)_3(x_i)}}{2}$, $F^n_{(N,M)_4(x_i)} = \frac{F^n_{(N)_4(x_i)} + F^n_{(M)_4(x_i)}}{2}$

i=1,…,j; n=1,…,p. (2)

and from Definition 3.5, (1) and (2), we get

$$M^{\ddot{C}} \breve{\cup}_A N^{\ddot{C}} = N^{\ddot{C}} \breve{\cup}_A M^{\ddot{C}}$$

Proof of {ii, iii} are obtained similar to i.

iv) From Definition 3.6 and ii, for $(M^{\ddot{C}} \breve{\cap}_A N^{\ddot{C}})$, we get

$(M,N)^n{}_1(x_i) = (M_1^n(x_i) \cap N_1^n(x_i))$, $(M,N)^n{}_2(x_i) = (M_2^n(x_i) \cap N_2^n(x_i))$,

$(M,N)^n{}_3(x_i) = (M_3^n(x_i) \cap N_3^n(x_i))$, $(M,N)^n{}_4(x_i) = (M_4^n(x_i) \cap N_4^n(x_i))$.

$T^n_{(M,N)_2(x_i)} = \frac{T^n_{(M)_2(x_i)} + T^n_{(N)_2(x_i)}}{2}$, $I^n_{(M,N)_3(x_i)} = \frac{I^n_{(M)_3(x_i)} + I^n_{(N)_3(x_i)}}{2}$, $F^n_{(M,N)_4(x_i)} = \frac{F^n_{(M)_4(x_i)} + F^n_{(N)_4(x_i)}}{2}$

i=1,…,j, n=1,…,p. (3)

From Definition 3.6 and ii, for $(N^{\ddot{C}} \breve{\cap}_A M^{\ddot{C}})$, we get

$(N,M)^n{}_1(x_i) = (N_1^n(x_i) \cap M_1^n(x_i))$, $(N,M)^n{}_2(x_i) = (N_2^n(x_i) \cap M_2^n(x_i))$,

$(N,M)^n{}_3(x_i) = (N_3^n(x_i) \cap M_3^n(x_i))$, $(N,M)^n{}_4(x_i) = (N_4^n(x_i) \cap M_4^n(x_i))$.

$$T^n_{(N,M)_2(x_i)} = \frac{T^n_{(N)_2(x_i)}+T^n_{(M)_2(x_i)}}{2},\ I^n_{(N,M)_3(x_i)} = \frac{I^n_{(N)_3(x_i)}+I^n_{(M)_3(x_i)}}{2},\ F^n_{(N,M)_4(x_i)} = \frac{F^n_{(N)_4(x_i)}+F^n_{(M)_4(x_i)}}{2}$$

i=1,…,j; n=1,…,p. (4)

and from Definition 3.5, (3) and (4), we get

Proof of {v, vi} are obtained similar to iv.

vii) We assume that

$$T^n_{(M)_2(x_i)} = T^n_{(P)_2(x_i)},\quad I^n_{(M)_2(x_i)} = I^n_{(P)_2(x_i)},\quad F^n_{(M)_2(x_i)} = F^n_{(P)_2(x_i)}$$

For Definition 3.6 $M^{\ddot{C}} \breve{\cup}_A (N^{\ddot{C}} \breve{\cup}_A P^{\ddot{C}})$,we get

$$\big(M,(N,P)\big)^n{}_1(x_i) = \Big(M^n_1(x_i) \cup \big(N^n_1(x_i) \cup P^n_1(x_i)\big)\Big),$$

$$\big(M,(N,P)\big)^n{}_2(x_i) = \Big(M^n_2(x_i) \cup \big(N^n_2(x_i) \cup P^n_2(x_i)\big)\Big),$$

$$\big(M,(N,P)\big)^n{}_3(x_i) = \Big(M^n_3(x_i) \cup \big(N^n_3(x_i) \cup P^n_3(x_i)\big)\Big),$$

$$\big(M,(N,P)\big)^n{}_4(x_i) = \Big(M^n_4(x_i) \cup \big(N^n_4(x_i) \cup P^n_4(x_i)\big)\Big).$$

$$T^n_{(M,(N,P))_2(x_i)} = \frac{T^n_{(M)_2(x_i)}+\frac{T^n_{(N)_2(x_i)}+T^n_{(P)_2(x_i)}}{2}}{2},\ I^n_{(M,(N,P))_3(x_i)} = \frac{I^n_{(M)_3(x_i)}+\frac{I^n_{(N)_3(x_i)}+I^n_{(P)_3(x_i)}}{2}}{2},$$

$$F^n_{(M,(N,P))_4(x_i)} = \frac{F^n_{(M)_4(x_i)} + \frac{F^n_{(N)_4(x_i)} + F^n_{(P)_4(x_i)}}{2}}{2}$$

i=1,…,j; n=1,…,p. (5)

For Definition 3.6, i. for $(M^{\ddot{C}} \breve{\cup}_A N^{\ddot{C}}) \breve{\cup}_A N^{\ddot{C}}$

$$\big((M,N),P\big)^n{}_1(x_i) = \Big(\big(M^n_1(x_i) \cup N^n_1(x_i)\big) \cup P^n_1(x_i)\Big),$$

$$\big((M,N),P\big)^n{}_2(x_i) = \Big(\big(M^n_2(x_i) \cup N^n_2(x_i)\big) \cup P^n_2(x_i)\Big),$$

$$\big((M,N),P\big)^n{}_3(x_i) = \Big(\big(M^n_3(x_i) \cup N^n_3(x_i)\big) \cup P^n_3(x_i)\Big),$$

$$\big((M,N),P\big)^n{}_4(x_i) = \Big(\big(M^n_4(x_i) \cup N^n_4(x_i)\big) \cup P^n_4(x_i)\Big).$$

$$T^n_{((M,N),P)_2(x_i)} = \frac{\frac{T^n_{(M)_2(x_i)}+T^n_{(N)_2(x_i)}}{2}+T^n_{(P)_2(x_i)}}{2},\ I^n_{((M,N),P)_3(x_i)} = \frac{\frac{I^n_{(M)_3(x_i)}+I^n_{(N)_3(x_i)}}{2}+I^n_{(P)_3(x_i)}}{2},$$

$$F^n_{((M,N),P)_4(x_i)} = \frac{\frac{F^n_{(M)_4(x_i)}+F^n_{(N)_4(x_i)}}{2}+F^n_{(P)_4(x_i)}}{2}$$

i=1,…,j; n=1,…,p. (6)

From Definition 3.5, (5) and (6) we get

$$M^{\ddot{C}} \ \breve{\cup}_A \ (N^{\ddot{C}} \ \breve{\cup}_A \ P^{\ddot{C}} \ ) = (M^{\ddot{C}} \ \breve{\cup}_A \ N^{\ddot{C}}) \ \breve{\cup}_A \ N^{\ddot{C}}$$

Proof of {viii, ix} are obtained similar to vii.

x) We assume that

$T^n_{(M)_2(x_i)} = T^n_{(P)_2(x_i)}$, $I^n_{(M)_2(x_i)} = I^n_{(P)_2(x_i)}$, $F^n_{(M)_2(x_i)} = F^n_{(P)_2(x_i)}$

From Definition 3.6 for $M^{\ddot{C}} \ \breve{\cap}_A \ (N^{\ddot{C}} \ \breve{\cap}_A \ P^{\ddot{C}} \ ) = (M^{\ddot{C}} \ \breve{\cap}_A \ N^{\ddot{C}}) \ \breve{\cap}_A \ P^{\ddot{C}}$, we get

$\big(M, (N, P)\big)^n{}_1(x_i) = \Big(M^n_1(x_i) \cap \big(N^n_1(x_i) \cap P^n_1(x_i)\big)\Big)$,

$\big(M, (N, P)\big)^n{}_2(x_i) = \Big(M^n_2(x_i) \cap \big(N^n_2(x_i) \cap P^n_2(x_i)\big)\Big)$,

$\big(M, (N, P)\big)^n{}_3(x_i) = \Big(M^n_3(x_i) \cap \big(N^n_3(x_i) \cap P^n_3(x_i)\big)\Big)$,

$\big(M, (N, P)\big)^n{}_4(x_i) = \Big(M^n_4(x_i) \cap \big(N^n_4(x_i) \cap P^n_4(x_i)\big)\Big)$.

$T^n_{(M,(N,P))_2(x_i)} = \frac{T^n_{(M)_2(x_i)} + \frac{T^n_{(N)_2(x_i)} + T^n_{(P)_2(x_i)}}{2}}{2}$, $I^n_{(M,(N,P))_3(x_i)} = \frac{I^n_{(M)_3(x_i)} + \frac{I^n_{(N)_3(x_i)} + I^n_{(P)_3(x_i)}}{2}}{2}$,

$F^n_{(M,(N,P))_4(x_i)} = \frac{F^n_{(M)_4(x_i)} + \frac{F^n_{(N)_4(x_i)} + F^n_{(P)_4(x_i)}}{2}}{2}$

i=1,…,j; n=1,…,p. (7)

For Definition 3.6 for $(M^{\ddot{C}} \ \breve{\cap}_A \ N^{\ddot{C}}) \ \breve{\cap}_A \ P^{\ddot{C}}$

$\big((M, N), P\big)^n{}_1(x_i) = \Big(\big(M^n_1(x_i) \cap N^n_1(x_i)\big) \cap P^n_1(x_i)\Big)$,

$\big((M, N), P\big)^n{}_2(x_i) = \Big(\big(M^n_2(x_i) \cap N^n_2(x_i)\big) \cap P^n_2(x_i)\Big)$

$\big((M, N), P\big)^n{}_3(x_i) = \Big(\big(M^n_3(x_i) \cap N^n_3(x_i)\big) \cap P^n_3(x_i)\Big)$,

$\big((M, N), P\big)^n{}_4(x_i) = \Big(\big(M^n_4(x_i) \cap N^n_4(x_i)\big) \cap P^n_4(x_i)\Big)$.

$T^n_{((M,N),P)_2(x_i)} = \frac{\frac{T^n_{(M)_2(x_i)} + T^n_{(N)_2(x_i)}}{2} + T^n_{(P)_2(x_i)}}{2}$, $I^n_{((M,N),P)_3(x_i)} = \frac{\frac{I^n_{(M)_3(x_i)} + I^n_{(N)_3(x_i)}}{2} + I^n_{(P)_3(x_i)}}{2}$,

$F^n_{((M,N),P)_4(x_i)} = \frac{\frac{F^n_{(M)_4(x_i)} + F^n_{(N)_4(x_i)}}{2} + F^n_{(P)_4(x_i)}}{2}$

i=1,…,j; n=1,…,p. (8)

From Definition 3.5, (7) and (8) we get

$$M^{\ddot{C}} \ \breve{\cap}_A \ (N^{\ddot{C}} \ \breve{\cap}_A \ P^{\ddot{C}} \ ) = (M^{\ddot{C}} \ \breve{\cap}_A \ N^{\ddot{C}}) \ \breve{\cap}_A \ P^{\ddot{C}}$$

Proof of {xi, xii} are obtained similar to x.

xiii) From Definition 3.6 for $M^{\ddot{C}} \ddot{\cap}_A (N^{\ddot{C}} \ddot{\cup}_A P^{\ddot{C}})$, we get

$$\big(M,(N,P)\big)^n{}_1(x_i) = \Big(M_1^n(x_i) \cap \big(N_1^n(x_i) \cup P_1^n(x_i)\big)\Big),$$

$$\big(M,(N,P)\big)^n{}_2(x_i) = \Big(M_2^n(x_i) \cap \big(N_2^n(x_i) \cup P_2^n(x_i)\big)\Big),$$

$$\big(M,(N,P)\big)^n{}_3(x_i) = \Big(M_3^n(x_i) \cap \big(N_3^n(x_i) \cup P_3^n(x_i)\big)\Big),$$

$$\big(M,(N,P)\big)^n{}_4(x_i) = \Big(M_4^n(x_i) \cap \big(N_4^n(x_i) \cup P_4^n(x_i)\big)\Big).$$

$$T^n_{(M,(N,P))_2(x_i)} = \frac{T^n_{(M)_2(x_i)} + \frac{T^n_{(N)_2(x_i)} + T^n_{(P)_2(x_i)}}{2}}{2},\ I^n_{(M,(N,P))_3(x_i)} = \frac{I^n_{(M)_3(x_i)} + \frac{I^n_{(N)_3(x_i)} + I^n_{(P)_3(x_i)}}{2}}{2},$$

$$F^n_{(M,(N,P))_4(x_i)} = \frac{F^n_{(M)_4(x_i)} + \frac{F^n_{(N)_4(x_i)} + F^n_{(P)_4(x_i)}}{2}}{2}$$

i=1,…,j; n=1,…,p. (9)

For Definition 3.6, $(M^{\ddot{C}} \ddot{\cap}_A N^{\ddot{C}}) \ddot{\cup}_A (M^{\ddot{C}} \ddot{\cap}_A P^{\ddot{C}})$, we get

$$\big((M,N),(M,P)\big)^n{}_1(x_i) = \Big(\big(M_1^n(x_i) \cap N_1^n(x_i)\big) \cup \big(M_1^n(x_i) \cap P_1^n(x_i)\big)\Big),$$

$$\big((M,N),(M,P)\big)^n{}_2(x_i) = \Big(\big(M_2^n(x_i) \cap N_2^n(x_i)\big) \cup \big(M_2^n(x_i) \cap P_2^n(x_i)\big)\Big),$$

$$\big((M,N),(M,P)\big)^n{}_3(x_i) = \Big(\big(M_3^n(x_i) \cap N_3^n(x_i)\big) \cup \big(M_3^n(x_i) \cap P_3^n(x_i)\big)\Big),$$

$$\big((M,N),(M,P)\big)^n{}_4(x_i) = \Big(\big(M_4^n(x_i) \cap N_4^n(x_i)\big) \cup \big(M_4^n(x_i) \cap P_4^n(x_i)\big)\Big),$$

$$T^n_{((M,N),(M,P))_2(x_i)} = \frac{\frac{T^n_{(M)_2(x_i)} + T^n_{(N)_2(x_i)}}{2} + \frac{T^n_{(M)_2(x_i)} + T^n_{(P)_2(x_i)}}{2}}{2},$$

$$I^n_{((M,N),(M,P))_3(x_i)} = \frac{\frac{I^n_{(M)_3(x_i)} + I^n_{(N)_3(x_i)}}{2} + \frac{I^n_{(M)_3(x_i)} + I^n_{(P)_3(x_i)}}{2}}{2},$$

$$F^n_{((M,N),(M,P))_4(x_i)} = \frac{\frac{F^n_{(M)_4(x_i)} + F^n_{(N)_4(x_i)}}{2} + \frac{F^n_{(M)_4(x_i)} + F^n_{(P)_4(x_i)}}{2}}{2}$$

i=1,…,j, n=1,…,p. (10)

From Definition 3.5, (9) and (10) we get

$$M^{\ddot{C}} \ddot{\cap}_A (N^{\ddot{C}} \ddot{\cup}_A P^{\ddot{C}}) = (M^{\ddot{C}} \ddot{\cap}_A N^{\ddot{C}}) \ddot{\cup}_A (M^{\ddot{C}} \ddot{\cap}_A P^{\ddot{C}})$$

Proof of {xiv, xv} are obtained similar to xiii.

xvi) From Definition 3.6 for $M^{\ddot{C}} \ddot{\cup}_A (N^{\ddot{C}} \ddot{\cap}_A P^{\ddot{C}})$, we get

$$\left(M,(N,P)\right)^{n}{}_{1}(x_i)=\left(M_1^n(x_i)\cup\left(N_1^n(x_i)\cap P_1^n(x_i)\right)\right),$$

$$\left(M,(N,P)\right)^{n}{}_{2}(x_i)=\left(M_2^n(x_i)\cup\left(N_2^n(x_i)\cap P_2^n(x_i)\right)\right)$$

$$\left(M,(N,P)\right)^{n}{}_{3}(x_i)=\left(M_3^n(x_i)\cup\left(N_3^n(x_i)\cap P_3^n(x_i)\right)\right),$$

$$\left(M,(N,P)\right)^{n}{}_{4}(x_i)=\left(M_4^n(x_i)\cup\left(N_4^n(x_i)\cap P_4^n(x_i)\right)\right)$$

$$T^n_{(M,(N,P))_2(x_i)}=\frac{T^n_{(M)_2(x_i)}+\frac{T^n_{(N)_2(x_i)}+T^n_{(P)_2(x_i)}}{2}}{2},\ I^n_{(M,(N,P))_3(x_i)}=\frac{I^n_{(M)_3(x_i)}+\frac{I^n_{(N)_3(x_i)}+I^n_{(P)_3(x_i)}}{2}}{2},$$

$$F^n_{(M,(N,P))_4(x_i)}=\frac{F^n_{(M)_4(x_i)}+\frac{F^n_{(N)_4(x_i)}+F^n_{(P)_4(x_i)}}{2}}{2}$$

i=1,…,j; n=1,…,p. (11)

For Definition 3.6, $(M^{\ddot{C}}\ \breve{\cup}_A\ N^{\ddot{C}})\cap'_O\ (M^{\ddot{C}}\ \breve{\cup}_A\ P^{\ddot{C}}\ )$, we get

$$\left((M,N),(M,P)\right)^{n}{}_{1}(x_i)=\left(\left(M_1^n(x_i)\cup N_1^n(x_i)\right)\cap\left(M_1^n(x_i)\cup P_1^n(x_i)\right)\right),$$

$$\left((M,N),(M,P)\right)^{n}{}_{2}(x_i)=\left(\left(M_2^n(x_i)\cup N_2^n(x_i)\right)\cap\left(M_2^n(x_i)\cup P_2^n(x_i)\right)\right),$$

$$\left((M,N),(M,P)\right)^{n}{}_{3}(x_i)=\left(\left(M_3^n(x_i)\cup N_3^n(x_i)\right)\cap\left(M_3^n(x_i)\cup P_3^n(x_i)\right)\right),$$

$$\left((M,N),(M,P)\right)^{n}{}_{4}(x_i)=\left(\left(M_4^n(x_i)\cup N_4^n(x_i)\right)\cap\left(M_4^n(x_i)\cup P_4^n(x_i)\right)\right),$$

$$T^n_{((M,N),(M,P))_2(x_i)}=\frac{\frac{T^n_{(M)_2(x_i)}+T^n_{(N)_2(x_i)}}{2}+\frac{T^n_{(M)_2(x_i)}+T^n_{(P)_2(x_i)}}{2}}{2},$$

$$I^n_{((M,N),(M,P))_3(x_i)}=\frac{\frac{I^n_{(M)_3(x_i)}+I^n_{(N)_3(x_i)}}{2}+\frac{I^n_{(M)_3(x_i)}+I^n_{(P)_3(x_i)}}{2}}{2},$$

$$F^n_{((M,N),(M,P))_4(x_i)}=\frac{\frac{F^n_{(M)_4(x_i)}+F^n_{(N)_4(x_i)}}{2}+\frac{F^n_{(M)_4(x_i)}+F^n_{(P)_4(x_i)}}{2}}{2}$$

i=1,…,j, n=1,…,p. (12)

From Definition 3.5, (11) and (12) we get

$$M^{\ddot{C}}\ \breve{\cup}_A\ (N^{\ddot{C}}\ \breve{\cap}_A\ P^{\ddot{C}}\ )=(M^{\ddot{C}}\ \breve{\cup}_A\ N^{\ddot{C}})\cap'_O\ (M^{\ddot{C}}\ \breve{\cup}_A\ P^{\ddot{C}}\ )$$

Proof of {xvii, xviii} are obtained similar to xvi.

xix) We assume that $M^{\ddot{C}}=N^{\ddot{C}}$. From Definition 3.5 and Definition 3.6, we obtain

$$T^n_{(M,N)_2(x_i)}=min\{T^n_{(M)_2(x_i)},T^n_{(N)_2(x_i)}\}=\frac{T^n_{(M)_2(x_i)}+T^n_{(N)_2(x_i)}}{2}=max\{T^n_{(M)_2(x_i)},T^n_{(N)_2(x_i)}\}$$

$$I^n_{(M,N)_3(x_i)} = max\{I^n_{(M)_3(x_i)}, I^n_{(N)_3(x_i)}\} = \frac{I^n_{(M)_3(x_i)} + I^n_{(N)_3(x_i)}}{2} = min\{I^n_{(M)_3(x_i)}, I^n_{(N)_3(x_i)}\}$$

$$F^n_{(M,N)_4(x_i)} = max\{F^n_{(M)_4(x_i)}, F^n_{(N)_4(x_i)}\} = \frac{F^n_{(M)_4(x_i)} + F^n_{(N)_4(x_i)}}{2} = min\{F^n_{(M)_4(x_i)}, F^n_{(N)_4(x_i)}\} \quad (13)$$

Also,

$$(M,N)^n{}_1(x_i) = M^n_1(x_i) \cup N^n_1(x_i),\ (M,N)^n{}_2(x_i) = M^n_2(x_i) \cup N^n_2(x_i)$$

$$(M,N)^n{}_3(x_i) = M^n_3(x_i) \cup N^n_3(x_i),\ (M,N)^n{}_4(x_i) = M^n_4(x_i) \cup N^n_4(x_i) \quad (14)$$

Thus from (13) and (14) we get

$$M^{\ddot{C}} \ \breve{\cup}_A \ N^{\ddot{C}} = M^{\ddot{C}} \ \breve{\cup}_O \ N^{\ddot{C}} = M^{\ddot{C}} \cup'_P \ N^{\ddot{C}}$$

Proof of xx are obtained similar to xix.

Example 3.8: Let

$$\ddot{C}_{n_1} = \left\{ \begin{array}{c} \{(\{M,G\},\{R,L\},\{M\},\{E,Ge\}),(\{R,L\},\{C\},\{G\}\{C,L\})(0.3,0.2,0.1,0.4), \\ (\{E,Ge\},\{M,G\},\{C,R\},\{B\})(0.5,0.5,0.3,0.4),(\{B\},\{Ge,B\},\{L,Ge\},\{M,Ge\})(0.1,0.3,0.3,0.3)\}, \\ \{(\{E\},\{E,M\},\{R,G\},\{B,M,G\}),(\{C,L\},\{Ge,G\},\{E\},\{L\}(0.2,0.4,0.3,0.1), \\ (\{M,G\},\{R\},\{C\},\{C\})(0.6,0.3,0.1,0.7),(\{R,B\},\{L,C\},\{L,Ge,M\},\{Ge,E,C\})(0.3,0.3,0.6,0.2)\}, \\ \{(\{B,Ge,E\},\{Ge,E\},\{L,C,B\},\{Ge\}),(\{L\},\{L,M\},\{M,G\},\{G,C,B\})(0.3,0.5,0.5,0.8), \\ (\{G,R\},\{G,R\},\{R\},\{E,M\})(0.5,0.3,0.2,0.1),(\{M\},\{B\},\{E\},\{R\})(0.2,0.2,0.3,0.1)\}, \\ \{(\{G,M\},\{C\},\{Ge,E,C\},\{M,C,G\}),(\{L,B\},\{Ge,C,B\},\{M,G\},\{L\})(0.3,0.7,0.5,0.2), \\ (\{R,E\},\{R\},\{R\},\{Ge,R\})(0.4,0.2,0.3,0.5),(\{Ge\},\{E\},\{B\},\{B\})(0.3,0.1,0.3,0.3)\}, \\ \{(\{G,E\},\{C\},\{E\},\{G\}),(\{Ge,M\},\{M\},\{L,G\},\{E,M\})(0.6,0.3,0.3,0.5), \\ (\{C,R\},\{B,L\},\{C,R\},\{C,R\})(0.2,0.4,0.1,0.3),(\{L,B\},\{R,E\},\{Ge,M,B\},\{L,B\})(0.3,0.3,0.6,0.2)\} \end{array} \right\}$$

and

$$\ddot{C}_{n_2} = \left\{ \begin{array}{c} \{(\{G,E\},\{R,K\},\{E\},\{M\}),(\{M,L\},\{E,Ge\},\{G\}\{G,C\})(0.6,0.2,0.1,0.3), \\ (\{M\},\{Ge,G\},\{C,R\},\{E\})(0.4,0.5,0.1,0.4),(\{B\},\{Ge,B\},\{L,Ge\},\{M,Ge\})(0.2,0.5,0.3,0.6)\}, \\ \{(\{Ge,E\},\{G,M\},\{L\},\{G,B\},(\{E,L\},\{M\},\{G,Ge,E\},\{R\}(0.5,0.3,0.4,0.1), \\ (\{C,G\},\{E,R\},\{M.Ge\},\{L\})(0.3,0.2,0.1,0.5),(\{R,E\},\{M\},\{L,C\},\{E,K,G\})(0.5,0.4,0.3,0.2)\}, \\ \{(\{G,E\},\{B,Ge,E\},\{C,B\},\{D\}),(\{E,Ge\},\{L,G,M\},\{R\},\{C,B\})(0.2,0.4,0.6,0.5), \\ (\{E,M\},\{C,Ge,G\},\{B\},\{L\})(0.2,0.4,0.3,0.5),(\{Ge,E\},\{B,G\},\{M,L\},\{R\})(0.5,0.1,0.3,0.2)\}, \\ \{(\{M\},\{C\},\{Ge,E,R\},\{B,L\}),(\{E,B,Ge\},\{C\},\{M\},\{G,L\})(0.4,0.1,0.3,0.6), \\ (\{B,R,E\},\{L\},\{Ge\},\{M\})(0.2,0.4,0.3,0.6),(\{B,Ge\},\{E,G\},\{M,L\},\{R\})(0.2,0.4,0.6,0.1)\}, \\ \{(\{G,E,Ge\},\{C\},\{R,M\},\{B\}),(\{Ge,E,M\},\{G\},\{L\},\{E,B,R\})(0.4,0.3,0.2,0.1), \\ (\{E,R\},\{B,Ge,L\},\{C,G\},\{R\})(0.3,0.1,0.2,0.4),(\{G\},\{E,B\},\{Ge,M\},\{L,R\})(0.4,0.4,0.2,0.3)\} \end{array} \right\}$$

be two MGSVNQSs such that in Example 3.4, the questions solved by a student from different textbooks for four consecutive weeks in two different months be represented by the MGSVNQSs.

i) For $\ddot{C}_{n_1}$ ve $\ddot{C}_{n_2}$, the 'Average $\cup$' operation is defined as follows;

$$\ddot{C}_{n_1} \quad \breve{\cup}_A \quad \ddot{C}_{n_2} = \left\{ \left\langle \begin{pmatrix} \{M,G,E\} = \{M,G\} \cup \{G,E\}, \\ \{R,L,C\} = \{R,L\} \cup \{R,C\}, \\ \{M,E\} = \{M\} \cup \{E\}, \\ \{E,G,M\} = \{E,G\} \cup \{M\} \end{pmatrix}, \begin{pmatrix} \{R,L,M\} = \{R,L\} \cup \{M,L\}, \\ \{C,E,Ge\} = \{C\} \cup \{E,Ge\}, \\ \{G\} = \{G\} \cup \{G\}, \\ \{C,L,G\} = \{C,L\} \cup \{G,C\} \end{pmatrix} T^1_{\left(\ddot{C}_{n_1},\ \ddot{C}_{n_2}\right)_2(x_1)}, \right.$$

$$\left. \begin{pmatrix} \{E,Ge,M\} = \{E,Ge\} \cup \{M\}, \\ \{M,G,Ge\} = \{M,G\} \cup \{Ge,G\}, \\ \{C,R\} = \{C,R\} \cup \{C,R\}, \\ \{B,E\} = \{B\} \cup \{E\} \end{pmatrix} I^1_{\left(\ddot{C}_{n_1},\ \ddot{C}_{n_2}\right)_2(x_1)}, \begin{pmatrix} \{B\} = \{B\} \cup \{B\}, \\ \{Ge,B\} = \{Ge,B\} \cup \{Ge,B\}, \\ \{L,Ge\} = \{L,Ge\} \cup \{L,Ge\}, \\ \{M,Ge\} = \{M,Ge\} \cup \{M,Ge\} \end{pmatrix} F^1_{\left(\ddot{C}_{n_1},\ \ddot{C}_{n_2}\right)_2(x_1)} \right\},$$

$$\left\langle \begin{pmatrix} \{E,Ge\} = \{E\} \cup \{Ge,E\}, \\ \{E,M,G\} = \{E,M\} \cup \{G,M\}, \\ \{R,G,L\} = \{R,G\} \cup \{L\}, \\ \{B,M,G\} = \{B,M,G\} \cup \{G,B\} \end{pmatrix}, \begin{pmatrix} \{C,L,E\} = \{C,L\} \cup \{E,L\}, \\ \{Ge,G,M\} = \{Ge,G\} \cup \{M\}, \\ \{E,G,Ge\} = \{E\} \cup \{G,Ge,E\}, \\ \{L,R\} = \{L\} \cup \{R\} \end{pmatrix} T^2_{\left(\ddot{C}_{n_1},\ \ddot{C}_{n_2}\right)_3(x_2)}\ , \right.$$

$$\left. \begin{pmatrix} \{M,G,C\} = \{M,G\} \cup \{C,G\}, \\ \{R,E\} = \{R\} \cup \{E,R\}, \\ \{C,M,Ge\} = \{C\} \cup \{M,Ge\}, \\ \{C\} = \{L\} \cup \{C,L\} \end{pmatrix} I^2_{\left(\ddot{C}_{n_1},\ \ddot{C}_{n_2}\right)_3(x_2)}, \begin{pmatrix} \{R,B,E\} = \{R,B\} \cup \{R,E\}, \\ \{L,C,M\} = \{L,C\} \cup \{M\}, \\ \{L,Ge,M,C\} = \{L,Ge,M\} \cup \{L,C\}, \\ \{Ge,E,C,G\} = \{Ge,E,C\} \cup \{E,C,G\} \end{pmatrix} F^2_{\left(\ddot{C}_{n_1},\ \ddot{C}_{n_2}\right)_3(x_2)} \right\},$$

$$\left\langle \begin{pmatrix} \{B,E,Ge,G\} = \{B,Ge,E\} \cup \{G,E\}, \\ \{Ge,E,B\} = \{Ge,E\} \cup \{B,Ge,E\}, \\ \{L,C,B\} = \{L,C,B\} \cup \{C,B\}, \\ \{Ge,R\} = \{Ge\} \cup \{R\} \end{pmatrix}, \begin{pmatrix} \{L,E,Ge\} = \{L\} \cup \{E,Ge\}, \\ \{L,M,G\} = \{L,M\} \cup \{L,G,M\}, \\ \{M,G,R\} = \{M,G\} \cup \{R\}, \\ \{G,C,B\} = \{G,C,B\} \cup \{C,B\} \end{pmatrix} T^3_{\left(\ddot{C}_{n_1},\ \ddot{C}_{n_2}\right)_4(x_3)}\ , \right.$$

$$\left. \begin{pmatrix} \{G,R,E,M\} = \{G,R\} \cup \{E,M\}, \\ \{G,R,C,Ge\} = \{G,R\} \cup \{C,Ge,G\}, \\ \{R,B\} = \{R\} \cup \{B\}, \\ \{E,M,L\} = \{E,M\} \cup \{L\} \end{pmatrix} I^3_{\left(\ddot{C}_{n_1},\ \ddot{C}_{n_2}\right)_4(x_3)}, \begin{pmatrix} \{M,Ge,E\} = \{M\} \cup \{Ge,E\}, \\ \{B,G\} = \{B\} \cup \{B,G\}, \\ \{E,M,L\} = \{E\} \cup \{M,L\}, \\ \{R\} = \{R\} \cup \{R\} \end{pmatrix} F^3_{\left(\ddot{C}_{n_1},\ \ddot{C}_{n_2}\right)_4(x_3)} \right\},$$

$$\left\langle \begin{pmatrix} \{G,M\} = \{G,M\} \cup \{M\}, \\ \{C\} = \{C\} \cup \{C\}, \\ \{Ge,E,C,R\} = \{Ge,E,C\} \cup \{Ge,E,R\}, \\ \{M,C,G,B,L\} = \{M,C,G\} \cup \{B,L\} \end{pmatrix}, \begin{pmatrix} \{L,B,E,Ge\} = \{L,B\} \cup \{E,B,Ge\}, \\ \{Ge,C,B\} = \{Ge,C,B\} \cup \{C\}, \\ \{M,G\} = \{M,G\} \cup \{M\}, \\ \{L,G\} = \{L\} \cup \{L,G\} \end{pmatrix} T^4_{\left(\ddot{C}_{n_1},\ \ddot{C}_{n_2}\right)_5(x_4)}, \right.$$

$$\left. \begin{pmatrix} \{R,E,B\} = \{R,E\} \cup \{B,R,E\}, \\ \{R,L\} = \{R\} \cup \{L\}, \\ \{R,Ge\} = \{R\} \cup \{Ge\}, \\ \{Ge,R,M\} = \{Ge,R\} \cup \{M\} \end{pmatrix} I^4_{\left(\ddot{C}_{n_1},\ \ddot{C}_{n_2}\right)_5(x_4)}, \begin{pmatrix} \{Ge,B\} = \{Ge\} \cup \{B,Ge\}, \\ \{E,G\} = \{E\} \cup \{E,G\}, \\ \{B,M,L\} = \{B\} \cup \{M,L\}, \\ \{B,R\} = \{B\} \cup \{R\} \end{pmatrix} F^4_{\left(\ddot{C}_{n_1},\ \ddot{C}_{n_2}\right)_5(x_4)} \right\},$$

$$\left\langle \begin{pmatrix} \{G,E,Ge\} = \{G,E\} \cup \{G,E,Ge\}, \\ \{C\} = \{C\} \cup \{C\}, \\ \{E,R,M\} = \{E\} \cup \{R,M\}, \\ \{G,B\} = \{G\} \cup \{B\} \end{pmatrix}, \begin{pmatrix} \{Ge,M,E\} = \{Ge,M\} \cup \{Ge,E,M\}, \\ \{M,G\} = \{M\} \cup \{G\}, \\ \{L,G\} = \{L,G\} \cup \{L\}, \\ \{E,M,B,R\} = \{E,M\} \cup \{E,B,R\} \end{pmatrix} T^5_{\left(\ddot{C}_{n_1},\ \ddot{C}_{n_2}\right)_6(x_5)}, \right.$$

$$\left. \begin{pmatrix} \{C,R,E\} = \{C,R\} \cup \{E,R\}, \\ \{B,L,Ge\} = \{B,L\} \cup \{B,Ge,L\}, \\ \{C,R,G\} = \{C,R\} \cup \{R,G\}, \\ \{C,R\} = \{C,R\} \cup \{R\} \end{pmatrix} I^5_{\left(\ddot{C}_{n_1},\ \ddot{C}_{n_2}\right)_6(x_5)}, \begin{pmatrix} \{L,B,G\} = \{L,B\} \cup \{G\}, \\ \{R,E,B\} = \{R,E\} \cup \{E,B\}, \\ \{Ge,M,B\} = \{Ge,M,B\} \cup \{Ge,M\}, \\ \{L,B,R\} = \{L,B\} \cup \{L,R\} \end{pmatrix} F^5_{\left(\ddot{C}_{n_1},\ \ddot{C}_{n_2}\right)_6(x_5)} \right\} \right\}$$

$$T^1_{(\ddot{C}_{n_1},\ddot{C}_{n_2})_2(x_1)} = \frac{T^1_{(\ddot{C}_{n_1})_2(x_1)}+T^1_{(\ddot{C}_{n_2})_2(x_1)}}{2} = \frac{(0.3+0.6)}{2}=0.45,$$

$$T^1_{(\ddot{C}_{n_1},\ddot{C}_{n_2})_2(x_2)} = \frac{T^1_{(\ddot{C}_{n_1})_2(x_2)}+T^1_{(\ddot{C}_{n_2})_2(x_2)}}{2} = \frac{(0.2+0.2)}{2}=0.2,$$

$$T^1_{(\ddot{C}_{n_1},\ddot{C}_{n_2})_2(x_3)} = \frac{T^1_{(\ddot{C}_{n_1})_2(x_3)}+T^1_{(\ddot{C}_{n_2})_2(x_3)}}{2} = \frac{(0.1+0.1)}{2}=0.1,$$

$$T^1_{(\ddot{C}_{n_1},\ddot{C}_{n_2})_2(x_4)} = \frac{T^1_{(\ddot{C}_{n_1})_2(x_4)}+T^1_{(\ddot{C}_{n_2})_2(x_4)}}{2} = \frac{(0.4+0.3)}{2}=0.35$$

$$I^1_{(\ddot{C}_{n_1},\ddot{C}_{n_2})_3(x_1)} = \frac{I^1_{(\ddot{C}_{n_1})_3(x_1)}+I^1_{(\ddot{C}_{n_2})_3(x_1)}}{2} = \frac{(0.5+0.4)}{2}=0.45,$$

$$I^1_{(\ddot{C}_{n_1},\ddot{C}_{n_2})_3(x_2)} = \frac{I^1_{(\ddot{C}_{n_1})_3(x_2)}+I^1_{(\ddot{C}_{n_2})_3(x_2)}}{2} = \frac{(0.5+0.5)}{2}=0.5,$$

$$I^1_{(\ddot{C}_{n_1},\ddot{C}_{n_2})_3(x_3)} = \frac{I^1_{(\ddot{C}_{n_1})_3(x_3)}+I^1_{(\ddot{C}_{n_2})_3(x_3)}}{2} = \frac{(0.3+0.1)}{2}=0.2,$$

$$I^1_{(\ddot{C}_{n_1},\ddot{C}_{n_2})_3(x_4)} = \frac{I^1_{(\ddot{C}_{n_1})_3(x_4)}+I^1_{(\ddot{C}_{n_2})_3(x_4)}}{2} = \frac{(0.4+0.4)}{2}=0.4$$

$$F^1_{(\ddot{C}_{n_1},\ddot{C}_{n_2})_4(x_1)} = \frac{F^1_{(\ddot{C}_{n_1})_4(x_1)}+F^1_{(\ddot{C}_{n_2})_4(x_1)}}{2} = \frac{(0.1+0.2)}{2}=0.15,$$

$$F^1_{(\ddot{C}_{n_1},\ddot{C}_{n_2})_4(x_2)} = \frac{F^1_{(\ddot{C}_{n_1})_4(x_2)}+F^1_{(\ddot{C}_{n_2})_4(x_2)}}{2} = \frac{(0.3+0.5)}{2}=0.4,$$

$$F^1_{(\ddot{C}_{n_1},\ddot{C}_{n_2})_4(x_3)} = \frac{F^1_{(\ddot{C}_{n_1})_4(x_3)}+F^1_{(\ddot{C}_{n_2})_4(x_3)}}{2} = \frac{(0.3+0.3)}{2}=0.3,$$

$$F^1_{(\ddot{C}_{n_1},\ddot{C}_{n_2})_4(x_4)} = \frac{F^1_{(\ddot{C}_{n_1})_4(x_4)}+F^1_{(\ddot{C}_{n_2})_4(x_4)}}{2} = \frac{(0.3+0.6)}{2}=0.45$$

$$T^2_{(\ddot{C}_{n_1},\ddot{C}_{n_2})_3(x_1)} = \frac{T^2_{(\ddot{C}_{n_1})_3(x_1)}+T^2_{(\ddot{C}_{n_2})_3(x_1)}}{2} = \frac{(0.2+0.5)}{2}=0.35,$$

$$T^2_{(\ddot{C}_{n_1},\ddot{C}_{n_2})_3(x_2)} = \frac{T^2_{(\ddot{C}_{n_1})_3(x_2)}+T^2_{(\ddot{C}_{n_2})_3(x_2)}}{2} = \frac{(0.4+0.3)}{2}=0.35,$$

$$T^2_{(\ddot{C}_{n_1},\ddot{C}_{n_2})_3(x_3)} = \frac{T^2_{(\ddot{C}_{n_1})_3(x_3)}+T^2_{(\ddot{C}_{n_2})_3(x_3)}}{2} = \frac{(0.3+0.4)}{2}=0.35,$$

$$T^2_{(\ddot{C}_{n_1},\ddot{C}_{n_2})_3(x_4)} = \frac{T^2_{(\ddot{C}_{n_1})_3(x_4)}+T^2_{(\ddot{C}_{n_2})_3(x_4)}}{2} = \frac{(0.1+0.1)}{2}=0.1$$

$$I^2_{(\ddot{C}_{n_1},\ddot{C}_{n_2})_3(x_1)} = \frac{I^2_{(\ddot{C}_{n_1})_3(x_1)}+I^2_{(\ddot{C}_{n_2})_3(x_1)}}{2} = \frac{(0.6+0.3)}{2}=0.45,$$

$$I^2_{(\ddot{C}_{n_1},\ddot{C}_{n_2})_3(x_2)} = \frac{I^2_{(\ddot{C}_{n_1})_3(x_2)}+I^2_{(\ddot{C}_{n_2})_3(x_2)}}{2} = \frac{(0.3+0.2)}{2}=0.25,$$

$$I^2_{(\ddot{C}_{n_1},\ddot{C}_{n_2})_3(x_3)} = \frac{I^2_{(\ddot{C}_{n_1})_3(x_3)}+I^2_{(\ddot{C}_{n_2})_3(x_3)}}{2} = \frac{(0.1+0.1)}{2}=0.1,$$

$$I^2_{(\ddot{C}_{n_1},\ddot{C}_{n_2})_3(x_4)} = \frac{I^2_{(\ddot{C}_{n_1})_3(x_4)}+I^2_{(\ddot{C}_{n_2})_3(x_4)}}{2} = \frac{(0.7+0.5)}{2}=0.6$$

$$F^2_{(\ddot{C}_{n_1},\ddot{C}_{n_2})_3(x_1)} = \frac{F^2_{(\ddot{C}_{n_1})_3(x_1)}+F^2_{(\ddot{C}_{n_2})_3(x_1)}}{2} = \frac{(0.3+0.5)}{2}=0.4,$$

$$F^2_{(\ddot{C}_{n_1},\ddot{C}_{n_2})_3(x_2)} = \frac{F^2_{(\ddot{C}_{n_1})_3(x_2)}+F^2_{(\ddot{C}_{n_2})_3(x_2)}}{2} = \frac{(0.3+0.4)}{2}=0.35,$$

$$F^2_{(\ddot{C}_{n_1},\ddot{C}_{n_2})_3(x_3)} = \frac{F^2_{(\ddot{C}_{n_1})_3(x_3)}+F^2_{(\ddot{C}_{n_2})_3(x_3)}}{2} = \frac{(0.6+0.3)}{2}=0.45,$$

$$F^2_{(\ddot{C}_{n_1},\ddot{C}_{n_2})_3(x_4)} = \frac{F^2_{(\ddot{C}_{n_1})_3(x_4)}+F^2_{(\ddot{C}_{n_2})_3(x_4)}}{2} = \frac{(0.2+0.2)}{2}=0.2$$

$$T^3_{(\ddot{C}_{n_1},\ddot{C}_{n_2})_4(x_1)} = \frac{T^3_{(\ddot{C}_{n_1})_4(x_1)}+T^3_{(\ddot{C}_{n_2})_4(x_1)}}{2} = \frac{(0.3+0.2)}{2}=0.25,$$

$$T^3_{(\ddot{C}_{n_1},\ddot{C}_{n_2})_4(x_2)} = \frac{T^3_{(\ddot{C}_{n_1})_4(x_2)}+T^3_{(\ddot{C}_{n_2})_4(x_2)}}{2} = \frac{(0.5+0.5)}{2}=0.5,$$

$$T^3_{(\ddot{C}_{n_1},\ddot{C}_{n_2})_4(x_3)} = \frac{T^3_{(\ddot{C}_{n_1})_4(x_3)}+T^3_{(\ddot{C}_{n_2})_4(x_3)}}{2} = \frac{(0.5+0.6)}{2}=0.55,$$

$$T^3_{(\ddot{C}_{n_1},\ddot{C}_{n_2})_4(x_4)} = \frac{T^3_{(\ddot{C}_{n_1})_4(x_4)}+T^3_{(\ddot{C}_{n_2})_4(x_4)}}{2} = \frac{(0.8+0.5)}{2}=0.65$$

$$I^3_{(\ddot{C}_{n_1},\ddot{C}_{n_2})_4(x_1)} = \frac{I^3_{(\ddot{C}_{n_1})_4(x_1)}+I^3_{(\ddot{C}_{n_2})_4(x_1)}}{2} = \frac{(0.5+0.2)}{2}=0.35,$$

$$I^3_{(\ddot{C}_{n_1},\ddot{C}_{n_2})_4(x_2)} = \frac{I^3_{(\ddot{C}_{n_1})_4(x_2)}+I^3_{(\ddot{C}_{n_2})_4(x_2)}}{2} = \frac{(0.3+0.4)}{2}=0.35$$

$$I^3_{(\ddot{C}_{n_1},\ddot{C}_{n_2})_4(x_3)} = \frac{I^3_{(\ddot{C}_{n_1})_4(x_3)}+I^3_{(\ddot{C}_{n_2})_4(x_3)}}{2} = \frac{(0.2+0.3)}{2}=0.25,$$

$$I^3_{(\ddot{C}_{n_1},\ddot{C}_{n_2})_4(x_4)} = \frac{I^3_{(\ddot{C}_{n_1})_4(x_4)}+I^3_{(\ddot{C}_{n_2})_4(x_4)}}{2} = \frac{(0.1+0.5)}{2}=0.3$$

$$F^3_{(\ddot{C}_{n_1},\ddot{C}_{n_2})_4(x_1)} = \frac{F^3_{(\ddot{C}_{n_1})_4(x_1)}+F^3_{(\ddot{C}_{n_2})_4(x_1)}}{2} = \frac{(0.2+0.2)}{2}=0.2,$$

$$F^3_{(\ddot{C}_{n_1},\ddot{C}_{n_2})_4(x_2)} = \frac{F^3_{(\ddot{C}_{n_1})_4(x_2)}+F^3_{(\ddot{C}_{n_2})_4(x_2)}}{2} = \frac{(0.2+0.1)}{2}=0.15,$$

$$F^3_{(\ddot{C}_{n_1},\ddot{C}_{n_2})_4(x_3)} = \frac{F^3_{(\ddot{C}_{n_1})_4(x_3)}+F^3_{(\ddot{C}_{n_2})_4(x_3)}}{2} = \frac{(0.3+0.3)}{2}=0.3,$$

$$F^3_{(\ddot{C}_{n_1},\ddot{C}_{n_2})_4(x_4)} = \frac{F^3_{(\ddot{C}_{n_1})_4(x_4)}+F^3_{(\ddot{C}_{n_2})_4(x_4)}}{2} = \frac{(0.1+0.2)}{2}=0.15$$

$$T^4_{(\ddot{C}_{n_1},\ddot{C}_{n_2})_5(x_1)} = \frac{T^4_{(\ddot{C}_{n_1})_5(x_1)}+T^5_{(\ddot{C}_{n_2})_5(x_1)}}{2} = \frac{(0.3+0.4)}{2}=0.35,$$

$$T^4_{(\ddot{C}_{n_1},\ddot{C}_{n_2})_5(x_2)} = \frac{T^4_{(\ddot{C}_{n_1})_5(x_2)}+T^4_{(\ddot{C}_{n_2})_5(x_2)}}{2} = \frac{(0.7+0.1)}{2}=0.4,$$

$$T^4_{(\ddot{C}_{n_1},\ddot{C}_{n_2})_5(x_3)} = \frac{T^4_{(\ddot{C}_{n_1})_5(x_3)}+T^4_{(\ddot{C}_{n_2})_5(x_3)}}{2} = \frac{(0.5+0.3)}{2}=0.4,$$

$$T^4_{(\ddot{C}_{n_1},\ddot{C}_{n_2})_5(x_4)} = \frac{T^4_{(\ddot{C}_{n_1})_5(x_4)}+T^4_{(\ddot{C}_{n_2})_5(x_4)}}{2} = \frac{(0.2+0.6)}{2}=0.4$$

$$I^4_{(\ddot{C}_{n_1},\ddot{C}_{n_2})_5(x_1)} = \frac{I^4_{(\ddot{C}_{n_1})_5(x_1)}+I^5_{(\ddot{C}_{n_2})_5(x_1)}}{2} = \frac{(0.4+0.2)}{2}=0.3,$$

$$I^4_{(\ddot{C}_{n_1},\ddot{C}_{n_2})_5(x_2)} = \frac{I^4_{(\ddot{C}_{n_1})_5(x_2)}+I^4_{(\ddot{C}_{n_2})_5(x_2)}}{2} = \frac{(0.2+0.4)}{2}=0.3,$$

$$I^4_{(\ddot{C}_{n_1},\ddot{C}_{n_2})_5(x_3)} = \frac{I^4_{(\ddot{C}_{n_1})_5(x_3)}+I^4_{(\ddot{C}_{n_2})_5(x_3)}}{2} = \frac{(0.3+0.3)}{2}=0.3,$$

$$I^4_{(\ddot{C}_{n_1},\ddot{C}_{n_2})_5(x_4)} = \frac{I^4_{(\ddot{C}_{n_1})_5(x_4)}+I^4_{(\ddot{C}_{n_2})_5(x_4)}}{2} = \frac{(0.5+0.6)}{2}=0.55$$

$$F^4_{(\ddot{C}_{n_1},\ddot{C}_{n_2})_5(x_1)} = \frac{F^4_{(\ddot{C}_{n_1})_5(x_1)}+F^5_{(\ddot{C}_{n_2})_5(x_1)}}{2} = \frac{(0.3+0.2)}{2}=0.25,$$

$$F^4_{(\ddot{C}_{n_1},\ddot{C}_{n_2})_5(x_2)} = \frac{F^4_{(\ddot{C}_{n_1})_5(x_2)}+F^4_{(\ddot{C}_{n_2})_5(x_2)}}{2} = \frac{(0.1+0.4)}{2}=0.25$$

$$F^4_{(\ddot{C}_{n_1},\ddot{C}_{n_2})_5(x_3)} = \frac{F^4_{(\ddot{C}_{n_1})_5(x_3)}+F^4_{(\ddot{C}_{n_2})_5(x_3)}}{2} = \frac{(0.3+0.6)}{2}=0.45,$$

$$F^4_{(\ddot{C}_{n_1},\ddot{C}_{n_2})_5(x_4)} = \frac{F^4_{(\ddot{C}_{n_1})_5(x_4)}+F^4_{(\ddot{C}_{n_2})_5(x_4)}}{2} = \frac{(0.3+0.1)}{2}=0.2$$

$$T^5_{(\ddot{C}_{n_1},\ddot{C}_{n_2})_6(x_1)} = \frac{T^5_{(\ddot{C}_{n_1})_6(x_1)}+T^5_{(\ddot{C}_{n_2})_6(x_1)}}{2} = \frac{(0.6+0.4)}{2}=0.5,$$

$$T^5_{(\ddot{C}_{n_1},\ddot{C}_{n_2})_6(x_2)} = \frac{T^5_{(\ddot{C}_{n_1})_6(x_2)}+T^5_{(\ddot{C}_{n_2})_6(x_2)}}{2} = \frac{(0.3+0.3)}{2}=0.3,$$

$$T^5_{(\ddot{C}_{n_1},\ddot{C}_{n_2})_6(x_3)} = \frac{T^5_{(\ddot{C}_{n_1})_6(x_3)}+T^5_{(\ddot{C}_{n_2})_6(x_3)}}{2} = \frac{(0.3+0.2)}{2}=0.25,$$

$$T^5_{(\ddot{C}_{n_1},\ddot{C}_{n_2})_6(x_4)} = \frac{T^5_{(\ddot{C}_{n_1})_6(x_4)}+T^5_{(\ddot{C}_{n_2})_6(x_4)}}{2} = \frac{(0.5+0.1)}{2}=0.3$$

$$I^5_{(\ddot{C}_{n_1},\ddot{C}_{n_2})_6(x_1)} = \frac{I^5_{(\ddot{C}_{n_1})_6(x_1)}+I^5_{(\ddot{C}_{n_2})_6(x_1)}}{2} = \frac{(0.2+0.3)}{2}=0.25,$$

$$I^5_{(\ddot{C}_{n_1},\ddot{C}_{n_2})_6(x_2)} = \frac{I^5_{(\ddot{C}_{n_1})_6(x_2)}+I^5_{(\ddot{C}_{n_2})_6(x_2)}}{2} = \frac{(0.4+0.1)}{2}=0.25,$$

$$I^5_{(\ddot{C}_{n_1},\ddot{C}_{n_2})_6(x_3)} = \frac{I^5_{(\ddot{C}_{n_1})_6(x_3)}+I^5_{(\ddot{C}_{n_2})_6(x_3)}}{2} = \frac{(0.1+0.2)}{2}=0.15,$$

$$I^5_{(\ddot{C}_{n_1},\ddot{C}_{n_2})_6(x_4)} = \frac{I^5_{(\ddot{C}_{n_1})_6(x_4)}+I^5_{(\ddot{C}_{n_2})_6(x_4)}}{2} = \frac{(0.3+0.4)}{2}=0.35$$

$$F^5_{(\ddot{C}_{n_1},\ddot{C}_{n_2})_6(x_1)} = \frac{F^5_{(\ddot{C}_{n_1})_6(x_1)}+F^5_{(\ddot{C}_{n_2})_6(x_1)}}{2} = \frac{(0.3+0.4)}{2}=0.35,$$

$F^5_{(\ \ddot{C}_{n_1},\ \ddot{C}_{n_2})_6(x_2)} = \frac{F^5_{(\ \ddot{C}_{n_1})_6(x_2)} + F^5_{(\ \ddot{C}_{n_2})_6(x_2)}}{2} = \frac{(0.3+0.4)}{2}$=0.35,

$F^5_{(\ \ddot{C}_{n_1},\ \ddot{C}_{n_2})_6(x_3)} = \frac{F^5_{(\ \ddot{C}_{n_1})_6(x_3)} + F^5_{(\ \ddot{C}_{n_2})_6(x_3)}}{2} = \frac{(0.6+0.2)}{2}$=0.4,

$F^5_{(\ \ddot{C}_{n_1},\ \ddot{C}_{n_2})_6(x_4)} = \frac{F^5_{(\ \ddot{C}_{n_1})_6(x_4)} + F^5_{(\ \ddot{C}_{n_2})_6(x_4)}}{2} = \frac{(0.2+0.3)}{2}$=0.25

Thus we obtain the set $\ddot{C}_{n_1} \ \cup_A \ \ddot{C}_{n_2}$ as

$\ddot{C}_{n_1} \cup_A \ddot{C}_{n_2} = \{\{(\{M,G,E\},\{R,L,C\},\{M,E\},\{E,G,M\}),$

$(\{R,L,M\},\{K,E,Ge\},\{G\}\{C,L,G\})(0.45,0.2,0.1,0.35),$

$(\{E,Ge,M\},\{M,G,Ge\},\{C,R\},\{B,E\})(0.45,0.5,0.2,0.4),$

$(\{B\},\{Ge,B\},\{L,Ge\},\{M,Ge\})(0.15,0.4,0.3,0.45)\},$

{({E,Ge},{E,M,G},{R,G,L},{B,M,G}),

$(\{C,L,E\},\{Ge,G,M\},\{E,G,Ge\},\{L,R\})(0.35,0.35,0.35,0.1),$

$(\{M,G,C\},\{R,E\},\{C,M,Ge\},\{C,L\})(0.45,0.25,0.1,0.6),$

$(\{R,B,E\},\{L,C,M\},\{L,Ge,M,C\},\{Ge,E,C,G\})(0.4,0.35,0.45,0.2)\},$

{({B,Ge,E,G},{Ge,E,B},{L,C,B},{Ge,R}),

$(\{L,E,Ge\},\{L,M,G\},\{M,G,R\}\{G,C,B\})(0.25,0.5,0.55,0.65),$

$(\{G,R,E,M\},\{G,R,C,Ge,G\},\{R,B\},\{E,M,L\})(0.35,0.35,0.35,0.3),$

$(\{M,Ge,E\},\{B,G\},\{E,M,L\},\{R\})(0.2,0.15,0.3,0.15),$

{({G,M},{C,M},{Ge,E,C,R},{M,C,G,B,L}),

$(\{L,B,E,Ge\},\{Ge,C,B\},\{M,G\}\{L,G\})(0.35,0.4,0.4,0.4),$

$(\{R,E,B\},\{R,E\},\{R,Ge\},\{Ge,R,M\})(0.3,0.3,0.3,0.55),$

$(\{Ge,B\},\{E,G\},\{B,M,L\},\{B,R\})(0.25,0.25,0.45,0.2)\},$

{({G,E,Ge},{C},{E,R,M},{G,B}),

$(\{Ge,M,E\},\{M,G\},\{L,G\}\{E,M,B,R\})(0.5,0.3,0.25,0.3),$

$(\{C,R,E\},\{B,L,Ge\},\{C,R,G\},\{C,R\})(0.25,0.25,0.15,0.35),$

$(\{L,B,G\},\{R,E,B\},\{Ge,M,B\},\{L,B,R\})(0.35,0.35,0.4,0.25)\}\}.$

ii) The 'Average $\cap$' operator for $\ddot{C}_{n_1}\ ve\ \ddot{C}_{n_2}$ is obtained from Definition 3.6 and similar to i such that

$\ddot{C}_{n_1} \ddot{\cap}_A \ddot{C}_{n_2} = \{\{(\{G\},\{R\},\{\emptyset\},\{\emptyset\}),(\{L\},\emptyset,\{G\}\{C\})(0.45,0.2,0.1,0.35),$

$(\emptyset,\{G\},\{C,R\},\emptyset)(0.45,0.5,0.2,0.4),(\{B\},\{Ge,B\},\{L,Ge\},\{M,Ge\})(0.15,0.4,0.3,0.45)\},$

$\{(\{\text{E}\},\{\text{M}\},\emptyset,\{\text{B, G}\}),(\{L\},\emptyset,\{E\},\emptyset)(0.35,0.35,0.35,0.1),(\{G\},\emptyset,\{C\},\emptyset)(0.45,0.25,0.1,0.6),$

$(\{R\},\emptyset,\{L\},\{E,C\})(0.4,0.35,0.45,0.2)\},$

$\{(\{E\},\{Ge,E\},\{C,B\},\emptyset),(\emptyset,\{L,M\},\{\emptyset\}\{C,B\})(0.25,0.5,0.55,0.65),$

$(\emptyset,\{G\},\emptyset,\emptyset)(0.35,0.35,0.35,0.3),(\emptyset,\{B\},\emptyset,\{R\})(0.2,0.15,0.3,0.15)\},$

$\{(\{\text{M}\},\{\text{C}\},\{Ge,\text{E,C}\},\emptyset),(\{B\},\{C\},\{M\},\{L\})\ \ (0.35,0.4,0.4,0.4),$

$(\{R,E\},\emptyset,\emptyset,\emptyset)(0.3,0.3,0.3,0.55),(\{Ge\},\{E\},\emptyset,\emptyset)(0.25,0.25,0.45,0.2)\},$

$\{(\{\text{G,E}\},\{\text{C}\},\emptyset,\emptyset),(\{Ge,M\},\emptyset,\{L,G\}\{E\})(0.5,0.3,0.25,0.3),$

$(\{R\},\{B\},\{C\},\{R\})(0.25,0.25,0.15,0.35),(\emptyset,\{E\},\{Ge,M\},\{L\})(0.35,0.35,0.4,0.25)\}.$

iii) For $\ddot{C}_{n_1}\ ve\ \ \ddot{C}_{n_2}$, the operation ''optimistic $\cup$'' is obtained from Definition 3.6 and similar to i such that

$\ddot{C}_{n_1} \ddot{\cup}_O \ddot{C}_{n_2} = \{(\{M,G,E\},\{R,L,C\},\{M,E\},\{M,E,G\}),$

$(\{R,L,M\},\{C,E,Ge\},\{G\},\{C,L,G\})(0.6,0.2,0.1,0.4),$

$(\{E,Ge,M\},\{M,G,Ge\},\{C,R\},\{B,E\})(0.4,0.5,0.1,0.4),$

$(\{B\},\{Ge,B\},\{L,Ge\},\{M,Ge\})(0.1,0.3,0.3,0.3)\},$

$\{(\{E,Ge\},\{E,M,G\},\{R,G,L\},\{B,M,G\}),$

$(\{C,L,E\},\{Ge,G,M\},\{E,G,Ge\}\{L,R\})(0.5,0.4,0.4,0.1),$

$(\{M,G,C\},\{R,E\},\{C,M,Ge\},\{C,L\})(0.3,0.2,0.1,0.5),$

$(\{R,B,E\},\{L,C,M\},\{L,Ge,M,C\},\{Ge,E,C,G\})(0.3,0.3,0.3,0.2)\},$

$\{(\{B,Ge,E,G\},\{Ge,E,B\},\{L,C,B\},\{Ge,R\}),$

$(\{L,E,Ge\},\{L,M,G\},\{M,G,R\},\{G,C,B\})(0.3,0.5,0.6,0.8),$

$(\{G,R,E,M\},\{G,R,C,Ge\},\{R,B\},\{E,M,L\})(0.2,0.3,0.2,0.1),$

$(\{M,Ge,E\},\{B,G\},\{E,M,L\},\{R\})(0.2,0.1,0.3,0.1)\},$

$\{(\{G,M\},\{C,M\},\{Ge,E,C,R\},\{M,C,G,B,L\}),$

$(\{L,B,E,Ge\},\{Ge,C,B\},\{M,G\},\{L,G\})(0.4,0.7,0.5,0.6),$

$(\{R,E,B\},\{R,L\},\{R,Ge\},\{Ge,R,M\})(0.2,0.2,0.3,0.5),$

$(\{Ge, B\}, \{E, G\}, \{B, M, L\}, \{B, R\})(0.2,0.1,0.3,0.1)$,

$\{(\{G, E, Ge\}, \{C\}, \{E, R, M\}, \{G, B\})$,

$(\{Ge, M, E\}, \{M, G\}, \{L, G\}\{E, M, B, R\})(0.6,0.3,0.3,0.5)$,

$(\{C, R, E\}, \{B, L, Ge\}, \{C, R, G\}, \{C, R\})(0.2,0.1,0.1,0.3)$,

$(\{L, B, G\}, \{R, E, B\}, \{Ge, M, B\}, \{L, B, R\})(0.3,0.3,0.2,0.2)\}\}$.

iv) The 'Optimistic $\cap$' operation for $\ddot{C}_{n_1}$ ve $\ddot{C}_{n_2}$ is obtained from Definition 3.6 and similar to i such that

$\ddot{C}_{n_1} \barwedge_O \ddot{C}_{n_2} = \{\{(\{G\}, \{R\}, \emptyset, \emptyset), (\{L\}, \emptyset, \{G\}, \{C\})(0.3,0.2,0.1,0.3)$,

$(\emptyset, \{G\}, \{C, R\}, \emptyset)(0.5,0.5,0.3,0.4)$,

$(\{B\}, \{Ge, B\}, \{L, Ge\}, \{M, Ge\})(0.2,0.5,0.3,0.6)\}$,

$\{(\{E\},\{M\},\emptyset,\{B, G\}),(\{L\}, \emptyset, \{E\}, \emptyset)(0.2,0.3,0.3,0.1)$,

$(\{G\}, \emptyset, \{C\}, \emptyset)(0.6,0.3,0.1,0.7)$,

$(\{R\}, \emptyset, \{L\}, \{E, C\})(0.5,0.4,0.6,0.2)\}$,

$\{(\{E\},\{Ge,E\},\{C,B\},\emptyset),(\emptyset, \{L, M\}, \emptyset\{C, B\})(0.2,0.4,0.5,0.5),(\emptyset, \{G\}, \emptyset, \emptyset)(0.5,0.4,0.3,0.5)$,

$(\emptyset, \{B\}, \emptyset, \{R\})(0.5,0.2,0.3,0.2)\}$,

$((\{M\},\{C\},\{Ge,E,C\},\emptyset),(\{B\}, \{C\}, \{M\}, \{L\})(0.3,0.1,0.3,0.2),(\{R, E\}, \emptyset, \emptyset, \emptyset)(0.4,0.4,0.3,0.6)$,

$(\{Ge\}, \{E\}, \emptyset, \emptyset)$ $(0.3,0.4,0.6,0.3)\}$,

$\{(\{G,E\},\{C\},\emptyset,\emptyset),(\{Ge, M\}, \emptyset, \{L, G\}, \{E\})(0.4,0.3,0.2,0.1)$,

$(\{R\}, \{B\}, \{C\}, \{R\})(0.3,0.4,0.2,0.4), (\emptyset, \{E\}, \{Ge, M\}, \{L\})(0.4,0.4,0.6,0.3)\}\}$.

v) The 'Pessimistic $\cup$' operation for $\ddot{C}_{n_1}$ and $\ddot{C}_{n_2}$ is obtained from Definition 3.6 and similar to i such that

$\ddot{C}_{n_1} \barvee_P \ddot{C}_{n_2} = \{(\{M, G, E\}, \{R, L, C\}, \{M, E\}, \{M, E, G\})$,

$(\{R, L, M\}, \{C, E, Ge\}, \{G\}, \{C, L, G\})(0.3,0.2,0.1,0.3)$,

$(\{E, Ge, M\}, \{M, G, Ge\}, \{C, R\}, \{B, E\})(0.5,0.5,0.3,0.4)$,

$(\{B\}, \{Ge, B\}, \{L, Ge\}, \{M, Ge\})(0.2,0.5,0.3,0.6))\}$,

$\{(\{E, Ge\}, \{E, M, G\}, \{R, G, L\}, \{B, M, G\})$,

$(\{C, L, E\}, \{Ge, G, M\}, \{E, G, Ge\}\{L, R\})(0.2,0.3,0.3,0.1)$,

$(\{M, G, C\}, \{R, E\}, \{C, M, Ge\}, \{C, L\})(0.6,0.3,0.1,0.7)$,

$(\{R,B,E\},\{L,C,M\},\{L,Ge,M,C\},\{Ge,E,C,G\})(0.5,0.4,0.6,0.2)\},$

$\{(\{B,Ge,E,G\},\{Ge,E,B\},\{L,C,B\},\{Ge,R\}),$

$(\{L,E,Ge\},\{L,M,G\},\{M,G,R\}\{G,C,B\})(0.2,0.4,0.5,0.5),$

$(\{G,R,E,M\},\{G,R,C,Ge,G\},\{R,B\},\{E,M,L\})(0.5,0.4,0.3,0.5),$

$(\{M,Ge,E\},\{B,G\},\{E,M,L\},\{R\})(0.5,0.2,0.3,0.2)\},$

$\{(\{G,M\},\{C,M\},\{Ge,E,C,R\},\{M,C,G,B,L\}),$

$(\{L,B,E,Ge\},\{Ge,C,B\},\{M,G\}\{L,G\})(0.3,0.1,0.3,0.2),$

$(\{R,E,B\},\{R,L\},\{R,Ge\},\{Ge,R,M\})(0.4,0.4,0.3,0.6),$

$(\{Ge,B\},\{E,G\},\{B,M,L\},\{B,R\})(0.3,0.4,0.6,0.3),$

$\{(\{G,E,Ge\},\{C\},\{E,R,M\},\{G,B\}),$

$(\{Ge,M,E\},\{M,G\},\{L,G\}\{E,M,B,R\})(0.4,0.3,0.2,0.1),$

$(\{K,R,E\},\{B,L,Ge\},\{C,R,G\},\{C,R\})(0.3,0.4,0.2,0.4),$

$(\{L,B,G\},\{R,E,B\},\{Ge,M,B\},\{L,B,R\})(0.4,0.4,0.6,0.3)\}.$

vi) The 'Pessimistic ∩' operation for $\ddot{C}_{n_1}\ ve\ \ddot{C}_{n_2}$ is obtained from Definition 3.6 and similar to i such that

$\ddot{C}_{n_1} \breve{\cap}_P \ddot{C}_{n_2} = \{\{(\{G\},\{R\},\emptyset,\emptyset),(\{L\},\emptyset,\{G\},\{C\})(0.3,0.2,0.1,0.3),(\emptyset,\{G\},\{C,R\},\emptyset)(0.5,0.5,0.3,0.4),$

$(\{B\},\{Ge,B\},\{L,Ge\},\{M,Ge\})(0.2,0.5,0.3,0.6)\},$

$\{(\{E\},\{M\},\emptyset,\{B,G\}),(\{L\},\emptyset,\{E\},\emptyset)(0.2,0.3,0.3,0.1),(\{G\},\emptyset,\{C\},\emptyset)(0.6,0.3,0.1,0.7),$

$(\{R\},\emptyset,\{L\},\{E,C\})(0.5,0.4,0.6,0.2)\},$

$\{(\{E\},\{Ge,E\},\{C,B\},\emptyset),(\emptyset,\{E,M\},\emptyset\{C,B\})(0.2,0.4,0.5,0.5),(\emptyset,\{G\},\emptyset,\emptyset)(0.5,0.4,0.3,0.5),$

$(\emptyset,\{B\},\emptyset,\{R\})\ \ (0.5,0.2,0.3,0.2)\},$

$\{(\{M\},\{C\},\{Ge,E,C\},\emptyset),(\{B\},\{C\},\{M\},\{L\})(0.3,0.1,0.3,0.2),$

$(\{R,E\},\emptyset,\emptyset,\emptyset)(0.4,0.4,0.3,0.6),(\{Ge\},\{E\},\emptyset,\emptyset)(0.3,0.4,0.6,0.3)\},$

$\{(\{G,E\},\{C\},\emptyset,\emptyset,),(\{Ge,M\},\emptyset,\{L,G\},\{E\})(0.4,0.3,0.2,0.1),$

$(\{R\},\{B\},\{C\},\{R\})(0.3,0.4,0.2,0.4),(\emptyset,\{E\},\{Ge,M\},\{L\})(0.4,0.4,0.6,0.3)\}.$

Abbreviations

NS: Neutrosophic Set

NQN: Neutrosophic Quadruple Number

NQS: Neutrosophic Quadruple Set

MVNS: Multi-Valued Neutrosophic Set

SVNQN: Set Valued Neutrosophic Quadruple Number

SVNQS: Set Valued Neutrosophic Quadruple Set

GSVNQN: Generalized Set Valued Neutrosophic Quintuple Number

GSVNQS: Generalized Set Valued Neutrosophic Quintuple Set

MGSVNQN: Multiple Generalized Set Valued Neutrosophic Quintuple Number

MGSVNQS: Multiple Generalized Set Valued Neutrosophic Quintuple Set

5. Conclusions

In this study, MGSVNQS and MGSVNQN are defined and their basic properties are given. For this reason, we have obtained a new structure for both MVNS and GSVNQSs. In addition, the operators of average union, average intersection, optimistic union, optimistic intersection, pessimistic union and pessimistic intersection are defined for this new structure. Thus, MGSVNQS can be used in decision-making applications by researchers. Also, researchers can be defined similarity and distance measure for MGSVNQSs.

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Received: March 23, 2024. Accepted: July 29, 2024

Neutrosophic Sets and Systems, Vol. 70, 2024
University of New Mexico

MAGDM Technique Based on Linguistic Neutrosophic Tangent Dombi Aggregation Operators and Its Application in Selecting New Energy Design Schemes

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Abstract: In linguistic decision theory, multi-attribute group decision-making (MAGDM) of linguistic neutrosophic numbers (LNNs) is one of the crucial research topics. Existing LNN aggregation algorithms do not consider trigonometric periodicity. In this case, they cannot perform MAGDM problems including periodic/multitemporal applications in LNN scenarios. To fill this gap, this article aims to present a MAGDM technique using the LNN tangent Dombi aggregation operators (TDAOs) for addressing MAGDM issues including multitemporal/periodic applications in LNN scenarios. First, we present the linguistic tangent Dombi t-norm and t-conorm and the LNN tangent Dombi operation laws. Second, we propose the LNN tangent Dombi weighted average and geometric operators and investigate their properties. Third, a MAGDM technique is built based on the two presented operators to tackle MAGDM problems in a LNN scenario. Lastly, the built MAGDM technique is applied to a choice of new energy design schemes in Ningbo City, China, and then its efficiency and appropriateness are verified by comparing with the existing MAGDM technique in the LNN scenario.

Keywords: linguistic neutrosophic number; linguistic tangent Dombi operation; linguistic neutrosophic number tangent Dombi aggregation operator; group decision making; new energy design scheme

1. Introduction

In multi-attribute (group) decision-making (MADM/MAGDM) problems, neutrosophic MADM methods have obtained wild applications [1-3]. Then, linguistic information can be adapted to human cognition and expression forms. Therefore, decision information is given in the form of linguistic or uncertain linguistic variables in MADM/MAGDM problems such as supply chain management, medical diagnosis, and risk investment. In the evaluation process, decision makers often provide qualitative information (e.g., bad, good, or other natural language forms) directly when evaluating decision objects such as the overall quality of people, performance of equipment, engineering projects, business partners. Linguistic decision theory and methods are not only of great academic value, but also have broad application prospects. Under linguistic scenarios, many scholars have proposed various MADM/MAGDM methods to tackle linguistic MADM/MAGDM issues [4–9]. In uncertain linguistic scenarios, some researchers also presented uncertain linguistic MADM/MAGDM models to carry out uncertain linguistic MADM/MAGDM applications [10–15]. Based on true and

false linguistic variables, some researchers [16–20] developed linguistic intuitionistic fuzzy numbers (LIFNs) and some LIFN aggregation operators to perform MADM/MAGDM issues in LIFN scenarios. Concerning true, indeterminate, and false linguistic variables, many scholars have introduced the linguistic neutrosophic number (LNN) aggregation operators (AOs) [21], the LNN Bonferroni mean AOs [22], LNN Hamy mean AOs [23], the LNN Hamacher AOs [24], the LNN power Heronian AOs [25], the LNN Einstein AO [26], the LNN Maclaurin symmetric mean AOs [27], the LNN Muirhead mean AO [28], and the LNN Dombi AOs [29], and then developed their MADM/MAGDM approaches in LNN scenarios. However, all these AOs and MADM/MAGDM approaches lack trigonometric AOs and periodic MADM/MAGDM capability in different linguistic scenarios.

Regarding the trigonometric AOs and periodic MADM issues of single-valued neutrosophic numbers (SVNNs), some researchers [30–32] have proposed the trigonometric AOs and trigonometric Dombi AOs of SVNNs and their MADM techniques to handle periodic/multitemporal MADM/MAGDM issues. However, all these existing SVNN trigonometric AOs and MADM/MAGDM techniques cannot deal with MADM/MAGDM issues under a linguistic neutrosophic scenario.

As mentioned above, neither the linguistic trigonometric operations and AOs nor their MADM/MAGDM methods/applications are present in the existing literature, which shows the obvious research gap. Since LNN contains a more universal linguistic framework including true, false and indeterminate linguistic variables in uncertainty, the existing linguistic value and LIFN can be viewed as special cases of LNN. Furthermore, Dombi operations show the merit of the flexible operations due to an adjusting parameter. Motivated by the trigonometric Dombi AOs of SVNNs [32], this study aims to develop a novel MAGDM technique using the LNN tangent Dombi AOs and apply it to a choice issue of new energy design schemes (NEDSs) to fill the research gap.

In this study, we first propose the linguistic tangent Dombi t-norm and t-conorm and the LNN tangent Dombi operation laws (LNNTDOLs). Next, we propose the LNN tangent Dombi weighted average and geometric (LNNTDWA and LNNTDWG) operators in terms of the LNNTDOLs. Based on the proposed LNNTDWA and LNNTDWG operators, a MAGDM technique is developed in a LNN scenario. Finally, the developed MAGDM technique is applied to a choice case of NEDSs. By comparing with the existing MAGDM technique, the efficiency and appropriateness of the developed MAGDM technique is investigated in the LNN scenario.

Generally, this original study mainly provides the following novel contributions:

- The novel linguistic tangent Dombi t-norm and t-conorm and LNNTDOLs are proposed to obtain the multitemporal/periodic operations in the LNN scenario.
- The LNNTDWA and LNNTDWG operators are presented to yield the periodic aggregation tools for MAGDM modeling in the LNN scenario.
- The MAGDM technique using the proposed LNNTDWA and LNNTDWG operators is built based on the multitemporal/periodic requirements to effectively solve the choice issue of NEDSs and in LNN scenarios.

The rest of the article is constructed as follows. Section 2 reviews some basic knowledge of LNNs. Section 3 presents the linguistic tangent Dombi t-norm and t-conorm and LNNTDOLs. Section 4 proposes the LNNTDWA and LNNTDWG operators based on LNNTDOLs. In Section 5, a MAGDM technique is built by the proposed LNNTDWA and LNNTDWG operators in a LNN scenario. In Section 6, the built MAGDM technique is applied in a choice case of NEDSs, and then its appropriateness and validity are examined by comparing it with the existing relative MAGDM technique in the LNN scenario. In Section 7 some conclusions and future work are mentioned.

2. Some Basic Knowledge of LNNs

Set $\Psi_\beta = \{\beta_1, \beta_2, \ldots, \beta_g\}$ as a set of linguistic terms (SLT) with odd cardinality $g+1$. Then LNN is defined as $\beta_L = <\beta_{Lt}, \beta_{Lu}, \beta_{Lf}>$ [18] over Ψ_β for $\beta_{Lt}, \beta_{Lu}, \beta_{Lf} \in \Psi_\beta$, where β_{Lt}, β_{Lu}, and β_{Lf} are the true, indeterminate, and false linguistic variables, respectively.

If two LNNs are $\beta_{L(1)} = <\beta_{Lt(1)}, \beta_{Lu(1)}, \beta_{Lf(1)}>$ and $\beta_{L(2)} = <\beta_{Lt(2)}, \beta_{Lu(2)}, \beta_{Lf(2)}>$ over Ψ_β and $\eta > 0$, then their Dombi operation laws [29] are introduced below:

$$(1)\ \beta_{L(1)} \oplus_D \beta_{L(2)} = \left\langle \beta_{g\left\{1-\frac{1}{1+\left\{\left(\frac{Lt(1)/g}{1-Lt(1)/g}\right)^{\varepsilon}+\left(\frac{Lt(2)/g}{1-Lt(2)/g}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}}, \beta_{g\left\{\frac{1}{1+\left\{\left(\frac{1-Lu(1)/g}{Lu(1)/g}\right)^{\varepsilon}+\left(\frac{1-Lu(2)/g}{Lu(2)/g}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}}, \beta_{g\left\{\frac{1}{1+\left\{\left(\frac{1-Lf(1)/g}{Lf(1)/g}\right)^{\varepsilon}+\left(\frac{1-Lf(2)/g}{Lf(2)/g}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}} \right\rangle;$$

$$(2)\ \beta_{L(1)} \otimes_D \beta_{L(2)} = \left\langle \beta_{g\left\{\frac{1}{1+\left\{\left(\frac{1-Lt(1)/g}{Lt(1)/g}\right)^{\varepsilon}+\left(\frac{1-Lt(2)/4g}{Lt(2)/g}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}}, \beta_{g\left\{1-\frac{1}{1+\left\{\left(\frac{Lu(1)/g}{1-Lu(1)/g}\right)^{\varepsilon}+\left(\frac{Lu(2)/g}{1-Lu(2)/g}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}}, \beta_{g\left\{1-\frac{1}{1+\left\{\left(\frac{Lf(1)/g}{1-Lf(1)/g}\right)^{\varepsilon}+\left(\frac{Lf(2)/g}{1-Lf(2)/g}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}} \right\rangle;$$

$$(3)\ \eta\beta_{L(1)} = \left\langle \beta_{g\left\{1-\frac{1}{1+\left\{\eta\left(\frac{Lt(1)/g}{1-Lt(1)/g}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}}, \beta_{g\left\{\frac{1}{1+\left\{\eta\left(\frac{1-Lu(1)/g}{Lu(1)/g}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}}, \beta_{g\left\{\frac{1}{1+\left\{\eta\left(\frac{1-Lf(1)/g}{Lf(1)/g}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}} \right\rangle;$$

$$(4)\ \beta_{L(1)}^{\eta} = \left\langle \beta_{g\left\{\frac{1}{1+\left\{\eta\left(\frac{1-Lt(1)/g}{Lt(1)/g}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}}, \beta_{g\left\{1-\frac{1}{1+\left\{\eta\left(\frac{Lu(1)/g}{1-Lu(1)/g}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}}, \beta_{g\left\{1-\frac{1}{1+\left\{\eta\left(\frac{Lf(1)/g}{1-Lf(1)/g}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}} \right\rangle.$$

If $\beta_{L(j)} = <\beta_{Lt(j)}, \beta_{Lu(j)}, \beta_{Lf(j)}>$ ($j = 1, 2, \ldots, e$) are e LNNs with their weights η_j and $\sum_{j=1}^{e}\eta_j = 1$, then the LNN Dombi weighted average and geometric (LNNDWA and LNNDWG) operators [29] are introduced below:

$$LNNDWA(\beta_{L(1)}, \beta_{L(2)}, \ldots, \beta_{L(e)}) = \bigoplus_{j=1}^{e}{}_D\, \eta_j\beta_{L(j)} = \left\langle \beta_{g\left\{1-\frac{1}{1+\left\{\sum_{j=1}^{e}\eta_j\left(\frac{Lt(j)/g}{1-Lt(j)/g}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}}, \beta_{g\left\{\frac{1}{1+\left\{\sum_{j=1}^{e}\eta_j\left(\frac{1-Lu(j)/g}{Lu(j)/g}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}}, \beta_{g\left\{\frac{1}{1+\left\{\sum_{j=1}^{e}\eta_j\left(\frac{1-Lf(j)/g}{Lf(j)/g}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}} \right\rangle. \quad (1)$$

$$LNNDWG(\beta_{L(1)}, \beta_{L(2)}, \ldots, \beta_{L(e)}) = \bigotimes_{j=1}^{e}{}_D\, \beta_{L(j)}^{\eta_j} = \left\langle \beta_{g\left\{\frac{1}{1+\left\{\sum_{j=1}^{e}\eta_j\left(\frac{1-Lt(j)/g}{Lt(j)/g}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}}, \beta_{g\left\{1-\frac{1}{1+\left\{\sum_{j=1}^{e}\eta_j\left(\frac{Lu(j)/g}{1-Lu(j)/g}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}}, \beta_{g\left\{1-\frac{1}{1+\left\{\sum_{j=1}^{e}\eta_j\left(\frac{Lf(j)/g}{1-Lf(j)/g}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}} \right\rangle. \quad (2)$$

Then, the score and accuracy equations of $\beta_{L(j)} = <\beta_{Lt(j)}, \beta_{Lu(j)}, \beta_{Lf(j)}>$ are introduced below [21]:

$$D(\beta_{L(j)}) = (2g + Lt(j) - Lu(j) - Lf(j))/(3g) \text{ for } D(\beta_{L(j)}) \in [0, 1]; \quad (3)$$

$$E(\beta_{L(j)}) = (Lt(j) - Lf(j))/g \text{ for } E(\beta_{L(j)}) \in [-1, 1]. \quad (4)$$

Comparative laws of two LNNs $\beta_{L(1)}$ and $\beta_{L(2)}$ are introduced below [21]:

(i) $\beta_{L(1)} > \beta_{L(2)}$ if $D(\beta_{L(1)}) > D(\beta_{L(2)})$;

(ii) $\beta_{L(1)} > \beta_{L(2)}$ if $D(\beta_{L(1)}) = D(\beta_{L(2)})$ and $E(\beta_{L(1)}) > E(\beta_{L(2)})$;

(iii) $\beta_{L(1)} = \beta_{L(2)}$ if $D(\beta_{L(1)}) = D(\beta_{L(2)})$ and $E(\beta_{L(1)}) = E(\beta_{L(2)})$.

3. Linguistic Tangent Dombi Operations and LNNTDOLs

This part proposes the linguistic tangent Dombi operations (LTDOs) and LNNTDOLs.

First, we propose LTDOs below.

Definition 1. Let $\beta_{Lt(1)}$ and $\beta_{Lt(2)}$ be two linguistic values on the SLT $\Psi_\beta = \{\beta_0, \beta_1, \ldots, \beta_g\}$ and $\varepsilon \geq 1$. Then the linguistic tangent Dombi t-norm and t-conorm of $\beta_{Lt(1)}$ and $\beta_{Lt(2)}$ are defined, respectively, by the LTDOs:

$$T_{TD}(\beta_{Lt(1)}, \beta_{Lt(2)}) = \beta_{\frac{4g}{\pi}\tan^{-1}\left\{\frac{1}{1+\left\{\left(\frac{1-\tan(k\pi+Lt(1)\pi/(4g))}{\tan(k\pi+Lt(1)\pi/(4g))}\right)^{\varepsilon}+\left(\frac{1-\tan(k\pi+Lt(2)\pi/(4g))}{\tan(k\pi+Lt(2)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}} = \beta_{\frac{4g}{\pi}\tan^{-1}\left\{\frac{1}{1+\left\{\left(\frac{1-\tan(Lt(1)\pi/(4g))}{\tan(Lt(1)\pi/(4g))}\right)^{\varepsilon}+\left(\frac{1-\tan(Lt(2)\pi/(4g))}{\tan(Lt(2)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}}, \quad (5)$$

$$T_{SD}(\beta_{Lt(1)}, \beta_{Lt(2)}) = \beta_{\frac{4g}{\pi}\tan^{-1}\left\{1-\frac{1}{1+\left\{\left(\frac{\tan(k\pi+Lt(1)\pi/4g)}{1-\tan(k\pi+Lt(1)\pi/4g)}\right)^{\varepsilon}+\left(\frac{\tan(k\pi+Lt(2)\pi/(4g))}{1-\tan(k\pi+Lt(2)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}} = \beta_{\frac{4g}{\pi}\tan^{-1}\left\{1-\frac{1}{1+\left\{\left(\frac{\tan(Lt(1)\pi/(4g))}{1-\tan(Lt(1)\pi/(4g))}\right)^{\varepsilon}+\left(\frac{\tan(Lt(2)\pi/(4g))}{1-\tan(Lt(2)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}}. \quad (6)$$

Especially when $\beta_{Lt(1)} = \beta_0$ or β_g and $\beta_{Lt(2)} = \beta_0$ or β_g in Eqs. (5) and (6), let $T_{TD}(\beta_0, \beta_0) = T_{TD}(\beta_0, \beta_g) = T_{TD}(\beta_g, \beta_0) = \beta_0$, $T_{TD}(\beta_g, \beta_g) = \beta_g$, $T_{SD}(\beta_0, \beta_g) = T_{SD}(\beta_g, \beta_0) = T_{SD}(\beta_g, \beta_g) = \beta_g$ and $T_{SD}(\beta_0, \beta_0) = \beta_0$. Clearly, the LTDOs of Eqs. (5) and (6) imply $k\pi$ periodicity for $k = 1, 2, \ldots, n$. According to Eqs. (5) and (6), LNNTDOLs are proposed below.

Definition 2. Let $\beta_{L(1)} = <\beta_{Lt(1)}, \beta_{Lu(1)}, \beta_{Lf(1)}>$ and $\beta_{L(2)} = <\beta_{Lt(2)}, \beta_{Lu(2)}, \beta_{Lf(2)}>$ be two LNNs, $\eta > 0$, and $\varepsilon \geq 1$. Then, the LNNTDOLs are defined below:

$$(1)\ \beta_{L(1)} \oplus_{TD} \beta_{L(2)} = \left\langle \begin{array}{l} \beta_{\frac{4g}{\pi}\tan^{-1}\left\{1-\frac{1}{1+\left\{\left(\frac{\tan(Lt(1)\pi/(4g))}{1-\tan(Lt(1)\pi/(4g))}\right)^{\varepsilon}+\left(\frac{\sin(Lt(2)\pi/(4g))}{1-\sin(Lt(2)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}}, \\ \beta_{\frac{4g}{\pi}\tan^{-1}\left\{\frac{1}{1+\left\{\left(\frac{1-\tan(Lu(1)\pi/(4g))}{\tan(Lu(1)\pi/(4g))}\right)^{\varepsilon}+\left(\frac{1-\tan(Lu(2)\pi/(4g))}{\tan(Lu(2)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}}, \\ \beta_{\frac{4g}{\pi}\tan^{-1}\left\{\frac{1}{1+\left\{\left(\frac{1-\tan(Lf(1)\pi/(4g))}{\tan(Lf(1)\pi/(4g))}\right)^{\varepsilon}+\left(\frac{1-\tan(Lf(2)\pi/(4g))}{\tan(Lf(2)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}} \end{array} \right\rangle;$$

$$(2)\ \beta_{L(1)} \otimes_{TD} \beta_{L(2)} = \left\langle \begin{array}{l} \beta_{\frac{4g}{\pi}\tan^{-1}\left\{\frac{1}{1+\left\{\left(\frac{1-\tan(Lt(1)\pi/(4g))}{\tan(Lt(1)\pi/(4g))}\right)^{\varepsilon}+\left(\frac{1-\tan(Lt(2)\pi/(4g))}{\tan(Lt(2)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}}, \\ \beta_{\frac{4g}{\pi}\tan^{-1}\left\{1-\frac{1}{1+\left\{\left(\frac{\tan(Lu(1)\pi/(4g))}{1-\tan(Lu(1)\pi/(4g))}\right)^{\varepsilon}+\left(\frac{\tan(Lu(2)\pi/(4g))}{1-\tan(Lu(2)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}}, \\ \beta_{\frac{4g}{\pi}\tan^{-1}\left\{1-\frac{1}{1+\left\{\left(\frac{\tan(Lf(1)\pi/(4g))}{1-\tan(Lf(1)\pi/(4g))}\right)^{\varepsilon}+\left(\frac{\tan(Lf(2)\pi/(4g))}{1-\tan(Lf(2)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}} \end{array} \right\rangle;$$

$$(3)\ \eta\beta_{L(1)} = \left\langle \beta_{\frac{4g}{\pi}\tan^{-1}\left\{1-\frac{1}{1+\left\{\eta\left(\frac{\tan(Lt(1)\pi/(4g))}{1-\tan(Lt(1)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}}, \beta_{\frac{4g}{\pi}\tan^{-1}\left\{\frac{1}{1+\left\{\eta\left(\frac{1-\tan(Lu(1)\pi/(4g))}{\tan(Lu(1)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}}, \beta_{\frac{4g}{\pi}\tan^{-1}\left\{\frac{1}{1+\left\{\eta\left(\frac{1-\tan(Lf(1)\pi/(4g))}{\tan(Lf(1)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}} \right\rangle;$$

$$(4)\ \beta_{L(1)}^{\eta} = \left\langle \beta_{\frac{4g}{\pi}\tan^{-1}\left\{\frac{1}{1+\left\{\eta\left(\frac{1-\tan(Lt(1)\pi/(4g))}{\tan(Lt(1)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}}, \beta_{\frac{4g}{\pi}\tan^{-1}\left\{1-\frac{1}{1+\left\{\eta\left(\frac{\tan(Lu(1)\pi/(4g))}{1-\tan(Lu(1)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}}, \beta_{\frac{4g}{\pi}\tan^{-1}\left\{1-\frac{1}{1+\left\{\eta\left(\frac{\tan(Lf(1)\pi/(4g))}{1-\tan(Lf(1)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}} \right\rangle.$$

Example 1. Set two LNNs on the SLT $\Psi_\beta = \{\beta_0, \beta_1, \ldots, \beta_8\}$ as $\beta_{L(1)} = \langle\beta_7, \beta_3, \beta_2\rangle$ and $\beta_{L(2)} = \langle\beta_7, \beta_2, \beta_4\rangle$ with $\eta = 0.8$ and $\varepsilon = 2$. Then, using the TDOLs (1)-(4) in Definition 2, their operations are shown below:

$$\beta_{L(1)} \oplus_{TD} \beta_{L(2)} = \left\langle \begin{array}{l} \beta_{\frac{32}{\pi}\tan^{-1}\left\{1-\frac{1}{1+\left\{\left(\frac{\tan(7\pi/32)}{1-\tan(7\pi/32)}\right)^{2}+\left(\frac{\tan(7\pi/32)}{1-\tan(7\pi/32)}\right)^{2}\right\}^{1/2}}\right\}}, \\ \beta_{\frac{32}{\pi}\tan^{-1}\left\{\frac{1}{1+\left\{\left(\frac{1-\tan(3\pi/32)}{\tan(3\pi/32)}\right)^{2}+\left(\frac{1-\tan(2\pi/32)}{\tan(2\pi/32)}\right)^{2}\right\}^{1/2}}\right\}}, \\ \beta_{\frac{32}{\pi}\tan^{-1}\left\{\frac{1}{1+\left\{\left(\frac{1-\tan(2\pi/32)}{\tan(2\pi/32)}\right)^{2}+\left(\frac{1-\tan(4\pi/32)}{\tan(4\pi/32)}\right)^{2}\right\}^{1/2}}\right\}} \end{array} \right\rangle = \langle\beta_{7.7626}, \beta_{0.865}, \beta_{1.0043}\rangle;$$

$$\beta_{L(1)} \otimes_{TD} \beta_{L(2)} = \left\langle \begin{array}{c} \beta_{\frac{32}{\pi}\tan^{-1}\left\{\frac{1}{1+\left\{\left(\frac{1-\tan(7\pi/32)}{\tan(7\pi/32)}\right)^2+\left(\frac{1-\tan(7\pi/32)}{\tan(7\pi/32)}\right)^2\right\}^{1/2}}\right\}}, \\ \beta_{\frac{32}{\pi}\tan^{-1}\left\{1-\frac{1}{1+\left\{\left(\frac{\tan(3\pi/32)}{1-\tan(3\pi/32)}\right)^2+\left(\frac{\tan(2\pi/32)}{1-\tan(2\pi/32)}\right)^2\right\}^{1/2}}\right\}}, \\ \beta_{\frac{32}{\pi}\tan^{-1}\left\{1-\frac{1}{1+\left\{\left(\frac{\tan(2\pi/32)}{1-\tan(2\pi/32)}\right)^2+\left(\frac{\tan(4\pi/32)}{1-\tan(4\pi/32)}\right)^2\right\}^{1/2}}\right\}} \end{array} \right\rangle = \langle \beta_{7.7626}, \beta_{1.1321}, \beta_{2.1985} \rangle;$$

$$\eta\beta_{L(1)} = \left\langle \beta_{\frac{32}{\pi}\tan^{-1}\left\{1-\frac{1}{1+\left\{0.8\left(\frac{\tan(7\pi/32)}{1-\tan(7\pi/32)}\right)^2\right\}^{1/2}}\right\}}, \beta_{\frac{32}{\pi}\tan^{-1}\left\{\frac{1}{1+\left\{0.8\left(\frac{1-\tan(3\pi/32)}{\tan(3\pi/32)}\right)^2\right\}^{1/2}}\right\}}, \beta_{\frac{32}{\pi}\tan^{-1}\left\{\frac{1}{1+\left\{0.8\left(\frac{1-\tan(2\pi/32)}{\tan(2\pi/32)}\right)^2\right\}^{1/2}}\right\}} \right\rangle;$$

$$= \langle \beta_{7.4269}, \beta_{3.1692}, \beta_{1.3523} \rangle$$

$$\beta_{L(1)}^{\eta} = \left\langle \beta_{\frac{32}{\pi}\tan^{-1}\left\{\frac{1}{1+\left\{0.8\left(\frac{1-\tan(7\pi/32)}{\tan(7\pi/32)}\right)^2\right\}^{1/2}}\right\}}, \beta_{\frac{32}{\pi}\tan^{-1}\left\{1-\frac{1}{1+\left\{0.8\left(\frac{\tan(3\pi/32)}{1-\tan(3\pi/32)}\right)^2\right\}^{1/2}}\right\}}, \beta_{\frac{32}{\pi}\tan^{-1}\left\{1-\frac{1}{1+\left\{0.8\left(\frac{\tan(2\pi/32)}{1-\tan(2\pi/32)}\right)^2\right\}^{1/2}}\right\}} \right\rangle.$$

$$= \langle \beta_{7.9037}, \beta_{0.7169}, \beta_{0.2451} \rangle$$

It is clear that the above LTDO results are still LNNs.

4. Tangent Dombi AOs of LNNs

According to the LNNTDOLs in Definition 2, the LNNTDWA and LNNTDWG operators of LNNs are built, and then their properties are investigated in this section.

Definition 3. Let $\beta_{L(j)} = \langle \beta_{Lt(j)}, \beta_{Lu(j)}, \beta_{Lf(j)} \rangle$ (j = 1, 2, …, e) be e LNNs with their weight vector $\boldsymbol{\eta} = (\eta_1, \eta_2, \ldots, \eta_e)$ for $\eta_j \in [0, 1]$ and $\sum_{j=1}^{e} \eta_j = 1$. Thus, the LNNTDWA and LNNTDWG operators of LNNs are defined below:

$$LNNTDWA(\beta_{L(1)}, \beta_{L(2)}, \ldots, \beta_{L(e)}) = \bigoplus_{j=1}^{e}{}_{TD}\, \eta_j \beta_{L(j)}, \tag{7}$$

$$LNNTDWG(\beta_{L(1)}, \beta_{L(2)}, \ldots, \beta_{L(e)}) = \bigotimes_{j=1}^{e}{}_{TD}\, \beta_{L(j)}^{\eta_j}. \tag{8}$$

Theorem 1. Let $\beta_{L(j)} = <\beta_{Lt(j)}, \beta_{Lu(j)}, \beta_{Lf(j)}>$ (j = 1, 2, …, e) be e LNNs with the weight vector $\boldsymbol{\eta} = (\eta_1, \eta_2, \ldots, \eta_e)$ with $\eta_j \in [0, 1]$ and $\sum_{j=1}^{e} \eta_j = 1$. Then, the LNNTDWA operator can obtain the aggregated LNN by the following equation:

$$LNNTDWA(\beta_{L(1)}, \beta_{L(2)}, \ldots, \beta_{L(e)}) = \oplus_{TD,\, j=1}^{e} \eta_j \beta_{L(j)}$$

$$= \left\langle \beta_{\frac{4g}{\pi}\tan^{-1}\left\{1-\frac{1}{1+\left\{\sum_{j=1}^{e}\eta_j\left(\frac{\tan(Lt(j)\pi/(4g))}{1-\tan(Lt(j)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}}, \beta_{\frac{4g}{\pi}\tan^{-1}\left\{\frac{1}{1+\left\{\sum_{j=1}^{e}\eta_j\left(\frac{1-\tan(Lu(j)\pi/(4g))}{\tan(Lu(j)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}}, \beta_{\frac{4g}{\pi}\tan^{-1}\left\{\frac{1}{1+\left\{\sum_{j=1}^{e}\eta_j\left(\frac{1-\tan(Lf(j)\pi/(4g))}{\tan(Lf(j)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}} \right\rangle. \quad (9)$$

Proof:

Here, Theorem 1 can be verified by mathematical induction.

Set e = 2. Based on the LNNTDOLs in Definition 2, we can give the result:

$$LNNTDWA(\beta_{L(1)}, \beta_{L(2)}) = \beta_{L(1)} \oplus_{TD} \beta_{L(2)}$$

$$= \left\langle \begin{array}{l} \beta_{\frac{4g}{\pi}\tan^{-1}\left\{1-\frac{1}{1+\left\{\eta_1\left(\frac{\tan(Lt(1)\pi/(4g))}{1-\tan(Lt(1)\pi/(4g))}\right)^{\varepsilon}+\eta_2\left(\frac{\tan(Lt(2)\pi/(4g))}{1-\tan(Lt(2)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}}, \\ \beta_{\frac{4g}{\pi}\tan^{-1}\left\{\frac{1}{1+\left\{\eta_1\left(\frac{1-\tan(Lu(1)\pi/(4g))}{\tan(Lu(1)\pi/(4g))}\right)^{\varepsilon}+\eta_2\left(\frac{1-\tan(Lu(2)\pi/(4g))}{\tan(Lu(2)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}}, \\ \beta_{\frac{4g}{\pi}\tan^{-1}\left\{\frac{1}{1+\left\{\eta_1\left(\frac{1-\tan(Lf(1)\pi/(4g))}{\tan(Lf(1)\pi/(4g))}\right)^{\varepsilon}+\eta_2\left(\frac{1-\tan(Lf(2)\pi/(4g))}{\tan(Lf(2)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}} \end{array} \right\rangle = \left\langle \begin{array}{l} \beta_{\frac{4g}{\pi}\tan^{-1}\left\{1-\frac{1}{1+\left\{\sum_{j=1}^{2}\eta_j\left(\frac{\tan(Lt(j)\pi/(4g))}{1-\tan(Lt(j)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}}, \\ \beta_{\frac{4g}{\pi}\tan^{-1}\left\{\frac{1}{1+\left\{\sum_{j=1}^{2}\eta_j\left(\frac{1-\tan(Lu(j)\pi/(4g))}{\tan(Lu(j)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}}, \\ \beta_{\frac{4g}{\pi}\tan^{-1}\left\{\frac{1}{1+\left\{\sum_{j=1}^{2}\eta_j\left(\frac{1-\tan(Lf(j)\pi/(4g))}{\tan(Lf(j)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}} \end{array} \right\rangle. \quad (10)$$

Set $e = n$. Eq. (9) can keep the equation:

$$LNNTDWA(\beta_{L(1)}, \beta_{L(2)}, \ldots, \beta_{L(n)}) = \oplus_{TD,\, j=1}^{n} \eta_j \beta_{L(j)}$$

$$= \left\langle \beta_{\frac{4g}{\pi}\tan^{-1}\left\{1-\frac{1}{1+\left\{\sum_{j=1}^{n}\eta_j\left(\frac{\tan(Lt(j)\pi/(4g))}{1-\tan(Lt(j)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}}, \beta_{\frac{4g}{\pi}\tan^{-1}\left\{\frac{1}{1+\left\{\sum_{j=1}^{n}\eta_j\left(\frac{1-\tan(Lu(j)\pi/(4g))}{\tan(Lu(j)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}}, \beta_{\frac{4g}{\pi}\tan^{-1}\left\{\frac{1}{1+\left\{\sum_{j=1}^{n}\eta_j\left(\frac{1-\tan(Lf(j)\pi/(4g))}{\tan(Lf(j)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}} \right\rangle. \quad (11)$$

Set $e = n + 1$. Based on Eqs. (10) and (11), there is the result:

$$LNNTDWA(\beta_{L(1)},\beta_{L(2)},...,\beta_{L(n)},\beta_{L(n+1)}) = \overset{n}{\underset{j=1}{\oplus_{TD}}} \eta_j\beta_{L(j)} \oplus_{TD} \eta_{n+1}\beta_{L(n+1)}$$

$$= \left\langle \begin{matrix} \beta_{\frac{4g}{\pi}\tan^{-1}\left\{1-\frac{1}{1+\left\{\sum_{j=1}^{n}\eta_j\left(\frac{\tan(Lt(j)\pi/(4g))}{1-\tan(Lt(j)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}}, \\ \beta_{\frac{4g}{\pi}\tan^{-1}\left\{\frac{1}{1+\left\{\sum_{j=1}^{n}\eta_j\left(\frac{1-\tan(Lu(j)\pi/(4g))}{\tan(Lu(j)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}}, \\ \beta_{\frac{4g}{\pi}\tan^{-1}\left\{\frac{1}{1+\left\{\sum_{k=1}^{n}\tau_k\left(\frac{1-\tan(Lf(j)\pi/(4g))}{\tan(Lf(j)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}} \end{matrix} \right\rangle \oplus_{TD} \eta_{n+1}\beta_{L(n+1)} = \left\langle \begin{matrix} \beta_{\frac{4g}{\pi}\tan^{-1}\left\{1-\frac{1}{1+\left\{\sum_{j=1}^{n+1}\eta_j\left(\frac{\tan(Lt(j)\pi/(4g))}{1-\tan(Lt(j)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}}, \\ \beta_{\frac{4g}{\pi}\tan^{-1}\left\{\frac{1}{1+\left\{\sum_{j=1}^{n+1}\eta_j\left(\frac{1-\tan(Lu(j)\pi/(4g))}{\tan(Lu(j)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}}, \\ \beta_{\frac{4g}{\pi}\tan^{-1}\left\{\frac{1}{1+\left\{\sum_{j=1}^{n+1}\eta_j\left(\frac{1-\tan(Lf(j)\pi/(4g))}{\tan(Lf(j)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}} \end{matrix} \right\rangle.$$

Therefore, Eq. (9) exists for any e. Thus, Theorem 1 is true. ■

Theorem 2. The LNNTDWA operator of Eq. (9) contains the properties below:

(i) Reducibility: When $\eta_j = 1/e$ ($j = 1, 2, \ldots, e$), the operation result of Eq. (9) is

$$LNNTDWA(\beta_{L(1)},\beta_{L(2)},...,\beta_{L(e)}) = \overset{e}{\underset{j=1}{\oplus_{TD}}} \eta_j\beta_{L(j)}$$

$$= \left\langle \beta_{\frac{4g}{\pi}\tan^{-1}\left\{1-\frac{1}{1+\left\{\sum_{j=1}^{e}\frac{1}{e}\left(\frac{\tan(Lt(j)\pi/(4g))}{1-\tan(Lt(j)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}}, \beta_{\frac{4g}{\pi}\tan^{-1}\left\{\frac{1}{1+\left\{\sum_{j=1}^{e}\frac{1}{e}\left(\frac{1-\tan(Lu(j)\pi/(4g))}{\tan(Lu(j)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}}, \beta_{\frac{4g}{\pi}\tan^{-1}\left\{\frac{1}{1+\left\{\sum_{j=1}^{e}\frac{1}{e}\left(\frac{1-\tan(Lf(j)\pi/(4g))}{\tan(Lf(j)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}} \right\rangle.$$

(ii) Idempotency: Set $\beta_{L(j)} = <\beta_{Lt(j)}, \beta_{Lu(j)}, \beta_{Lf(j)}>$ ($j = 1, 2, ..., e$) as e LNNs. When $\beta_{L(j)} = \beta_L$ ($j = 1, 2, \ldots, e$), $LNNTDWA(\beta_{L(1)}, \beta_{L(2)}, \ldots, \beta_{L(e)}) = \beta_L$.

(iii) Commutativity: Set $(\beta'_{L(1)}, \beta'_{L(2)}, ..., \beta'_{L(e)})$ as any permutation of ($\beta_{L(1)}, \beta_{L(2)}, \ldots, \beta_{L(e)}$). Then $LNNTDWA(\beta'_{L(1)}, \beta'_{L(2)}, ..., \beta'_{L(e)}) = LNNTDWA\left(\beta_{L(1)},\beta_{L(2)},\ldots,\beta_{L(e)}\right)$.

(iv) Boundedness: Set the minimum and maximum LNNs as $\beta_{L\min} = \left\langle \min\limits_j(\beta_{Lt(j)}), \max\limits_j(\beta_{Lu(j)}), \max\limits_j(\beta_{Lf(j)}) \right\rangle$ and $\beta_{L\max} = \left\langle \max\limits_j(\beta_{Lt(j)}), \min\limits_j(\beta_{Lu(j)}), \min\limits_j(\beta_{Lf(j)}) \right\rangle$. Then $\beta_{L\min} \le LNNTDWA(\beta_{L(1)}, \beta_{L(2)}, \ldots, \beta_{L(e)}) \le \beta_{L\max}$.

Proof:

(i) The property of Eq. (9) is obvious.

(ii) Using Eq. (9) for $\beta_{L(j)} = <\beta_{Lt(j)}, \beta_{Lu(j)}, \beta_{Lf(j)}> = \beta_L = <\beta_{Lt}, \beta_{Lu}, \beta_{Lf}>$ ($j = 1, 2, ..., e$), the result of Eq. (9) is given below:

$$LNNTDWA(\beta_{L(1)},\beta_{L(2)},...,\beta_{L(e)})=\overset{e}{\underset{j=1}{\oplus_{TD}}}\eta_j\beta_{L(j)}$$

$$=\left\langle \beta_{\frac{4g}{\pi}\tan^{-1}\left\{1-\frac{1}{1+\left\{\sum_{j=1}^{e}\eta_j\left(\frac{\tan(Lt(j)\pi/(4g))}{1-\tan(Lt(j)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}},\beta_{\frac{4g}{\pi}\tan^{-1}\left\{\frac{1}{1+\left\{\sum_{j=1}^{e}\eta_j\left(\frac{1-\tan(Lu(j)\pi/(4g))}{\tan(Lu(j)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}},\beta_{\frac{4g}{\pi}\tan^{-1}\left\{\frac{1}{1+\left\{\sum_{j=1}^{e}\eta_j\left(\frac{1-\tan(Lf(j)\pi/(4g))}{\tan(Lf(j)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}}\right\rangle$$

$$=\left\langle \beta_{\frac{4g}{\pi}\tan^{-1}\left\{1-\frac{1}{1+\left\{\left(\frac{\tan(Lt\pi/(4g))}{1-\tan(Lt\pi/(4g))}\right)^{\varepsilon}\sum_{j=1}^{e}\eta_j\right\}^{1/\varepsilon}}\right\}},\beta_{\frac{4g}{\pi}\tan^{-1}\left\{\frac{1}{1+\left\{\left(\frac{1-\tan(Lu\pi/(4g))}{\tan(Lu\pi/(4g))}\right)^{\varepsilon}\sum_{j=1}^{e}\eta_j\right\}^{1/\varepsilon}}\right\}},\beta_{\frac{4g}{\pi}\tan^{-1}\left\{\frac{1}{1+\left\{\left(\frac{1-\tan(Lu\pi/(4g))}{\tan(Lu\pi/(4g))}\right)^{\varepsilon}\sum_{j=1}^{e}\eta_j\right\}^{1/\varepsilon}}\right\}}\right\rangle$$

$$=\left\langle \beta_{\frac{4g}{\pi}\tan^{-1}\left\{1-\frac{1}{1+\left\{\left(\frac{\tan(Lt\pi/(4g))}{1-\tan(Lt\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}},\beta_{\frac{4g}{\pi}\tan^{-1}\left\{\frac{1}{1+\left\{\left(\frac{1-\tan(Lu\pi/(4g))}{\tan(Lu\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}},\beta_{\frac{4g}{\pi}\tan^{-1}\left\{\frac{1}{1+\left\{\left(\frac{1-\tan(Lf\pi/(4g))}{\tan(Lf\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}}\right\rangle$$

$$=\left\langle \beta_{\frac{4g}{\pi}\tan^{-1}\left\{1-\frac{1}{1+\frac{\tan(Lt\pi/(4g))}{1-\tan(Lt\pi/(4g))}}\right\}},\beta_{\frac{4g}{\pi}\tan^{-1}\left\{\frac{1}{1+\frac{1-\tan(Lu\pi/(4g))}{\tan(Lu\pi/(4g))}}\right\}},\beta_{\frac{4g}{\pi}\tan^{-1}\left\{\frac{1}{1+\frac{1-\tan(Lf\pi/(4g))}{\tan(Lf\pi/(4g))}}\right\}}\right\rangle=\left\langle \beta_{Lt},\beta_{Lu},\beta_{Lf}\right\rangle=\beta_L.$$

Subsequently, $LNNTDWA(\beta_{L(1)}, \beta_{L(2)},\ldots, \beta_{L(e)}) = \beta_L$ exists.

(iii) It is clear that this feature is true.

(iv) Since $\min_j(\beta_{Lt(j)})\le\beta_{Lt(j)}\le\max_j(\beta_{Lt(j)})$, $\min_j(\beta_{Lu(j)})\le\beta_{Lu(j)}\le\max_j(\beta_{Lu(j)})$, and $\min_j(\beta_{Lf(j)})\le\beta_{Lf(j)}\le\max_j(\beta_{Lf(j)})$, there are the inequalities:

$$1-\frac{1}{1+\left\{\sum_{j=1}^{e}\eta_j\left(\frac{\min_j(\tan(Lt(j)\pi/(4g)))}{1-\min_j(\tan(Lt(j)\pi/(4g)))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\le1-\frac{1}{1+\left\{\sum_{j=1}^{e}\eta_j\left(\frac{(\tan(Lt(j)\pi/(4g)))}{1-\tan(Lt(j)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\le1-\frac{1}{1+\left\{\sum_{j=1}^{e}\eta_j\left(\frac{\max_j(\tan(Lt(j)\pi/(4g)))}{1-\max_j(\tan(Lt(j)\pi/(4g)))}\right)^{\varepsilon}\right\}^{1/\varepsilon}},$$

$$\frac{1}{1+\left\{\sum_{j=1}^{e}\eta_j\left(\frac{1-\min_j(\tan(Lu(j)\pi/(4g)))}{\min_j(\tan(Lu(j)\pi/(4g)))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\le\frac{1}{1+\left\{\sum_{j=1}^{e}\eta_j\left(\frac{1-\tan(Lu(j)\pi/(4g))}{\tan(Lu(j)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\le\frac{1}{1+\left\{\sum_{j=1}^{e}\beta_j\left(\frac{1-\max_j(\tan(Lu(j)\pi/(4g)))}{\max_j(\tan(Lu(j)\pi/(4g)))}\right)^{\varepsilon}\right\}^{1/\varepsilon}},$$

$$\frac{1}{1+\left\{\sum_{j=1}^{e}\eta_j\left(\frac{1-\min_j(\tan(Lf(j)\pi/(4g)))}{\min_j(\tan(Lf(j)\pi/(4g)))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\le\frac{1}{1+\left\{\sum_{j=1}^{e}\eta_j\left(\frac{1-\tan(Lf(j)\pi/(4g))}{\tan(Lf(j)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\le\frac{1}{1+\left\{\sum_{j=1}^{e}\eta_j\left(\frac{1-\max_j(\tan(Lf(j)\pi/(4g)))}{\max_j(\tan(Lf(j)\pi/(4g)))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}.$$

Then, we also have the result:

$$\beta_{\frac{4g}{\pi}\tan^{-1}\left\{1-\frac{1}{1+\left\{\sum_{j=1}^{e}\eta_j\left(\frac{\min_j(\tan(Lt(j)\pi/(4g)))}{1-\min_j(\tan(Lt(j)\pi/(4g)))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}} \le \beta_{\frac{4g}{\pi}\tan^{-1}\left\{1-\frac{1}{1+\left\{\sum_{j=1}^{e}\eta_j\left(\frac{(\tan(Lt(j)\pi/(4g)))}{1-\tan(Lt(j)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}} \le \beta_{\frac{4g}{\pi}\tan^{-1}\left\{1-\frac{1}{1+\left\{\sum_{j=1}^{e}\eta_j\left(\frac{\max_j(\tan(Lt(j)\pi/(4g)))}{1-\max_j(\tan(Lt(j)\pi/(4g)))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}},$$

$$\beta_{\frac{4g}{\pi}\tan^{-1}\left\{\frac{1}{1+\left\{\sum_{j=1}^{e}\eta_j\left(\frac{1-\min_j(\tan(Lu(j)\pi/(4g)))}{\min_j(\tan(Lu(j)\pi/(4g)))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}} \le \beta_{\frac{4g}{\pi}\tan^{-1}\left\{\frac{1}{1+\left\{\sum_{j=1}^{e}\eta_j\left(\frac{1-\tan(Lu(j)\pi/(4g))}{\tan(Lu(j)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}} \le \beta_{\frac{4g}{\pi}\tan^{-1}\left\{\frac{1}{1+\left\{\sum_{j=1}^{e}\eta_j\left(\frac{1-\max_j(\tan(Lu(j)\pi/(4g)))}{\max_j(\tan(Lu(j)\pi/(4g)))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}},$$

$$\beta_{\frac{4g}{\pi}\tan^{-1}\left\{\frac{1}{1+\left\{\sum_{j=1}^{e}\eta_j\left(\frac{1-\min_j(\tan(Lf(j)\pi/(4g)))}{\min_j(\tan(Lf(j)\pi/(4g)))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}} \le \beta_{\frac{4g}{\pi}\tan^{-1}\left\{\frac{1}{1+\left\{\sum_{k=1}^{e}\tau_k\left(\frac{1-\tan(Lf(j)\pi/(4g))}{\tan(Lf(j)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}} \le \beta_{\frac{4g}{\pi}\tan^{-1}\left\{\frac{1}{1+\left\{\sum_{j=1}^{e}\eta_j\left(\frac{1-\max_j(\tan(Lf(j)\pi/(4g)))}{\max_j(\tan(Lf(j)\pi/(4g)))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}}.$$

According to the property (ii), we have the inequalities:

$$\min_j(\beta_{Lt(j)}) \le \beta_{\frac{4g}{\pi}\tan^{-1}\left\{1-\frac{1}{1+\left\{\sum_{j=1}^{e}\eta_j\left(\frac{(\tan(Lt(j)\pi/(4g)))}{1-\tan(Lt(j)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}} \le \max_j(\beta_{Lt(j)})',$$

$$\min_j(\beta_{Lu(j)}) \le \beta_{\frac{4g}{\pi}\tan^{-1}\left\{\frac{1}{1+\left\{\sum_{j=1}^{e}\eta_j\left(\frac{1-\tan(Lu(j)\pi/(4g))}{\tan(Lu(j)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}} \le \max_j(\beta_{Lu(j)k})',$$

$$\min_j(\beta_{Lf(j)}) \le \beta_{\frac{4g}{\pi}\tan^{-1}\left\{\frac{1}{1+\left\{\sum_{j=1}^{e}\eta_j\left(\frac{1-\tan(Lf(j)\pi/(4g))}{\tan(Lf(j)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}} \le \max_j(\beta_{Lf(j)})'.$$

Thus, $\beta_{L\min} \le LNNTDWA(\beta_{L(1)}, \beta_{L(2)}, \ldots, \beta_{L(e)}) \le \beta_{L\max}$ can also held.

Based on the proof of the above properties, Theorem 2 is true. ■

Theorem 3. Let $\beta_{L(j)} = <\beta_{Lt(j)}, \beta_{Lu(j)}, \beta_{Lf(j)}>$ for $j = 1, 2, \ldots, e$ be e LNNs with the weight vector $\boldsymbol{\eta} = (\eta_1, \eta_2, \ldots, \eta_e)$, subject to $\eta_j \in [0, 1]$ and $\sum_{j=1}^{e}\eta_j = 1$. Then, the LNNTDWG operator can obtain the aggregated LNN by the following equation:

$$LNNTDWG(\beta_{L(1)},\beta_{L(2)},...,\beta_{L(e)}) = \overset{h}{\underset{j=1}{\otimes_{TD}}}\ \beta_{L(j)}^{\eta_j}$$

$$= \left\langle \beta_{\frac{4g}{\pi}\tan^{-1}\left\{\frac{1}{1+\left\{\sum_{j=1}^{e}\eta_j\left(\frac{1-\tan(Lt(j)\pi/(4g))}{\tan(Lt(j)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}}, \beta_{\frac{4g}{\pi}\tan^{-1}\left\{1-\frac{1}{1+\left\{\sum_{j=1}^{e}\eta_j\left(\frac{\tan(Lu(j)\pi/(4g))}{1-\tan(Lu(j)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}}, \beta_{\frac{4g}{\pi}\tan^{-1}\left\{1-\frac{1}{1+\left\{\sum_{j=1}^{e}\eta_j\left(\frac{\tan(Lf(j)\pi/(4g))}{1-\tan(Lf(j)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}} \right\rangle. \quad (12)$$

A similar proof of Theorem 1 can also be used for Theorem 3 (omitted).

Theorem 4. The LNNTDWG operator of Eq. (12) has the properties:

(i) Reducibility: When $\eta_j = 1/e$ ($j = 1, 2, \ldots, e$), Eq. (12) has the result:

$$LNNTDWG(\beta_{L(1)},\beta_{L(2)},...,\beta_{L(e)}) = \overset{e}{\underset{j=1}{\otimes_{TD}}}\ \beta_{L(j)}^{\eta_j}$$

$$= \left\langle \beta_{\frac{4g}{\pi}\tan^{-1}\left\{\frac{1}{1+\left\{\sum_{j=1}^{e}\frac{1}{e}\left(\frac{1-\tan(Lt(j)\pi/(4g))}{\tan(Lt(j)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}}, \beta_{\frac{4g}{\pi}\tan^{-1}\left\{1-\frac{1}{1+\left\{\sum_{j=1}^{e}\frac{1}{e}\left(\frac{\tan(Lu(j)\pi/(4g))}{1-\tan(Lu(j)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}}, \beta_{\frac{4g}{\pi}\tan^{-1}\left\{1-\frac{1}{1+\left\{\sum_{j=1}^{e}\frac{1}{e}\left(\frac{\tan(Lf(j)\pi/(4g))}{1-\tan(Lf(j)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}} \right\rangle.$$

(ii) Idempotency: Set $\beta_{L(j)} = <\beta_{Lt(j)}, \beta_{Lu(j)}, \beta_{Lf(j)}>$ ($j = 1, 2, ..., e$) as e LNNs. If $\beta_{L(j)} = \beta_L$ ($j = 1, 2, \ldots, e$), then $LNNTDWG(\beta_{L(1)}, \beta_{L(2)},\ldots, \beta_{L(e)}) = \beta_L$.

(iii) Commutativity: Set $(\beta'_{L(1)}, \beta'_{L(2)}, ..., \beta'_{L(e)})$ as any permutation of $(\beta_{L(1)}, \beta_{L(2)}, \ldots, \beta_{L(e)})$. Then $LNNTDWG\ (\beta'_{L(1)}, \beta'_{L(2)}, ..., \beta'_{L(e)}) = LNNTDWG\ \left(\beta_{L(1)},\beta_{L(2)},\ldots,\beta_{L(e)}\right)$.

(iv) Boundedness: Set the minimum and maximum LNNs as $\beta_{L\min} = \left\langle \min_j(\beta_{Lt(j)}),\ \max_j(\beta_{Lu(j)}),\ \max_j(\beta_{Lf(j)}) \right\rangle$ and $\beta_{L\max} = \left\langle \max_j(\beta_{Lt(j)}),\ \min_j(\beta_{Lu(j)}),\ \min_j(\beta_{Lf(j)}) \right\rangle$. Then $\beta_{L\min} \le LNNTDWG(\beta_{L(1)}, \beta_{L(2)}, \ldots, \beta_{L(e)}) \le \beta_{L\max}$.

A similar proof of Theorem 2 can be also applied to Theorem 4 (omitted).

Example 2. Set two LNNs on the SLT $\Psi_\beta = \{\beta_0, \beta_1, \ldots, \beta_8,\}$ as $\beta_{L(1)} = <\beta_7, \beta_2, \beta_3>$ and $\beta_{L(2)} = <\beta_6, \beta_3, \beta_4>$ with $\eta_1 = 0.3$, $\eta_2 = 0.7$, and $\varepsilon = 2$. Then, using Eqs. (9) and (12), their aggregation results are given by the following calculation:

$$LNNTDWA(\beta_{L1},\beta_{L2}) = \bigoplus_{j=1}^{2}{}_{TD}\ \eta_j\beta_{L(j)}$$

$$= \left\langle \begin{array}{l} \beta_{\frac{32}{\pi}\tan^{-1}\left\{1-\frac{1}{1+\left\{0.3\left(\frac{\tan(7\pi/32)}{1-\tan(7\pi/32)}\right)^2+0.7\left(\frac{\tan(6\pi/32)}{1-\tan(6\pi/32)}\right)^2\right\}^{1/2}}\right\}}, \\ \beta_{\frac{32}{\pi}\tan^{-1}\left\{\frac{1}{1+\left\{0.3\left(\frac{1-\tan(2\pi/32)}{\tan(2\pi/32)}\right)^2+0.7\left(\frac{1-\tan(3\pi/32)}{\tan(3\pi/32)}\right)^2\right\}^{1/2}}\right\}}, \\ \beta_{\frac{32}{\pi}\tan^{-1}\left\{\frac{1}{1+\left\{0.3\left(\frac{1-\tan(3\pi/32)}{\tan(3\pi/32)}\right)^2+0.7\left(\frac{1-\tan(4\pi/32)}{\tan(4\pi/32)}\right)^2\right\}^{1/2}}\right\}} \end{array} \right\rangle = \langle \beta_{6.9969}, \beta_{1.9070}, \beta_{3.8883} \rangle.$$

$$LNNTDWG(\beta_{L(1)},\beta_{L(2)}) = \bigotimes_{j=1}^{2}{}_{TD}\ \beta_{L(j)}^{\eta_j}$$

$$= \left\langle \begin{array}{l} \beta_{\frac{32}{\pi}\tan^{-1}\left\{\frac{1}{1+\left\{0.3\left(\frac{1-\tan(7\pi/28)}{\tan(7\pi/28)}\right)^2+0.7\left(\frac{1-\tan(6\pi/28)}{\tan(6\pi/28)}\right)^2\right\}^{1/2}}\right\}}, \\ \beta_{\frac{32}{\pi}\tan^{-1}\left\{1-\frac{1}{1+\left\{0.3\left(\frac{\tan(2\pi/28)}{1-\tan(2\pi/28)}\right)^2+0.7\left(\frac{\tan(3\pi/28)}{1-\tan(3\pi/28)}\right)^2\right\}^{1/2}}\right\}}, \\ \beta_{\frac{4g}{\pi}\tan^{-1}\left\{1-\frac{1}{1+\left\{0.3\left(\frac{\tan(3\pi/28)}{1-\tan(3\pi/28)}\right)^2+0.7\left(\frac{\tan(4\pi/28)}{1-\tan(4\pi/28)}\right)^2\right\}^{1/2}}\right\}} \end{array} \right\rangle = \langle \beta_{7.5455}, \beta_{0.7148}, \beta_{1.7058} \rangle.$$

5. MAGDM Technique Using the LNNTDWA and LNNTDWG Operators

In terms of the LNNTDWA and LNNTDWG operators, this part establishes a MAGDM technique to perform MAGDM issues in the LNN scenario.

Regarding a MAGDM issue, there usually are a set of alternatives $\xi_A = \{\xi_{A1}, \xi_{A2}, ..., \xi_{Aq}\}$ and a set of attributes $\zeta_B = \{\zeta_{B1}, \zeta_{B2}, ..., \zeta_{Be}\}$, subject to the weight $\eta_j \in [0, 1]$ of each attribute ζ_{Bj} with $\sum_{j=1}^{e}\eta_j = 1$. In the MAGDM process, a team of decision makers, denoted as $\zeta_D = \{\zeta_{D1}, \zeta_{D2}, ..., \zeta_{Ds}\}$ with the weight $\delta_j \in [0, 1]$ of each ζ_{Dk} with $\sum_{k=1}^{s}\delta_k = 1$, will provide their suitability evaluation of the alternatives ξ_{Ai} on the attributes ζ_{Bj} by the LNNs $\beta_{L(ij)}^{k} =< \beta_{Lt^k(ij)}, \beta_{Lu^k(ij)}, \beta_{Lf^k(ij)} >$ (j = 1, 2, ..., e; i = 1, 2, ..., q; k = 1, 2,

…, *s*) yielded from the SLT $\Psi_\beta = \{\beta_0, \beta_1, \ldots, \beta_g\}$. Thus, the *k*-th LNN decision matrix is represented as $M_\beta^k = \left(\beta_{L(ij)}^k\right)_{q\times e}$.

Based on the LNNTDWA and LNNTDWG operators, a MAGDM technique is built to tackle a MAGDM issue in the LNN scenario by the following algorithm.

Step 1: Obtain the aggregated matrix $M_\beta = (\beta_{L(ij)})_{q\times e}$ by one of the LNNTDWA and LNNTDWA operators:

$$\beta_{L(ij)} = LNNTDWA(\beta_{L(ij)}^1, \beta_{L(ij)}^2, \ldots, \beta_{L(ij)}^s) = \overset{s}{\underset{k=1}{\oplus_{TD}}} \delta_k \beta_{L(ij)}^k$$

$$= \left\langle \beta_{\frac{4g}{\pi}\tan^{-1}\left\{1-\frac{1}{1+\left\{\sum_{k=1}^{s}\delta_k\left(\frac{\tan(Lt^k(ij)\pi/(4g))}{1-\tan(Lt^k(ij)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}}, \beta_{\frac{4g}{\pi}\tan^{-1}\left\{\frac{1}{1+\left\{\sum_{k=1}^{s}\delta_k\left(\frac{1-\tan(Lu^k(ij)\pi/(4g))}{\tan(Lu^k(ij)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}}, \beta_{\frac{4g}{\pi}\tan^{-1}\left\{\frac{1}{1+\left\{\sum_{k=1}^{s}\delta_k\left(\frac{1-\tan(Lf^k(ij)\pi/(4g))}{\tan(Lf^k(ij)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}} \right\rangle. \quad (13)$$

$$\beta_{L(ij)} = LNNTDWG(\beta_{L(ij)}^1, \beta_{L(ij)}^2, \ldots, \beta_{L(ij)}^s) = \overset{s}{\underset{k=1}{\otimes_{TD}}} \left(\beta_{L(ij)}^k\right)^{\delta_k}$$

$$= \left\langle \beta_{\frac{4g}{\pi}\tan^{-1}\left\{\frac{1}{1+\left\{\sum_{k=1}^{s}\delta_k\left(\frac{1-\tan(Lt^k(ij)\pi/(4g))}{\tan(Lt^k(ij)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}}, \beta_{\frac{4g}{\pi}\tan^{-1}\left\{1-\frac{1}{1+\left\{\sum_{k=1}^{s}\delta_k\left(\frac{\tan(Lu^k(ij)\pi/(4g))}{1-\tan(Lu^k(ij)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}}, \beta_{\frac{4g}{\pi}\tan^{-1}\left\{1-\frac{1}{1+\left\{\sum_{k=1}^{s}\delta_k\left(\frac{\tan(Lf^k(j)\pi/(4g))}{1-\tan(Lf^k(j)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}} \right\rangle. \quad (14)$$

Step 2: Yield each aggregated LNN $\beta_{L(i)}$ for each alternative ξ_{Ai} (i = 1, 2, …, q) by one of the following LNNTDWA and LNNTDWG operators with $\varepsilon \geq 1$:

$$\beta_{L(i)} = LNNTDWA(\beta_{L(i1)}, \beta_{L(i2)}, \ldots, \beta_{L(ie)}) = \overset{e}{\underset{j=1}{\oplus_{TD}}} \eta_j \beta_{L(ij)}$$

$$= \left\langle \beta_{\frac{4g}{\pi}\tan^{-1}\left\{1-\frac{1}{1+\left\{\sum_{j=1}^{e}\eta_j\left(\frac{\tan(Lt(ij)\pi/(4g))}{1-\tan(Lt(ij)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}}, \beta_{\frac{4g}{\pi}\tan^{-1}\left\{\frac{1}{1+\left\{\sum_{j=1}^{e}\eta_j\left(\frac{1-\tan(Lu(ij)\pi/(4g))}{\tan(Lu(ij)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}}, \beta_{\frac{4g}{\pi}\tan^{-1}\left\{\frac{1}{1+\left\{\sum_{j=1}^{e}\eta_j\left(\frac{1-\tan(Lf(ij)\pi/(4g))}{\tan(Lf(ij)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}} \right\rangle, \quad (15)$$

$$\beta_{L(i)} = LNNTDWG(\beta_{L(i1)}, \beta_{L(i2)}, \ldots, \beta_{L(ie)}) = \overset{e}{\underset{j=1}{\otimes_{TD}}} \beta_{L(ij)}^{\eta_j}$$

$$= \left\langle \beta_{\frac{4g}{\pi}\tan^{-1}\left\{\frac{1}{1+\left\{\sum_{j=1}^{e}\eta_j\left(\frac{1-\tan(Lt(ij)\pi/(4g))}{\tan(Lt(ij)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}}, \beta_{\frac{4g}{\pi}\tan^{-1}\left\{1-\frac{1}{1+\left\{\sum_{j=1}^{e}\eta_j\left(\frac{\tan(Lu(ij)\pi/(4g))}{1-\tan(Lu(ij)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}}, \beta_{\frac{4g}{\pi}\tan^{-1}\left\{1-\frac{1}{1+\left\{\sum_{j=1}^{e}\eta_j\left(\frac{1-\tan(Lf(ij)\pi/(4g))}{1-\tan(Lf(ij)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}} \right\rangle. \quad (16)$$

Step 2: Get the score (accuracy if necessary) values of $D(\beta_{L(i)})$ ($E(\beta_{L(i)})$) (i = 1, 2, ..., q) by Eq. (3) (Eq. (4)).

Step 3: Sort the alternatives and decide the optimal one.

Step 4: End.

6. MAGDM Application

6.1 Selection of NEDSs

This section applies the built MAGDM technique in the selection of NEDSs for an investment project in Ningbo City, China, in the LNN scenario.

New energy development is becoming increasingly important and necessary to diversify energy sources and reduce carbon emissions. To effectively develop new energy sources, an energy company wants to develop some new energy generation as an investment project in Ningbo City. Therefore, the energy engineers provide four potential design schemes: solar energy (ξ_{A1}), wind energy (ξ_{A2}), wave energy (ξ_{A3}), tidal energy (ξ_{A4}), which are denoted as their set ξ_A = {ξ_{A1}, ξ_{A2}, ξ_{A3}, ξ_{A4}}. In the assessment process, they must meet the requirements of important factors/attributes: technical performance (ζ_{D1}), cost-effectiveness in terms of benefits and return on investment (ζ_{D2}), environmental condition (ζ_{D3}), and operational adaptability (ζ_{D4}), then their importance is specified by the known weight vector $\boldsymbol{\eta}$ = (0.3, 0.25, 0.23, 0.22). The energy company invites a panel of three decision makers/experts, referred to as their set ζ_D = {ζ_{D1}, ζ_{D2}, ζ_{D3}}, subject to their specified weight vector $\boldsymbol{\delta}$= (0.37, 0.33, 0.3), to evaluate the suitability degrees of each scheme ξ_{Ai} (i = 1, 2, 3, 4) over the factors ζ_{Dj} (j = 1, 2, 3, 4) in terms of the SLT Ψ_β = {β_0 (Extremely unsuitable), β_1 (Very unsuitable), β_2 (Unsuitable), β_3 (Slightly unsuitable), β_4 (Medium), β_5 (Slightly suitable), β_6 (Suitable), β_7 (Very suitable), β_8 (Extremely suitable)}. Thus, their true, false and indeterminate linguistic assessment values can be constructed as the LNNs $\beta_{L(ij)}^k = < \beta_{Lt^k(ij)}, \beta_{Lu^k(ij)}, \beta_{Lf^k(ij)} >$ (j, i = 1, 2, 3, 4; k = 1, 2, 3). Subsequently, the LNNs provided by the three decision makers/experts can be constructed as the three LNN evaluation matrices:

$$M_\beta^1 = \left(\beta_{L(ij)}^1\right)_{4\times4} = \begin{bmatrix} \langle \beta_6, \beta_1, \beta_1 \rangle & \langle \beta_7, \beta_2, \beta_2 \rangle & \langle \beta_5, \beta_1, \beta_2 \rangle & \langle \beta_6, \beta_2, \beta_2 \rangle \\ \langle \beta_7, \beta_1, \beta_1 \rangle & \langle \beta_6, \beta_1, \beta_3 \rangle & \langle \beta_7, \beta_2, \beta_2 \rangle & \langle \beta_6, \beta_1, \beta_2 \rangle \\ \langle \beta_6, \beta_2, \beta_3 \rangle & \langle \beta_5, \beta_2, \beta_3 \rangle & \langle \beta_5, \beta_3, \beta_1 \rangle & \langle \beta_4, \beta_1, \beta_2 \rangle \\ \langle \beta_5, \beta_1, \beta_2 \rangle & \langle \beta_5, \beta_3, \beta_2 \rangle & \langle \beta_5, \beta_2, \beta_2 \rangle & \langle \beta_6, \beta_2, \beta_3 \rangle \end{bmatrix},$$

$$M_\beta^2 = \left(\beta_{L(ij)}^2\right)_{4\times4} = \begin{bmatrix} \langle \beta_6, \beta_2, \beta_3 \rangle & \langle \beta_7, \beta_2, \beta_2 \rangle & \langle \beta_5, \beta_3, \beta_2 \rangle & \langle \beta_6, \beta_3, \beta_2 \rangle \\ \langle \beta_7, \beta_2, \beta_1 \rangle & \langle \beta_7, \beta_1, \beta_3 \rangle & \langle \beta_7, \beta_1, \beta_1 \rangle & \langle \beta_6, \beta_1, \beta_2 \rangle \\ \langle \beta_6, \beta_2, \beta_3 \rangle & \langle \beta_6, \beta_3, \beta_3 \rangle & \langle \beta_5, \beta_2, \beta_2 \rangle & \langle \beta_5, \beta_2, \beta_3 \rangle \\ \langle \beta_5, \beta_2, \beta_2 \rangle & \langle \beta_6, \beta_1, \beta_2 \rangle & \langle \beta_5, \beta_3, \beta_2 \rangle & \langle \beta_5, \beta_4, \beta_3 \rangle \end{bmatrix},$$

$$M_\beta^3 = \left(\beta_{L(ij)}^3\right)_{4\times4} = \begin{bmatrix} \langle \beta_5, \beta_1, \beta_2 \rangle & \langle \beta_7, \beta_2, \beta_2 \rangle & \langle \beta_6, \beta_1, \beta_2 \rangle & \langle \beta_6, \beta_2, \beta_3 \rangle \\ \langle \beta_6, \beta_1, \beta_2 \rangle & \langle \beta_7, \beta_1, \beta_2 \rangle & \langle \beta_6, \beta_2, \beta_3 \rangle & \langle \beta_7, \beta_2, \beta_4 \rangle \\ \langle \beta_5, \beta_2, \beta_1 \rangle & \langle \beta_6, \beta_3, \beta_3 \rangle & \langle \beta_5, \beta_3, \beta_3 \rangle & \langle \beta_6, \beta_3, \beta_2 \rangle \\ \langle \beta_6, \beta_3, \beta_4 \rangle & \langle \beta_5, \beta_3, \beta_4 \rangle & \langle \beta_6, \beta_3, \beta_2 \rangle & \langle \beta_5, \beta_2, \beta_1 \rangle \end{bmatrix}.$$

Thus, the proposed MAGDM technique is utilized to tackle the MAGDM issue in the LNN scenario by the following decision algorithm.

First, set ε = 1 as a calculational example in this MAGDM application. By Eq. (13) or (14), the aggregated LNN matrix is obtained below:

$$M_\beta = \left(\beta_{L(ij)}\right)_{4\times 4}$$
$$= \begin{bmatrix} \langle \beta_{5.7765}, \beta_{1.1982}, \beta_{1.5903} \rangle & \langle \beta_7, \beta_2, \beta_2 \rangle & \langle \beta_{5.3893}, \beta_{1.2841}, \beta_2 \rangle & \langle \beta_6, \beta_{2.2485}, \beta_{2.2235} \rangle \\ \langle \beta_{6.823}, \beta_{1.1982}, \beta_{1.177} \rangle & \langle \beta_{6.7723}, \beta_1, \beta_{2.6107} \rangle & \langle \beta_{6.823}, \beta_{1.505}, \beta_{1.629} \rangle & \langle \beta_{6.4603}, \beta_{1.177}, \beta_{2.3574} \rangle \\ \langle \beta_{5.7765}, \beta_2, \beta_{1.7032} \rangle & \langle \beta_{5.7171}, \beta_{2.5337}, \beta_3 \rangle & \langle \beta_5, \beta_{2.5771}, \beta_{1.5777} \rangle & \langle \beta_{5.157}, \beta_{1.5777}, \beta_{2.2485} \rangle \\ \langle \beta_5, \beta_{1.461}, \beta_{1.8098} \rangle & \langle \beta_{5.4229}, \beta_{1.8129}, \beta_{2.3574} \rangle & \langle \beta_{5.3893}, \beta_{2.5337}, \beta_2 \rangle & \langle \beta_{5.4663}, \beta_{2.4002}, \beta_{1.881} \rangle \end{bmatrix}$$

$$M_\beta = \left(\beta_{L(ij)}\right)_{4\times 4}$$
$$\text{or} = \begin{bmatrix} \langle \beta_{5.6645}, \beta_{1.3599}, \beta_{2.0642} \rangle & \langle \beta_7, \beta_2, \beta_2 \rangle & \langle \beta_{5.2665}, \beta_{1.7966}, \beta_2 \rangle & \langle \beta_6, \beta_{2.3672}, \beta_{2.3355} \rangle \\ \langle \beta_{6.6715}, \beta_{1.3599}, \beta_{1.3285} \rangle & \langle \beta_{6.5987}, \beta_1, \beta_{2.7335} \rangle & \langle \beta_{6.6715}, \beta_{1.6988}, \beta_{2.0639} \rangle & \langle \beta_{6.2728}, \beta_{1.3285}, \beta_{2.7664} \rangle \\ \langle \beta_{5.6645}, \beta_2, \beta_{2.1688} \rangle & \langle \beta_{5.591}, \beta_{2.6675}, \beta_3 \rangle & \langle \beta_5, \beta_{2.7054}, \beta_{2.0294} \rangle & \langle \beta_{4.8063}, \beta_{2.0294}, \beta_{2.3672} \rangle \\ \langle \beta_5, \beta_{1.6605}, \beta_{2.8167} \rangle & \langle \beta_{5.2946}, \beta_{2.4653}, \beta_{2.7664} \rangle & \langle \beta_{5.2665}, \beta_{2.6675}, \beta_2 \rangle & \langle \beta_{5.3325}, \beta_{2.8333}, \beta_{2.5181} \rangle \end{bmatrix}.$$

Second, the aggregated LNNs $\beta_{L(i)}$ for the schemes ξ_{Ai} (i = 1, 2, …, q) are yielded by Eqs. (14) or (15) as follows:

$\beta_{L(1)} = <\beta_{6.2825}, \beta_{1.5338}, \beta_{1.8958}>$, $\beta_{L(2)} = <\beta_{6.7449}, \beta_{1.1904}, \beta_{1.7107}>$, $\beta_{L(3)} = <\beta_{5.4896}, \beta_{2.0959}, \beta_{1.9893}>$, and $\beta_{L(4)} = <\beta_{5.3110}, \beta_{1.9035}, \beta_{1.9853}>$.

Or $\beta_{L(1)} = <\beta_{5.9241}, \beta_{1.8615}, \beta_{2.0957}>$, $\beta_{L(2)} = <\beta_{6.5622}, \beta_{1.3485}, \beta_{2.2201}>$, $\beta_{L(3)} = <\beta_{5.2808}, \beta_{2.3536}, \beta_{2.4116}>$, and $\beta_{L(4)} = <\beta_{5.2047}, \beta_{2.3846}, \beta_{2.5668}>$.

Lastly, using Eq. (3), the score values of $D(\beta_L(i))$ (i = 1, 2, ..., q) are in the following:

$D(\beta_{L(1)}) = 0.7855$, $D(\beta_{L(2)}) = 0.8268$, $D(\beta_{L(3)}) = 0.7252$, and $D(\beta_{L(4)}) = 0.7259$.

Or $D(\beta_{L(1)}) = 0.7486$, $D(\beta_{L(2)}) = 0.7914$, $D(\beta_{L(3)}) = 0.6882$, and $D(\beta_{L(4)}) = 0.6772$.

Subsequently, the sorting orders of the four NEDSs are $\xi_{A2} > \xi_{A1} > \xi_{A4} > \xi_{A3}$ or $\xi_{A2} > \xi_{A1} > \xi_{A3} > \xi_{A4}$, then the optimal one is ξ_{A2}.

Similarly to the above calculation process, all the evaluation results with $\varepsilon = \{1, 3, 5, 7, 9, 11\}$ are shown in Table 1 and Table 2.

Table 1. Evaluation results based on the proposed MAGDM technique using the LNNTDWA operator

ε	Score value	Sorting	Optimal one
$\varepsilon = 1$	0.7855, 0.8268, 0.7252, 0.7259	$\xi_{A2} > \xi_{A1} > \xi_{A4} > \xi_{A3}$	ξ_{A2}
$\varepsilon = 3$	0.8174, 0.8482, 0.7627, 0.7653	$\xi_{A2} > \xi_{A1} > \xi_{A4} > \xi_{A3}$	ξ_{A2}
$\varepsilon = 5$	0.8372, 0.8581, 0.7868, 0.7876	$\xi_{A2} > \xi_{A1} > \xi_{A4} > \xi_{A3}$	ξ_{A2}
$\varepsilon = 7$	0.8481, 0.8629, 0.8001, 0.8000	$\xi_{A2} > \xi_{A1} > \xi_{A3} > \xi_{A4}$	ξ_{A2}
$\varepsilon = 9$	0.8544, 0.8657, 0.8078, 0.8075	$\xi_{A2} > \xi_{A1} > \xi_{A3} > \xi_{A4}$	ξ_{A2}
$\varepsilon = 11$	0.8584, 0.8674, 0.8127, 0.8123	$\xi_{A2} > \xi_{A1} > \xi_{A3} > \xi_{A4}$	ξ_{A2}

Table 2. Evaluation results based on the proposed MAGDM technique using the LNNTDWG operator

ε	Score value	Sorting	Optimal one
$\varepsilon = 1$	0.7486, 0.7914, 0.6882, 0.6772	$\xi_{A2} > \xi_{A1} > \xi_{A3} > \xi_{A4}$	ξ_{A2}
$\varepsilon = 3$	0.7184, 0.7568, 0.6620, 0.6391	$\xi_{A2} > \xi_{A1} > \xi_{A3} > \xi_{A4}$	ξ_{A2}
$\varepsilon = 5$	0.6962, 0.7325, 0.6438, 0.6139	$\xi_{A2} > \xi_{A1} > \xi_{A3} > \xi_{A4}$	ξ_{A2}
$\varepsilon = 7$	0.6805, 0.7170, 0.6309, 0.5974	$\xi_{A2} > \xi_{A1} > \xi_{A3} > \xi_{A4}$	ξ_{A2}
$\varepsilon = 9$	0.6696, 0.7068, 0.6218, 0.5863	$\xi_{A2} > \xi_{A1} > \xi_{A3} > \xi_{A4}$	ξ_{A2}
$\varepsilon = 11$	0.6620, 0.6999, 0.6153, 0.5786	$\xi_{A2} > \xi_{A1} > \xi_{A3} > \xi_{A4}$	ξ_{A2}

From the results of Table 1, we see that the different values of ε can affect the ranking of the four schemes, but the optimal one is always ξ_{A2}.

From the results of Table 2, we see that the four schemes corresponding to the different values of ε show the ranking robustness, and then the optimal one is always ξ_{A2}.

According to the sorting results in Tables 1 and 2, the energy company should select ξ_{A2} (wind energy) as the investment project in Ningbo City. The wind power generation is reasonable because Ningbo City is closer to the east see.

6.2 Comparison with the Existing MAGDM Technique Using the LNNDWA and LNNDWG Operators

In this part, the proposed MAGDM technique is compared with the existing MAGDM technique using the LNNDWA and LNNDWG operators [29] by the selection case of NEDSs to demonstrate the validity and appropriateness of the former.

Based on the existing MAGDM technique using the LNNDWA and LNNDWG operators [29], it can be applied to the selection case of NEDSs. Using Eqs. (1) – (3), all the evaluation results with ε = {1, 3, 5, 7, 9, 11} are shown in Tables 3 and 4.

Table 3. Evaluation results based on the existing MAGDM technique using the LNNDWA operator [29]

ε	Score value	Sorting	Optimal one
$\varepsilon = 1$	0.7858, 0.8270, 0.7256, 0.7264	$\xi_{A2} > \xi_{A1} > \xi_{A4} > \xi_{A3}$	ξ_{A2}
$\varepsilon = 3$	0.8184, 0.8487, 0.7640, 0.7666	$\xi_{A2} > \xi_{A1} > \xi_{A4} > \xi_{A3}$	ξ_{A2}
$\varepsilon = 5$	0.8381, 0.8585, 0.7881, 0.7888	$\xi_{A2} > \xi_{A1} > \xi_{A4} > \xi_{A3}$	ξ_{A2}
$\varepsilon = 7$	0.8488, 0.8632, 0.8011, 0.8010	$\xi_{A2} > \xi_{A1} > \xi_{A3} > \xi_{A4}$	ξ_{A2}
$\varepsilon = 9$	0.8549, 0.8659, 0.8086, 0.8083	$\xi_{A2} > \xi_{A1} > \xi_{A3} > \xi_{A4}$	ξ_{A2}
$\varepsilon = 11$	0.8588, 0.8676, 0.8133, 0.8130	$\xi_{A2} > \xi_{A1} > \xi_{A3} > \xi_{A4}$	ξ_{A2}

Table 4. Evaluation results based on the existing MAGDM technique using the LNNDWG operator [29]

ε	Score value	Sorting	Optimal one
$\varepsilon = 1$	0.7476, 0.7903, 0.6872, 0.6758	$\xi_{A2} > \xi_{A1} > \xi_{A3} > \xi_{A4}$	ξ_{A2}
$\varepsilon = 3$	0.7164, 0.7547, 0.6603, 0.6366	$\xi_{A2} > \xi_{A1} > \xi_{A3} > \xi_{A4}$	ξ_{A2}
$\varepsilon = 5$	0.6939, 0.7303, 0.6418, 0.6112	$\xi_{A2} > \xi_{A1} > \xi_{A3} > \xi_{A4}$	ξ_{A2}
$\varepsilon = 7$	0.6783, 0.7150, 0.6290, 0.5949	$\xi_{A2} > \xi_{A1} > \xi_{A3} > \xi_{A4}$	ξ_{A2}
$\varepsilon = 9$	0.6676, 0.7051, 0.6200, 0.5842	$\xi_{A2} > \xi_{A1} > \xi_{A3} > \xi_{A4}$	ξ_{A2}
$\varepsilon = 11$	0.6603, 0.6984, 0.6138, 0.5768	$\xi_{A2} > \xi_{A1} > \xi_{A3} > \xi_{A4}$	ξ_{A2}

According to the sorting results in Tables 1–4, it is obvious that the sorting orders between the proposed MAGDM technique and the existing MAGDM technique [29] are very consistent in this decision application, which investigates the validity and appropriateness of the proposed MAGDM technique in the LNN scenario. However, the proposed MAGDM technique contains the periotic/multitemporal MAGDM capability, while the existing MAGDM technique [29] cannot imply it and shows its research gap in the LNN scenario. Therefore, the former fill the research gap of the latter in the LNN scenario.

7. Conclusions

This article first presented the linguistic tangent Dombi t-norm and t-conorm and the LNNDOLs, which imply the trigonometric periodicity. To overcome the inadequacy of existing LNN aggregation operators without periodicity, the LNNTDWA and LNNTDWG operators were developed to provide periodic aggregation tools for building periodic MAGDM techniques in the LNN scenario. Next, the

developed MAGDM technique using the LNNTDWA and LNNTDWG operators can reasonably tackle MAGDM problems with periodic/multitemporal requirements in LNN scenarios. Finally, the developed MAGDM technique was applied to the decision case of NEDSs and compared with the existing MAGDM technique using the LNNDWA and LNNDWG operators in the LNN scenario. Then the decision results demonstrated that the developed MAGDM technique was effective and applicable. Meanwhile, the developed MAGDM technique has filled the research gap of the existing MAGDM techniques, which lack periodic/multitemporal capability in LNN scenarios.

Generally, this paper has developed the LNNTDWA and LNNTDWG operators and their MAGDM technique and used them for the first time in the choice of NEDSs. To present much more research work in the future, we need to further propose LNN trigonometric Einstein, Heronian, and Bonferroni operators and to apply them in supply chain management, medical diagnosis, and risk investment in LNN scenarios.

Funding: This research received no external funding.

Conflicts of Interest: The authors declare no conflict of interest.

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Received: March 20, 2024. Accepted: July 30, 2024

Neutrosophic Sets and Systems, Vol. 70, 2024
University of New Mexico

Some Operators For Interval Generalized Set Valued Neutrosophic Quintuple Numbers And Sets

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Abstract: In this study, some constructions related to interval generalized set-valued neutrosophic quintuple sets and score and certainty functions are given. Moreover, optimistic union, pessimistic union, mean union, optimistic intersection, pessimistic intersection and mean intersection operators are defined for interval generalized set-valued neutrosophic quintuple set. Thus, interval generalized set-valued neutrosophic quintuple set are effectively used in decision making applications. An example application was realised using these operators and functions. With this application, the use of operators and functions in decision making methods was demonstrated. In addition, it was concluded that similar or different results can be obtained by using different operators. In addition, the importance of choosing the appropriate operator for the problem in decision making applications was mentioned for the decision making process. Researchers can obtain more objective results in decision making processes by using the operators and functions in this study.

Keywords: Interval Generalized Set Valued Neutrosophic Quintuple set, Interval Generalized Set Valued Neutrosophic Quintuple set, Operators, Score and Accuracy Functions, Decision Making Applications.

1. Introduction

The theory of neutrosophic sets and neutrosophic logic, developed by Florentin Smarandache in 1998, provides an important mathematical framework for handling uncertain concepts more effectively [1]. Furthermore, Wang et al. in 2010 defined single-valued neutrosophic sets (SVNS) [2]. This theory determines the degrees of truth (T), uncertainty (I) and falsity (F) of each element in the interval [0,1]. Thus, it recognises that each element has a degree of uncertainty, without being bound to a precise degree of truth or falsity. Neutrosophic set and logic theory is an important tool for understanding and solving complex situations full of uncertainties and finds applications in many different disciplines. Kargın and Şahin defined SuperHyper Groups and Neutro-SuperHyper Groups in 2023 [3]. Sahin et al. 2023 defined metric spaces and normed spaces for neutrosophic numbers [4]. J. Jakhar et al. In 2024, he studied to find the stability of Cauchy additive functional equation in neutrosophic normed space [6]. S.N. Suber Bathusha and S. Angelin Kavitha Raj did a study on a new approach on the energy of $\boldsymbol{\lambda J}$-dominated single-valued neutrosophic graph structure in 2024 [7]. B. Banik et al. conducted a study on the Analysis of Economic Decline of Different Countries due to COVID-19 Surge in 2024 with Advanced Multiple MOORA Strategy under

Pentagonal Neutrosophic Area [8]. P. Agarwal et al. 2024 on Ideals and Filters in Neutrosophic Topologies Generated by Neutrosophic Relations [9].

Wang et al. In 2005, Wang et al. defined interval neutrosophic sets (INS) as an important step in the development of neutrosophic theory [9]. In INS; accuracy, uncertainty and inaccuracy values are used as intervals instead of a single value. This approach provides a more useful structure in decision-making applications because it is often not possible to make a definite judgement in a decision-making situation. In offer a more flexible approach when dealing with decision-making situations fraught with uncertainty. In 2022, Kargın and Şahin defined interval generalized set-valued neutrosophic quadruple sets and numbers [10]. In 2023, Sahin and Kargin defined interval generalized set-valued Pythagorean neutrosophic quadruple set [11]. In 2024, Lo, H. W. conducted a study on a new interval neutrosophic-based group decision-making approach for sustainable development assessment in the computer manufacturing industry [12]. Rosli, S. N. I. I., and Zulkifly, M. I. E. Z. 2024 on Interval Neutrosophic Cubic Bézier Curve Approximation Model for Complex Data [13]. Hamad, F. A., and Algamudi, B. M. In 2024, he did a study on N-Cylindrical Fuzzy Interval Neutrosophic set [14]. Li, H. In 2024, he conducted a study on an improved group decision-making framework for financial performance evaluation of high-tech enterprises in an interval neutrosophic environment [15].

Smarandache defined neutrosophic quadruple sets and numbers in 2015 [16].Neutrosophic quadruple set have truth (T), uncertainty (I) and falsity (F) values as in neutrosophic sets, but unlike neutrosophic sets, they also have known and unknown parts such that (i)+(jT+kI+lF) (i,j,k,l∈ ℝ or ℂ) [17]. Thus, this set is named with neutrosophic quadruple set because of four components (i, jT, kI, lF). Therefore, neutrosophic quadruple sets are a generalization of neutrosophic sets. Sahin et al. in 2020 defined generalized set-valued neutrosophic quadruple sets and numbers [18].Thanks to this new structure, neutrosophic quadruple sets can be used in decision-making applications. Şahin et al. In 2021, Şahin et al. defined generalized Euclidean measures based on generalized set-valued neutrosophic quadruple numbers and multi-criteria decision-making applications [19]. Kargın et al. In 2021, he studied on Generalized Hamming Similarity Measure Based on Neutrosophic Quadruple Sets and Applications to Legal Sciences [20]. In 2022, Şahin and Kargın defined interval generalized set-valued neutrosophic quadruple sets [21]. A new structure was obtained using interval neutrosophic sets and generalized set-valued neutrosophic quadruple sets together. Sahin et al. In 2022, Pythagorean neutrosophic quadruple set and set-valued Pythagorean neutrosophic quadruple set were defined [22]. In 2023, Kargın and Şahin defined neutrosophic triple normed spaces based on set-valued neutrosophic quadruple numbers [23]. Sahin et al. in 2023 defined interval generalized set-valued neutrosophic quadruple graphs [24]. Şahin and Kargın defined interval generalized set-valued Pythagorean neutrosophic quadruple numbers in 2023 [25].

Sahin et al. In 2022, neutrosophic quintuple sets and numbers were defined [26]. Neutrosophic quintuple set have truth (T), uncertainty (U), contradiction (C) and falsity (F) values as in quadripartitioned neutrosophic sets, but unlike quadripartitioned neutrosophic sets, they also have known and unknown parts such that (i)+(jT+kU+lC+mF) (i,j,k,l,m∈ ℝ or ℂ) [26]. Thus, this set is named with neutrosophic quadruple set because of five components (i, jT, kU, lC, mF). Thus, this new structure, which is a generalization of neutrosophic quadruple sets and quadripartitioned neutrosophic sets, provides the properties of both neutrosophic quadruple sets and quadripartitioned neutrosophic sets. In 2022, Sahin and Kargin defined interval set-valued neutrosophic quintuple sets (ISVNQS) and numbers (ISVNQN) [27]. These new sets, which are a generalization of neutrosophic quintuple sets and interval quadruple neutrosophic sets, satisfy the properties of both neutrosophic quintuple sets and interval quadruple neutrosophic sets. In 2022, Şahin and Kargın defined bipolar neutrosophic quintuple set and bipolar set-valued neutrosophic

quintuple set [28]. Sahin et al. defined generalized set-valued neutrosophic quintuple set in 2023 [29]. This new structure, which is a generalized form of set-valued neutrosophic quintuple set, provides the properties of set-valued neutrosophic quintuple set. Şahin et al. in 2024 defined single-valued neutrosophic quintuple graphs [30]. Thus, a new structure using generalized set-valued neutrosophic quintuple set and graph theory is obtained. Sahin et al. 2024 defined interval generalized set-valued neutrosophic quintuple sets (IGSVNQS) and numbers (IGSVNQN) and (IGSVNQS) graphs [31]. Thus, a new structure was added to both neutrosophic quintuple set theory and graph theory. Also, Chatterjee et al. defined quadripartitioned neutrosophic set [32] in 2016. Pramanik obtained interval quadripartitioned neutrosophic set (IQNS) [33] in 2022.

In this study, in the Preliminaries Section, the basic definitions and properties to be used in the study are given. In the Research and Fundigs Section, some constructions related to (IGSVNQS) and score and certainty functions are defined. Moreover, optimistic union, pessimistic union, mean union, optimistic intersection, pessimistic intersection and mean intersection operators are defined for IGSVNQS. Thus, IGSVNQS are made available for decision making applications. In the Application Section, a sample application was made using the functions and some operators defined for IGSVNQS. With this application, the use of operators and functions in decision making processes is concretely demonstrated. In addition, it was shown that similar or different results can be obtained by using different operators and the importance of choosing the appropriate operator for the problem in the decision making process was emphasised. In the Conclusion Section, the findings of the study are summarised and suggestions for future studies are presented.

2. Preliminaries

Definition 2.1: [2] Let $\mathcal{P}$ be a universal set. For $\forall \breve{p} \in \mathcal{P}$,

a SVNS $\tilde{A}$ over $\mathcal{P}$ with functions; is defined as

$$\tilde{A} = \{\langle \breve{p}, T_{\tilde{A}(\breve{p})}, I_{\tilde{A}(\breve{p})}, F_{\tilde{A}(\breve{p})}\rangle : \breve{p} \in \mathcal{P}\}$$

such that

$$0 \leq T_{\tilde{A}(\breve{p})} + I_{\tilde{A}(\breve{p})} + F_{\tilde{A}(\breve{p})} \leq 3$$

$$T_{\tilde{A}}: \mathcal{P} \to [0,1] \ , \ I_{\tilde{A}}: \mathcal{P} \to [0,1] \quad \text{and} \quad F_{\tilde{A}}: \mathcal{P} \to [0,1]$$

Where $T_{\tilde{A}(\breve{p})}, I_{\tilde{A}(\breve{p})}$ and $F_{\tilde{A}(\breve{p})}$ are the degree of truth, degree of indeterminancy and degree of falsity of $\breve{p} \in \mathcal{P}$ respectively.

Definition 2.2:[10] Let $\boldsymbol{\mathcal{P}} \neq \emptyset$. An INS is defined as

$$\widetilde{\boldsymbol{A}} = \{< \breve{\boldsymbol{p}}, [\boldsymbol{T^L}_{\tilde{\boldsymbol{A}}(\breve{\boldsymbol{p}})}, \boldsymbol{T^U}_{\tilde{\boldsymbol{A}}(\breve{\boldsymbol{p}})}], [\boldsymbol{I^L}_{\tilde{\boldsymbol{A}}(\breve{\boldsymbol{p}})}, \boldsymbol{I^U}_{\tilde{\boldsymbol{A}}(\breve{\boldsymbol{p}})}], [\boldsymbol{F^L}_{\tilde{\boldsymbol{A}}(\breve{\boldsymbol{p}})}, \boldsymbol{F^U}_{\tilde{\boldsymbol{A}}(\breve{\boldsymbol{p}})}] >: \breve{\boldsymbol{p}} \in \boldsymbol{\mathcal{P}}\}.$$

Here,

$[\boldsymbol{T^L}_{\tilde{\boldsymbol{A}}(\breve{\boldsymbol{p}})}, \boldsymbol{T^U}_{\tilde{\boldsymbol{A}}(\breve{\boldsymbol{p}})}], [\boldsymbol{I^L}_{\tilde{\boldsymbol{A}}(\breve{\boldsymbol{p}})}, \boldsymbol{I^U}_{\tilde{\boldsymbol{A}}(\breve{\boldsymbol{p}})}]$ and $[\boldsymbol{F^L}_{\tilde{\boldsymbol{A}}(\breve{\boldsymbol{p}})}, \boldsymbol{F^U}_{\tilde{\boldsymbol{A}}(\breve{\boldsymbol{p}})}]$ are intervals

$\boldsymbol{T^L}_{\tilde{\boldsymbol{A}}}: \boldsymbol{\mathcal{P}} \to [0,1]$, $\boldsymbol{T^U}_{\tilde{\boldsymbol{A}}}: \boldsymbol{\mathcal{P}} \to [0,1]$ truth functions,

$\boldsymbol{I^L}_{\tilde{\boldsymbol{A}}}: \boldsymbol{\mathcal{P}} \to [0,1]$, $\boldsymbol{I^U}_{\tilde{\boldsymbol{A}}}: \boldsymbol{\mathcal{P}} \to [0,1]$ indeterminancy functions,

$\boldsymbol{F^L}_{\tilde{\boldsymbol{A}}}: \boldsymbol{\mathcal{P}} \to [0,1]$, $\boldsymbol{F^U}_{\tilde{\boldsymbol{A}}}: \boldsymbol{\mathcal{P}} \to [0,1]$ are falsity functions.

Definition 2.3: [32] Let $\mathcal{P} \neq \emptyset$. An IQNS is defined as

$$\widetilde{A} = \{< \breve{p}: \left[T^L_{\widetilde{A}(\breve{p})}, T^U_{\widetilde{A}(\breve{p})}\right], \left[U^L_{\widetilde{A}(\breve{p})}, U^U_{\widetilde{A}(\breve{p})}\right], \left[C^L_{\widetilde{A}(\breve{p})}, C^U_{\widetilde{A}(\breve{p})}\right], \left[F^L_{\widetilde{A}(\breve{p})}, F^U_{\widetilde{A}(\breve{p})}\right] >; \breve{p} \in \mathcal{P}\}$$

Here,

$\left[T^L_{\widetilde{A}(\breve{p})}, T^U_{\widetilde{A}(\breve{p})}\right], \left[U^L_{\widetilde{A}(\breve{p})}, U^U_{\widetilde{A}(\breve{p})}\right], \left[C^L_{\widetilde{A}(\breve{p})}, C^U_{\widetilde{A}(\breve{p})}\right]$ and $\left[F^L_{\widetilde{A}(\breve{p})}, F^U_{\widetilde{A}(\breve{p})}\right]$ are intervals

$T^L_{\widetilde{A}} : \mathcal{P} \to [0,1]$, $T^U_{\widetilde{A}} : \mathcal{P} \to [0,1]$ truth functions,

$U^L_{\widetilde{A}} : \mathcal{P} \to [0,1]$ $U^U_{\widetilde{A}} : \mathcal{P} \to [0,1]$ unknownability functions,

$C^L_{\widetilde{A}} : \mathcal{P} \to [0,1]$, $C^U_{\widetilde{A}} : \mathcal{P} \to [0,1]$ contradiction functions,

$F^L_{\widetilde{A}} : \mathcal{P} \to [0,1]$, $F^U_{\widetilde{A}} : \mathcal{P} \to [0,1]$ falsity functions.

Definition 2.4:[19] Let $\mathcal{N}$ be a set and $\boldsymbol{P}(\mathcal{N})$ be the power set of $\mathcal{N}$. An ISVNQN is represented as

$$\widetilde{\mathcal{R}}_N = (\breve{\mathcal{P}},\ \breve{\mathcal{P}}_T[T^L,\ T^U],\ \breve{\mathcal{P}}_U[U^L,\ U^U],\ \breve{\mathcal{P}}_C[C^L,\ C^U],\ \breve{\mathcal{P}}_F[F^L,\ F^U])$$

Here, the components $T^L,\ T^U, U^L,\ U^U, C^L,\ C^U, F^L,\ F^U$ are the truth, unknownability, contradiction and falsity functions in the IQNS. In addition,

$$\breve{\mathcal{P}},\ \breve{\mathcal{P}}_T, \breve{\mathcal{P}}_U,\ \breve{\mathcal{P}}_C, \breve{\mathcal{P}}_F \in \mathbf{P}(\mathcal{N})$$

Also, an ISVNQS

$$\widetilde{\mathcal{R}}=\{(\breve{\mathcal{P}},\ \breve{\mathcal{P}}_T[T^L,\ T^U],\ \breve{\mathcal{P}}_U[U^L,\ U^U],\ \breve{\mathcal{P}}_C[C^L,\ C^U],\ \breve{\mathcal{P}}_F[F^L,\ F^U]):\breve{\mathcal{P}},\ \breve{\mathcal{P}}_T, \breve{\mathcal{P}}_U,\ \breve{\mathcal{P}}_C, \breve{\mathcal{P}}_F \in \mathbf{P}(\mathcal{N})\}$$

is shown in the form. Here it is,

$$(\breve{\mathcal{P}},\ \breve{\mathcal{P}}_T[T^L,\ T^U],\ \breve{\mathcal{P}}_U[U^L,\ U^U],\ \breve{\mathcal{P}}_C[C^L,\ C^U],\ \breve{\mathcal{P}}_F[F^L,\ F^U])$$

represents a number, idea, object, etc. Also

$$\text{“}\breve{\mathcal{P}}\text{”}$$

is called known part and

$$\text{‘‘}(\breve{\mathcal{P}}_T[T^L,\ T^U],\ \breve{\mathcal{P}}_U[U^L,\ U^U],\ \breve{\mathcal{P}}_C[C^L,\ C^U],\ \breve{\mathcal{P}}_F[F^L,\ F^U])\text{’’}$$

is called the unknown part.

Definition 2.5:[19] Let

$$(\widetilde{\mathcal{R}}_N)_1 = (\breve{\mathcal{P}}_1,\ \breve{\mathcal{P}}_{1_T}[T^l,\ T^u],\ \breve{\mathcal{P}}_{1_U}[U^l,\ U^u],\ \breve{\mathcal{P}}_{1_C}[C^l,\ C^u],\ \breve{\mathcal{P}}_{1_F}[F^l,\ F^u])$$

and

$$(\widetilde{\mathcal{R}}_N)_2 = (\breve{\mathcal{P}}_2,\ \breve{\mathcal{P}}_{2_T}[T^l,\ T^u],\ \breve{\mathcal{P}}_{2_U}[U^l,\ U^u],\ \breve{\mathcal{P}}_{2_C}[C^l,\ C^u],\ \breve{\mathcal{P}}_{2_F}[F^l,\ F^u])$$

be two ISVNQSs.

The operations $\cup$ and $\cap$ are defined for ISVNQSs as following

i)$(\widetilde{\mathcal{R}}_N)_1 \cup (\widetilde{\mathcal{R}}_N)_2$= $(\breve{\mathcal{P}}_1 \cup \breve{\mathcal{P}}_2$,$(\breve{\mathcal{P}}_{1_T} \cup \breve{\mathcal{P}}_{2_T})[T^l,\ T^u]$,$(\breve{\mathcal{P}}_{1_U} \cup \breve{\mathcal{P}}_{2_U})[U^l,\ U^u]$,$(\breve{\mathcal{P}}_{1_C} \cup \breve{\mathcal{P}}_{2_C})[C^l,\ C^u]$,

$(\breve{\mathcal{P}}_{1_F} \cup \breve{\mathcal{P}}_{2_F})[F^l,\ F^u]$)

ii)$(\widetilde{\mathcal{R}}_N)_1 \cap (\widetilde{\mathcal{R}}_N)_2$= $(\breve{\mathcal{P}}_1 \cap \breve{\mathcal{P}}_2$,$(\breve{\mathcal{P}}_{1_T} \cap \breve{\mathcal{P}}_{2_T})[T^l,\ T^u]$,$(\breve{\mathcal{P}}_{1_U} \cap \breve{\mathcal{P}}_{2_U})[U^l,\ U^u]$,$(\breve{\mathcal{P}}_{1_C} \cap \breve{\mathcal{P}}_{2_C})[C^l,\ C^u]$,

$(\breve{\mathcal{P}}_{1_F} \cap \breve{\mathcal{P}}_{2_F})[F^l,\ F^u]$)

Definition 2.6: [31] Let $\mathcal{N} \neq \emptyset$ and $\boldsymbol{P}(\mathcal{N})$ be the power set of $\mathcal{N}$. An IGSVNQS

$\widetilde{\mathcal{R}} = \{< \breve{\mathcal{P}}_{1_{\widetilde{\mathcal{R}}_1}},\ \breve{\mathcal{P}}_{1_{T_{\widetilde{\mathcal{R}}_1}}}[\ T^L_{\widetilde{\mathcal{R}}_1}, T^U_{\widetilde{\mathcal{R}}_1}],\ \breve{\mathcal{P}}_{1_{U_{\widetilde{\mathcal{R}}_1}}}[U^L_{\widetilde{\mathcal{R}}_1},\ U^U_{\widetilde{\mathcal{R}}_1}],\ \breve{\mathcal{P}}_{1_{C_{\widetilde{\mathcal{R}}_1}}}[C^L_{\widetilde{\mathcal{R}}_1},\ C^U_{\widetilde{\mathcal{R}}_1}],\breve{\mathcal{P}}_{1_{F_{\widetilde{\mathcal{R}}_1}}}[F^L_{\widetilde{\mathcal{R}}_1}, F^U_{\widetilde{\mathcal{R}}_1}];$

$\breve{\mathcal{P}}_{2_{\widetilde{\mathcal{R}}_2}},\ \breve{\mathcal{P}}_{2_{T_{\widetilde{\mathcal{R}}_2}}}[\ T^L_{\widetilde{\mathcal{R}}_2}, T^U_{\widetilde{\mathcal{R}}_2}],\ \breve{\mathcal{P}}_{2_{U_{\widetilde{\mathcal{R}}_2}}}[U^L_{\widetilde{\mathcal{R}}_2},\ U^U_{\widetilde{\mathcal{R}}_2}],\ \breve{\mathcal{P}}_{2_{C_{\widetilde{\mathcal{R}}_2}}}[C^L_{\widetilde{\mathcal{R}}_2},\ C^U_{\widetilde{\mathcal{R}}_2}],\breve{\mathcal{P}}_{2_{F_{\widetilde{\mathcal{R}}_2}}}[F^L_{\widetilde{\mathcal{R}}_2}, F^U_{\widetilde{\mathcal{R}}_2}];$

...

$\breve{\mathcal{P}}_{s_{\widetilde{\mathcal{R}}_s}},\ \breve{\mathcal{P}}_{s_{T_{\widetilde{\mathcal{R}}_s}}}[\ T^L_{\widetilde{\mathcal{R}}_s}, T^U_{\widetilde{\mathcal{R}}_s}],\ \breve{\mathcal{P}}_{s_{U_{\widetilde{\mathcal{R}}_s}}}[U^L_{\widetilde{\mathcal{R}}_s},\ U^U_{\widetilde{\mathcal{R}}_s}],\ \breve{\mathcal{P}}_{s_{C_{\widetilde{\mathcal{R}}_s}}}[C^L_{\widetilde{\mathcal{R}}_s},\ C^U_{\widetilde{\mathcal{R}}_s}],\breve{\mathcal{P}}_{s_{F_{\widetilde{\mathcal{R}}_s}}}[F^L_{\widetilde{\mathcal{R}}_s}, F^U_{\widetilde{\mathcal{R}}_s}] >,$

$\breve{\mathcal{P}}_{l_{\widetilde{\mathcal{R}}_l}}, \breve{\mathcal{P}}_{l_{T_{\widetilde{\mathcal{R}}_l}}}, \breve{\mathcal{P}}_{l_{U_{\widetilde{\mathcal{R}}_l}}}, \breve{\mathcal{P}}_{l_{C_{\widetilde{\mathcal{R}}_l}}}, \breve{\mathcal{P}}_{l_{F_{\widetilde{\mathcal{R}}_l}}} \in \boldsymbol{P}(\mathcal{N}); \boldsymbol{l} = \boldsymbol{1}, \boldsymbol{2}, \boldsymbol{3}, \dots, \boldsymbol{s}\}$

is shown in the form. Where $T^L_{\widetilde{\mathcal{R}}_l},\ T^U_{\widetilde{\mathcal{R}}_l}, U^L_{\widetilde{\mathcal{R}}_l},\ U^U_{\widetilde{\mathcal{R}}_l},\ C^L_{\widetilde{\mathcal{R}}_l},\ C^U_{\widetilde{\mathcal{R}}_l},\ F^L_{\widetilde{\mathcal{R}}_l}, F^U_{\widetilde{\mathcal{R}}_l}$ the components are the degree of truth, degree of uncertainty, degree of contradiction and degree of falsity.

Also, an IGSVNQN

$$\left(\widetilde{\mathcal{R}}_N\right)_1 = \{< \breve{\mathcal{P}}_{1_{\widetilde{\mathcal{R}}_1}},\ \breve{\mathcal{P}}_{1_{T_{\widetilde{\mathcal{R}}_1}}}[\ T^L_{\widetilde{\mathcal{R}}_1}, T^U_{\widetilde{\mathcal{R}}_1}],\ \breve{\mathcal{P}}_{1_{U_{\widetilde{\mathcal{R}}_1}}}[U^L_{\widetilde{\mathcal{R}}_1},\ U^U_{\widetilde{\mathcal{R}}_1}],\ \breve{\mathcal{P}}_{1_{C_{\widetilde{\mathcal{R}}_1}}}[C^L_{\widetilde{\mathcal{R}}_1},\ C^U_{\widetilde{\mathcal{R}}_1}],\ \breve{\mathcal{P}}_{1_{F_{\widetilde{\mathcal{R}}_1}}}[F^L_{\widetilde{\mathcal{R}}_1}, F^U_{\widetilde{\mathcal{R}}_1}] >\}$$

is shown in the form. Here, it is shown as

$$\text{“}\breve{\mathcal{P}}_{1_{\widetilde{\mathcal{R}}_1}}\text{”}$$

known part and

$$\text{“}\breve{\mathcal{P}}_{1_{T_{\widetilde{\mathcal{R}}_1}}}[T^L_{\widetilde{\mathcal{R}}_1}, T^U_{\widetilde{\mathcal{R}}_1}],\ \breve{\mathcal{P}}_{1_{U_{\widetilde{\mathcal{R}}_1}}}[U^L_{\widetilde{\mathcal{R}}_1},\ U^U_{\widetilde{\mathcal{R}}_1}],\ \breve{\mathcal{P}}_{1_{C_{\widetilde{\mathcal{R}}_1}}}[C^L_{\widetilde{\mathcal{R}}_1},\ C^U_{\widetilde{\mathcal{R}}_1}],\ \breve{\mathcal{P}}_{1_{F_{\widetilde{\mathcal{R}}_1}}}[F^L_{\widetilde{\mathcal{R}}_1}, F^U_{\widetilde{\mathcal{R}}_1}]\text{”}$$

is called the unknown part.

It can also be represented as

$$\widetilde{\mathcal{R}} = \{\left(\widetilde{\mathcal{R}}_N\right)_l : \boldsymbol{l} = \boldsymbol{1}, \boldsymbol{2}, \dots, \boldsymbol{s}\}.$$

3. Research and Findings

3.1. Some Structures Related to Interval Generalized Set Valued Neutrosophic Quintuple Sets

Definition 3.1.1: Let

$$\tilde{\mathcal{R}} = \{< \breve{\mathcal{P}}_{1\tilde{\mathcal{R}}_1},\ \breve{\mathcal{P}}_{1T\tilde{\mathcal{R}}_1}[\ T^L_{\tilde{\mathcal{R}}_1}, T^U_{\tilde{\mathcal{R}}_1}],\ \breve{\mathcal{P}}_{1U\tilde{\mathcal{R}}_1}[U^L_{\tilde{\mathcal{R}}_1},\ U^U_{\tilde{\mathcal{R}}_1}],\ \breve{\mathcal{P}}_{1C\tilde{\mathcal{R}}_1}[C^L_{\tilde{\mathcal{R}}_1},\ C^U_{\tilde{\mathcal{R}}_1}], \breve{\mathcal{P}}_{1F\tilde{\mathcal{R}}_1}[F^L_{\tilde{\mathcal{R}}_1}, F^U_{\tilde{\mathcal{R}}_1}];$$

$$\breve{\mathcal{P}}_{2\tilde{\mathcal{R}}_2},\ \breve{\mathcal{P}}_{2T\tilde{\mathcal{R}}_2}[\ T^L_{\tilde{\mathcal{R}}_2}, T^U_{\tilde{\mathcal{R}}_2}],\ \breve{\mathcal{P}}_{2U\tilde{\mathcal{R}}_2}[U^L_{\tilde{\mathcal{R}}_2},\ U^U_{\tilde{\mathcal{R}}_2}],\ \breve{\mathcal{P}}_{2C\tilde{\mathcal{R}}_2}[C^L_{\tilde{\mathcal{R}}_2},\ C^U_{\tilde{\mathcal{R}}_2}], \breve{\mathcal{P}}_{2F\tilde{\mathcal{R}}_2}[F^L_{\tilde{\mathcal{R}}_2}, F^U_{\tilde{\mathcal{R}}_2}];$$

$$\dots$$

$$\breve{\mathcal{P}}_{s\tilde{\mathcal{R}}_s},\ \breve{\mathcal{P}}_{sT\tilde{\mathcal{R}}_s}[\ T^L_{\tilde{\mathcal{R}}_s}, T^U_{\tilde{\mathcal{R}}_s}],\ \breve{\mathcal{P}}_{sU\tilde{\mathcal{R}}_s}[U^L_{\tilde{\mathcal{R}}_s},\ U^U_{\tilde{\mathcal{R}}_s}],\ \breve{\mathcal{P}}_{sC\tilde{\mathcal{R}}_s}[C^L_{\tilde{\mathcal{R}}_s},\ C^U_{\tilde{\mathcal{R}}_s}], \breve{\mathcal{P}}_{sF\tilde{\mathcal{R}}_s}[F^L_{\tilde{\mathcal{R}}_s}, F^U_{\tilde{\mathcal{R}}_s}] >,$$

$$\breve{\mathcal{P}}_{l\tilde{\mathcal{R}}_l}, \breve{\mathcal{P}}_{lT\tilde{\mathcal{R}}_l}, \breve{\mathcal{P}}_{lU\tilde{\mathcal{R}}_l}, \breve{\mathcal{P}}_{lC\tilde{\mathcal{R}}_l}, \breve{\mathcal{P}}_{lF\tilde{\mathcal{R}}_l} \in P(\mathcal{N}); l = 1,2,3,\dots,s\}$$

and

$$\tilde{\mathcal{H}} = \{< \breve{\mathcal{P}}_{1\tilde{\mathcal{H}}_1},\ \breve{\mathcal{P}}_{1T\tilde{\mathcal{H}}_1}[\ T^L_{\tilde{\mathcal{H}}_1}, T^U_{\tilde{\mathcal{H}}_1}],\ \breve{\mathcal{P}}_{1U\tilde{\mathcal{H}}_1}[U^L_{\tilde{\mathcal{H}}_1},\ U^U_{\tilde{\mathcal{H}}_1}],\ \breve{\mathcal{P}}_{1C\tilde{\mathcal{H}}_1}[C^L_{\tilde{\mathcal{H}}_1},\ C^U_{\tilde{\mathcal{H}}_1}], \breve{\mathcal{P}}_{1F\tilde{\mathcal{H}}_1}[F^L_{\tilde{\mathcal{H}}_1}, F^U_{\tilde{\mathcal{H}}_1}];$$

$$\breve{\mathcal{P}}_{2\tilde{\mathcal{H}}_2},\ \breve{\mathcal{P}}_{2T\tilde{\mathcal{H}}_2}[\ T^L_{\tilde{\mathcal{H}}_2}, T^U_{\tilde{\mathcal{H}}_2}],\ \breve{\mathcal{P}}_{2U\tilde{\mathcal{H}}_2}[U^L_{\tilde{\mathcal{H}}_2},\ U^U_{\tilde{\mathcal{H}}_2}],\ \breve{\mathcal{P}}_{2C\tilde{\mathcal{H}}_2}[C^L_{\tilde{\mathcal{H}}_2},\ C^U_{\tilde{\mathcal{H}}_2}], \breve{\mathcal{P}}_{2F\tilde{\mathcal{H}}_2}[F^L_{\tilde{\mathcal{H}}_2}, F^U_{\tilde{\mathcal{H}}_2}];$$

$$\dots$$

$$\breve{\mathcal{P}}_{s\tilde{\mathcal{H}}_s},\ \breve{\mathcal{P}}_{sT\tilde{\mathcal{H}}_s}[\ T^L_{\tilde{\mathcal{H}}_s}, T^U_{\tilde{\mathcal{H}}_s}],\ \breve{\mathcal{P}}_{sU\tilde{\mathcal{H}}_s}[U^L_{\tilde{\mathcal{H}}_s},\ U^U_{\tilde{\mathcal{H}}_s}],\ \breve{\mathcal{P}}_{sC\tilde{\mathcal{H}}_s}[C^L_{\tilde{\mathcal{H}}_s},\ C^U_{\tilde{\mathcal{H}}_s}], \breve{\mathcal{P}}_{sF\tilde{\mathcal{H}}_s}[F^L_{\tilde{\mathcal{H}}_s}, F^U_{\tilde{\mathcal{H}}_s}] >,$$

$$\breve{\mathcal{P}}_{l\tilde{\mathcal{H}}_l}, \breve{\mathcal{P}}_{lT\tilde{\mathcal{H}}_l}, \breve{\mathcal{P}}_{lU\tilde{\mathcal{H}}_l}, \breve{\mathcal{P}}_{lC\tilde{\mathcal{H}}_l}, \breve{\mathcal{P}}_{lF\tilde{\mathcal{H}}_l} \in P(\mathcal{N}); l = 1,2,3,\dots,s\}$$

be IGSVNQSs.

i) $\tilde{\mathcal{H}}$ is a subset of $\tilde{\mathcal{R}}$ $\left(\tilde{\mathcal{R}} \subset' \tilde{\mathcal{H}}\right)$ if and only if

$$\breve{\mathcal{P}}_{l\tilde{\mathcal{R}}_l} \subset \breve{\mathcal{P}}_{l\tilde{\mathcal{H}}_l},\ \breve{\mathcal{P}}_{lT\tilde{\mathcal{R}}_l} \subset \breve{\mathcal{P}}_{lT\tilde{\mathcal{H}}_l},\ \breve{\mathcal{P}}_{lU\tilde{\mathcal{R}}_l} \subset \breve{\mathcal{P}}_{lU\tilde{\mathcal{H}}_l},\ \breve{\mathcal{P}}_{lC\tilde{\mathcal{R}}_l} \subset \breve{\mathcal{P}}_{lC\tilde{\mathcal{H}}_l},\ \breve{\mathcal{P}}_{lF\tilde{\mathcal{R}}_l} \subset \breve{\mathcal{P}}_{lF\tilde{\mathcal{H}}_l}$$

$$T^L_{\tilde{\mathcal{R}}_l} \le T^L_{\tilde{\mathcal{H}}_l},\ T^U_{\tilde{\mathcal{R}}_l} \le T^U_{\tilde{\mathcal{H}}_l},\ U^L_{\tilde{\mathcal{R}}_l} \ge U^L_{\tilde{\mathcal{H}}_l},\ U^U_{\tilde{\mathcal{R}}_l} \ge U^U_{\tilde{\mathcal{H}}_l}$$

$$C^L_{\tilde{\mathcal{R}}_l} \ge C^L_{\tilde{\mathcal{H}}_l},\ C^U_{\tilde{\mathcal{R}}_l} \ge C^U_{\tilde{\mathcal{H}}_l},\ F^L_{\tilde{\mathcal{R}}_l} \ge F^L_{\tilde{\mathcal{H}}_l},\ F^U_{\tilde{\mathcal{R}}_l} \ge F^U_{\tilde{\mathcal{H}}_l}$$

ii) $\tilde{\mathcal{R}}$ is equal to $\tilde{\mathcal{H}}$ $(\tilde{\mathcal{R}} =' \tilde{\mathcal{H}})$ if and only if

$$\breve{\mathcal{P}}_{l\tilde{\mathcal{R}}_l} = \breve{\mathcal{P}}_{l\tilde{\mathcal{H}}_l},\ \breve{\mathcal{P}}_{lT\tilde{\mathcal{R}}_l} = \breve{\mathcal{P}}_{lT\tilde{\mathcal{H}}_l},\ \breve{\mathcal{P}}_{lU\tilde{\mathcal{R}}_l} = \breve{\mathcal{P}}_{lU\tilde{\mathcal{H}}_l},\ \breve{\mathcal{P}}_{lC\tilde{\mathcal{R}}_l} = \breve{\mathcal{P}}_{lC\tilde{\mathcal{H}}_l},\ \breve{\mathcal{P}}_{lF\tilde{\mathcal{R}}_l} = \breve{\mathcal{P}}_{lF\tilde{\mathcal{H}}_l}$$

$$T^L_{\tilde{\mathcal{R}}_l} = T^L_{\tilde{\mathcal{H}}_l},\ T^U_{\tilde{\mathcal{R}}_l} = T^U_{\tilde{\mathcal{H}}_l},\ U^L_{\tilde{\mathcal{R}}_l} = U^L_{\tilde{\mathcal{H}}_l},\ U^U_{\tilde{\mathcal{R}}_l} = U^U_{\tilde{\mathcal{H}}_l}$$

$$C^L_{\tilde{\mathcal{R}}_l} = C^L_{\tilde{\mathcal{H}}_l},\ C^U_{\tilde{\mathcal{R}}_l} = C^U_{\tilde{\mathcal{H}}_l},\ F^L_{\tilde{\mathcal{R}}_l} = F^L_{\tilde{\mathcal{H}}_l},\ F^U_{\tilde{\mathcal{R}}_l} = F^U_{\tilde{\mathcal{H}}_l}$$

Example 3.1.2: Let $\acute{A} = \{\acute{n}, \acute{t}, \acute{r}, \acute{s}, \acute{f}, \acute{k}, \acute{b}, \acute{l}\}$ be a set and

$\widehat{\mathcal{X}} = \{< \{\acute{n}, \acute{t}, \acute{l}\}, \{\acute{r}, \acute{f}, \acute{b}\}[0,0.6], \{\acute{n}\}[0.3,0,6], \{\acute{s}, \acute{k}\}[0.6,0.7], \{\acute{f}\}[0.6,0.9];$

$\{\acute{k}\}, \{\acute{t}, \acute{f}\}[0,0.5], \{\acute{b}\}[0.4,0.7], \{\acute{r}, \acute{s}\}[0.5,0.9], \{\acute{l}\}[0.2,0.6] >\},$

$\hat{\mathcal{Z}} = \{< \{\acute{n}, \acute{t}, \acute{s}, \acute{k}, \acute{l}\}, \{\acute{r}, \acute{f}, \acute{b}, \acute{l}\}[0,0.8], \{\acute{n}, \acute{l}\}[0.2,0.5], \{\acute{r}, \acute{s}, \acute{k}, \acute{l}\}[0.5,0.7], \{\acute{n}, \acute{f}\}[0.5,0.8];$

$\{k, \acute{l}\}, \{\acute{t}, \acute{f}, \acute{l}\}[0.1,0.6], \{\acute{b}, \acute{l}\}[0.3,0.4], \{\acute{n}, \acute{r}, \acute{s}\}[0.5,0.8], \{\acute{r}, \acute{l}\}[0.2,0.3] >\}$

be two IGSVNQSs.

$\left(\widehat{\mathcal{X}}_N\right)_1$ and $(\hat{\mathcal{Z}}_N)_1$ satisfy the conditions

$$\left((\widehat{\mathcal{X}}_N)_1 \subset' (\hat{\mathcal{Z}}_N)_1\right)$$

such that

$$\{\acute{n}, \acute{t}, \acute{l}\} \subset \{\acute{n}, \acute{t}, \acute{s}, \acute{k}, \acute{l}\},\ \{\acute{r}, \acute{f}, \acute{b}\} \subset \{\acute{r}, \acute{f}, \acute{b}, \acute{l}\},\ \{\acute{n}\} \subset \{\acute{n}, \acute{l}\},\ \{\acute{s}, \acute{k}\} \subset \{\acute{r}, \acute{s}, \acute{k}, \acute{l}\},\ \{\acute{f}\} \subset \{\acute{n}, \acute{f}\}$$

$$0 \leq 0,\ \ 0.6 \leq 0.8,\ 0.3 \geq 0{,}2,\ 0.6 \geq 0.5,\ 0{,}6 \geq 0{,}5,\ 0.7 \geq 0.7,\ 0.6 \geq 0.5,\ 0.9 \geq 0.8.$$

Similarly, the $\left(\widehat{\mathcal{X}}_N\right)_2$and $(\hat{\mathcal{Z}}_N)_2$ satisfy the conditions

$$\left((\widehat{\mathcal{X}}_N)_2 \subset' (\hat{\mathcal{Z}}_N)_2\right)$$

such that

$$\{k\} \subset \{k, \acute{l}\},\ \{\acute{t}, \acute{f}\} \subset \{\acute{t}, \acute{f}, \acute{l}\},\ \{\acute{b}\} \subset \{\acute{b}, \acute{l}\},\ \{\acute{r}, \acute{s}\} \subset \{\acute{n}, \acute{r}, \acute{s}\},\ \{\acute{r}\} \subset \{\acute{r}, \acute{l}\}$$

$$0 \leq 0.1,\ 0.5 \leq 0.6,\ 0.4 \geq 0.3,\ 0.7 \geq 0.4,\ 0.5 \geq 0.5,\ 0.9 \geq 0.8,\ 0.2 \geq 0.2,\ 0.6 \geq 0.3.$$

Thus, we obtain that

$$\widehat{\mathcal{X}} \subset' \hat{\mathcal{Z}}$$

Definition 3.1.3: Let

$\left(\tilde{\mathcal{R}}_N\right)_1$=$\{< \breve{\mathcal{P}}_{1_{\tilde{\mathcal{R}}_1}},\ \breve{\mathcal{P}}_{1_{T_{\tilde{\mathcal{R}}_1}}}[\ T^L_{\tilde{\mathcal{R}}_1}, T^U_{\tilde{\mathcal{R}}_1}],\ \breve{\mathcal{P}}_{1_{U_{\tilde{\mathcal{R}}_1}}}[U^L_{\tilde{\mathcal{R}}_1},\ U^U_{\tilde{\mathcal{R}}_1}],\ \breve{\mathcal{P}}_{1_{C_{\tilde{\mathcal{R}}_1}}}[C^L_{\tilde{\mathcal{R}}_1},\ C^U_{\tilde{\mathcal{R}}_1}],\ \breve{\mathcal{P}}_{1_{F_{\tilde{\mathcal{R}}_1}}}[F^L_{\tilde{\mathcal{R}}_1}, F^U_{\tilde{\mathcal{R}}_1}] >\}$

be an IGSVNQN. Score and accuracy functions for IGSVNQN is defined as

$$S_D((\mathcal{R}_N)_1) = \frac{\left|\ s\left(\breve{\mathcal{P}}_{1_{\tilde{\mathcal{R}}_1}} \cap \breve{\mathcal{P}}_{1_{T_{\tilde{\mathcal{R}}_1}}}\right) - \ s\left(\breve{\mathcal{P}}_{1_{\tilde{\mathcal{R}}_1}} \cap \breve{\mathcal{P}}_{1_{U_{\tilde{\mathcal{R}}_1}}}\right) - \ s\left(\breve{\mathcal{P}}_{1_{\tilde{\mathcal{R}}_1}} \cap \breve{\mathcal{P}}_{1_{C_{\tilde{\mathcal{R}}_1}}}\right) - \ s\left(\breve{\mathcal{P}}_{1_{\tilde{\mathcal{R}}_1}} \cap \breve{\mathcal{P}}_{1_{F_{\tilde{\mathcal{R}}_1}}}\right)\right|}{s\left(\breve{\mathcal{P}}_{1_{\tilde{\mathcal{R}}_1}}\right)} +$$

$$\frac{\left|T^L_{\tilde{\mathcal{R}}_1} - U^L_{\tilde{\mathcal{R}}_1} - C^L_{\tilde{\mathcal{R}}_1} - F^L_{\tilde{\mathcal{R}}_1}\right|}{3} - \frac{\left|T^U_{\tilde{\mathcal{R}}_1} - U^U_{\tilde{\mathcal{R}}_1} - C^U_{\tilde{\mathcal{R}}_1} - F^U_{\tilde{\mathcal{R}}_1}\right|}{3} + 1$$

$$K_D((\mathcal{R}_N)_1) = \frac{s\left(\breve{\mathcal{P}}_{1_{T_{\tilde{\mathcal{R}}_1}}} / \breve{\mathcal{P}}_{1_{U_{\tilde{\mathcal{R}}_1}}}\right) + s\left(\breve{\mathcal{P}}_{1_{T_{\tilde{\mathcal{R}}_1}}} / \breve{\mathcal{P}}_{1_{C_{\tilde{\mathcal{R}}_1}}}\right) + s\left(\breve{\mathcal{P}}_{1_{T_{\tilde{\mathcal{R}}_1}}} / \breve{\mathcal{P}}_{1_{F_{\tilde{\mathcal{R}}_1}}}\right)}{2.s(\breve{\mathcal{P}}_{1_{T_{\tilde{\mathcal{R}}_1}}})} + T^L_{\tilde{\mathcal{R}}_1} + T^U_{\tilde{\mathcal{R}}_1}$$

Where $s\left(\breve{\mathcal{P}}_{1_{\tilde{\mathcal{R}}_1}}\right)$ is the number of elements of the set $\breve{\mathcal{P}}_{1_{\tilde{\mathcal{R}}_1}}$,

$s\left(\breve{\mathcal{P}}_{1_{\tilde{\mathcal{R}}_1}} \cap \breve{\mathcal{P}}_{1_{T_{\tilde{\mathcal{R}}_1}}}\right)$ is the number of elements of the intersection of the sets $\breve{\mathcal{P}}_{1_{\tilde{\mathcal{R}}_1}}$ and $\breve{\mathcal{P}}_{1_{T_{\tilde{\mathcal{R}}_1}}}$,

$s\left(\breve{\mathcal{P}}_{1_{T_{\tilde{\mathcal{R}}_1}}}/\breve{\mathcal{P}}_{1_{U_{\tilde{\mathcal{R}}_1}}}\right)$ is the number of elements of the difference of the sets $\breve{\mathcal{P}}_{1_{T_{\tilde{\mathcal{R}}_1}}}$ and $\breve{\mathcal{P}}_{1_{U_{\tilde{\mathcal{R}}_1}}}$.

3.2. Operators for Interval Generalized Set Values Neutrosophic Quintuple Sets

Definition 3.2.1: Let

$$\tilde{\mathcal{R}} = \{< \breve{\mathcal{P}}_{1_{\tilde{\mathcal{R}}_1}},\ \breve{\mathcal{P}}_{1_{T_{\tilde{\mathcal{R}}_1}}}[\ T^L_{\tilde{\mathcal{R}}_1}, T^U_{\tilde{\mathcal{R}}_1}],\ \breve{\mathcal{P}}_{1_{U_{\tilde{\mathcal{R}}_1}}}[U^L_{\tilde{\mathcal{R}}_1},\ U^U_{\tilde{\mathcal{R}}_1}],\ \breve{\mathcal{P}}_{1_{C_{\tilde{\mathcal{R}}_1}}}[C^L_{\tilde{\mathcal{R}}_1},\ C^U_{\tilde{\mathcal{R}}_1}], \breve{\mathcal{P}}_{1_{F_{\tilde{\mathcal{R}}_1}}}[F^L_{\tilde{\mathcal{R}}_1}, F^U_{\tilde{\mathcal{R}}_1}];$$

$$\breve{\mathcal{P}}_{2_{\tilde{\mathcal{R}}_2}},\ \breve{\mathcal{P}}_{2_{T_{\tilde{\mathcal{R}}_2}}}[\ T^L_{\tilde{\mathcal{R}}_2}, T^U_{\tilde{\mathcal{R}}_2}],\ \breve{\mathcal{P}}_{2_{U_{\tilde{\mathcal{R}}_2}}}[U^L_{\tilde{\mathcal{R}}_2},\ U^U_{\tilde{\mathcal{R}}_2}],\ \breve{\mathcal{P}}_{2_{C_{\tilde{\mathcal{R}}_2}}}[C^L_{\tilde{\mathcal{R}}_2},\ C^U_{\tilde{\mathcal{R}}_2}], \breve{\mathcal{P}}_{2_{F_{\tilde{\mathcal{R}}_2}}}[F^L_{\tilde{\mathcal{R}}_2}, F^U_{\tilde{\mathcal{R}}_2}];$$

$$\ldots$$

$$\breve{\mathcal{P}}_{s_{\tilde{\mathcal{R}}_s}},\ \breve{\mathcal{P}}_{s_{T_{\tilde{\mathcal{R}}_s}}}[\ T^L_{\tilde{\mathcal{R}}_s}, T^U_{\tilde{\mathcal{R}}_s}],\ \breve{\mathcal{P}}_{s_{U_{\tilde{\mathcal{R}}_s}}}[U^L_{\tilde{\mathcal{R}}_s},\ U^U_{\tilde{\mathcal{R}}_s}],\ \breve{\mathcal{P}}_{s_{C_{\tilde{\mathcal{R}}_s}}}[C^L_{\tilde{\mathcal{R}}_s},\ C^U_{\tilde{\mathcal{R}}_s}], \breve{\mathcal{P}}_{s_{F_{\tilde{\mathcal{R}}_s}}}[F^L_{\tilde{\mathcal{R}}_s}, F^U_{\tilde{\mathcal{R}}_s}] >,$$

$$\breve{\mathcal{P}}_{l_{\tilde{\mathcal{R}}_l}}, \breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{R}}_l}}}, \breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{R}}_l}}}, \breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{R}}_l}}}, \breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{R}}_l}}} \in P(\mathcal{N}); l = 1,2,3,\ldots,s\}$$

and

$$\tilde{\mathcal{H}} = \{< \breve{\mathcal{P}}_{1_{\tilde{\mathcal{H}}_1}},\ \breve{\mathcal{P}}_{1_{T_{\tilde{\mathcal{H}}_1}}}[\ T^L_{\tilde{\mathcal{H}}_1}, T^U_{\tilde{\mathcal{H}}_1}],\ \breve{\mathcal{P}}_{1_{U_{\tilde{\mathcal{H}}_1}}}[U^L_{\tilde{\mathcal{H}}_1},\ U^U_{\tilde{\mathcal{H}}_1}],\ \breve{\mathcal{P}}_{1_{C_{\tilde{\mathcal{H}}_1}}}[C^L_{\tilde{\mathcal{H}}_1},\ C^U_{\tilde{\mathcal{H}}_1}], \breve{\mathcal{P}}_{1_{F_{\tilde{\mathcal{H}}_1}}}[F^L_{\tilde{\mathcal{H}}_1}, F^U_{\tilde{\mathcal{H}}_1}];$$

$$\breve{\mathcal{P}}_{2_{\tilde{\mathcal{H}}_2}},\ \breve{\mathcal{P}}_{2_{T_{\tilde{\mathcal{H}}_2}}}[\ T^L_{\tilde{\mathcal{H}}_2}, T^U_{\tilde{\mathcal{H}}_2}],\ \breve{\mathcal{P}}_{2_{U_{\tilde{\mathcal{H}}_2}}}[U^L_{\tilde{\mathcal{H}}_2},\ U^U_{\tilde{\mathcal{H}}_2}],\ \breve{\mathcal{P}}_{2_{C_{\tilde{\mathcal{H}}_2}}}[C^L_{\tilde{\mathcal{H}}_2},\ C^U_{\tilde{\mathcal{H}}_2}], \breve{\mathcal{P}}_{2_{F_{\tilde{\mathcal{H}}_2}}}[F^L_{\tilde{\mathcal{H}}_2}, F^U_{\tilde{\mathcal{H}}_2}];$$

$$\ldots$$

$$\breve{\mathcal{P}}_{s_{\tilde{\mathcal{H}}_s}},\ \breve{\mathcal{P}}_{s_{T_{\tilde{\mathcal{H}}_s}}}[\ T^L_{\tilde{\mathcal{H}}_s}, T^U_{\tilde{\mathcal{H}}_s}],\ \breve{\mathcal{P}}_{s_{U_{\tilde{\mathcal{H}}_s}}}[U^L_{\tilde{\mathcal{H}}_s},\ U^U_{\tilde{\mathcal{H}}_s}],\ \breve{\mathcal{P}}_{s_{C_{\tilde{\mathcal{H}}_s}}}[C^L_{\tilde{\mathcal{H}}_s},\ C^U_{\tilde{\mathcal{H}}_s}], \breve{\mathcal{P}}_{s_{F_{\tilde{\mathcal{H}}_s}}}[F^L_{\tilde{\mathcal{H}}_s}, F^U_{\tilde{\mathcal{H}}_s}] >,$$

$$\breve{\mathcal{P}}_{l_{\tilde{\mathcal{H}}_l}}, \breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{H}}_l}}}, \breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{H}}_l}}}, \breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{H}}_l}}}, \breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{H}}_l}}} \in P(\mathcal{N}); l = 1,2,3,\ldots,s\}$$

be two IGSVNQSs.

i. The "mean $\cup$'' operation for $\tilde{\mathcal{R}}$ and $\tilde{\mathcal{H}}$ is defined as

$$\tilde{\mathcal{R}} \cup'_M \tilde{\mathcal{H}} = \{< \breve{\mathcal{P}}_{1_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}},\ \breve{\mathcal{P}}_{1_{T_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}}}[\ T^L_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}, T^U_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}],\ \breve{\mathcal{P}}_{1_{U_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}}}[U^L_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1},\ U^U_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}],$$

$$\breve{\mathcal{P}}_{1_{C_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}}}[C^L_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}, C^U_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}], \breve{\mathcal{P}}_{1_{F_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}}}[F^L_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}, F^U_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}];$$

$$\breve{\mathcal{P}}_{2_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}},\ \breve{\mathcal{P}}_{2_{T_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}}}[\ T^L_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}, T^U_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}],\ \breve{\mathcal{P}}_{2_{U_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}}}[U^L_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2},\ U^U_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}],$$

$$\breve{\mathcal{P}}_{2_{C_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}}}[C^L_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}, C^U_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}], \breve{\mathcal{P}}_{2_{F_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}}}[F^L_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}, F^U_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}];$$

$$\ldots$$

$$\breve{\mathcal{P}}_{s_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}},\ \breve{\mathcal{P}}_{s_{T\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}}[\ T^L_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}, T^U_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}],\ \breve{\mathcal{P}}_{s_{U\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}}[U^L_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s},\ U^U_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}],$$

$$\breve{\mathcal{P}}_{s_{C\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}}[C^L_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}, C^U_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}], \breve{\mathcal{P}}_{s_{F\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}}[F^L_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}, F^U_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}] >,$$

$$\breve{\mathcal{P}}_{l_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}, \breve{\mathcal{P}}_{l_{T\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}, \breve{\mathcal{P}}_{l_{U\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}, \breve{\mathcal{P}}_{l_{C\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}, \breve{\mathcal{P}}_{l_{F\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}} \in P(\mathcal{N}); l = 1,2,3,\dots,s\}.$$

Here, for $l = 1,2,3,\dots,s$;

$$\breve{\mathcal{P}}_{l_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}} = \left(\breve{\mathcal{P}}_{l_{\tilde{\mathcal{R}}_l}} \cup \breve{\mathcal{P}}_{l_{\tilde{\mathcal{H}}_l}}\right),\ \breve{\mathcal{P}}_{l_{T\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}} = \left(\breve{\mathcal{P}}_{l_{T\tilde{\mathcal{R}}_l}} \cup \breve{\mathcal{P}}_{l_{T\tilde{\mathcal{H}}_l}}\right),\ \breve{\mathcal{P}}_{l_{U\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}} = \left(\breve{\mathcal{P}}_{l_{U\tilde{\mathcal{R}}_l}} \cup \breve{\mathcal{P}}_{l_{U\tilde{\mathcal{H}}_l}}\right)$$

$$\breve{\mathcal{P}}_{l_{C\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}} = \left(\breve{\mathcal{P}}_{l_{C\tilde{\mathcal{R}}_l}} \cup \breve{\mathcal{P}}_{l_{C\tilde{\mathcal{H}}_l}}\right),\ \breve{\mathcal{P}}_{l_{F\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}} = \left(\breve{\mathcal{P}}_{l_{F\tilde{\mathcal{R}}_l}} \cup \breve{\mathcal{P}}_{l_{F\tilde{\mathcal{H}}_l}}\right)$$

$$T^L_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \frac{T^L_{\tilde{\mathcal{R}}_l} + T^L_{\tilde{\mathcal{H}}_l}}{2},\ T^U_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \frac{T^U_{\tilde{\mathcal{R}}_l} + T^U_{\tilde{\mathcal{H}}_l}}{2},\ U^L_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \frac{U^L_{\tilde{\mathcal{R}}_l} + U^L_{\tilde{\mathcal{H}}_l}}{2},\ U^U_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \frac{U^U_{\tilde{\mathcal{R}}_l} + U^U_{\tilde{\mathcal{H}}_l}}{2}$$

$$C^L_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \frac{C^L_{\tilde{\mathcal{R}}_l} + C^L_{\tilde{\mathcal{H}}_l}}{2},\ C^U_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \frac{C^U_{\tilde{\mathcal{R}}_l} + C^U_{\tilde{\mathcal{H}}_l}}{2},\ F^L_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \frac{F^L_{\tilde{\mathcal{R}}_l} + F^L_{\tilde{\mathcal{H}}_l}}{2},\ F^U_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \frac{F^U_{\tilde{\mathcal{R}}_l} + F^U_{\tilde{\mathcal{H}}_l}}{2}$$

ii. For $\mathcal{R}$ and $\mathcal{H}$, the operation "mean $\cap$ '' is defined as

$$\tilde{\mathcal{R}} \cap'_M \tilde{\mathcal{H}} = \{< \breve{\mathcal{P}}_{1_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}},\ \breve{\mathcal{P}}_{1_{T\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}}[\ T^L_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}, T^U_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}],\ \breve{\mathcal{P}}_{1_{U\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}}[U^L_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1},\ U^U_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}],$$

$$\breve{\mathcal{P}}_{1_{C\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}}[C^L_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}, C^U_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}], \breve{\mathcal{P}}_{1_{F\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}}[F^L_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}, F^U_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}];$$

$$\breve{\mathcal{P}}_{2_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}},\ \breve{\mathcal{P}}_{2_{T\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}}[\ T^L_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}, T^U_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}],\ \breve{\mathcal{P}}_{2_{U\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}}[U^L_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2},\ U^U_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}],$$

$$\breve{\mathcal{P}}_{2_{C\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}}[C^L_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}, C^U_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}], \breve{\mathcal{P}}_{2_{F\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}}[F^L_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}, F^U_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}];$$

...

$$\breve{\mathcal{P}}_{s_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}},\ \breve{\mathcal{P}}_{s_{T\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}}[\ T^L_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}, T^U_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}],\ \breve{\mathcal{P}}_{s_{U\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}}[U^L_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s},\ U^U_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}],$$

$$\breve{\mathcal{P}}_{s_{C\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}}[C^L_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}, C^U_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}], \breve{\mathcal{P}}_{s_{F\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}}[F^L_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}, F^U_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}] >,$$

$$\breve{\mathcal{P}}_{l_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}, \breve{\mathcal{P}}_{l_{T\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}, \breve{\mathcal{P}}_{l_{U\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}, \breve{\mathcal{P}}_{l_{C\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}, \breve{\mathcal{P}}_{l_{F\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}} \in P(\mathcal{N}); l = 1,2,3,\dots,s\}$$

Here, for $l = 1,2,3,\dots,s$;

$$\breve{\mathcal{P}}_{l_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}} = \left(\breve{\mathcal{P}}_{l_{\tilde{\mathcal{R}}_l}} \cap \breve{\mathcal{P}}_{l_{\tilde{\mathcal{H}}_l}}\right),\ \breve{\mathcal{P}}_{l_{T\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}} = \left(\breve{\mathcal{P}}_{l_{T\tilde{\mathcal{R}}_l}} \cap \breve{\mathcal{P}}_{l_{T\tilde{\mathcal{H}}_l}}\right),\ \breve{\mathcal{P}}_{l_{U\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}} = \left(\breve{\mathcal{P}}_{l_{U\tilde{\mathcal{R}}_l}} \cap \breve{\mathcal{P}}_{l_{U\tilde{\mathcal{H}}_l}}\right)$$

$$\breve{\mathcal{P}}_{l_{C\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}} = \left(\breve{\mathcal{P}}_{l_{C\tilde{\mathcal{R}}_l}} \cap \breve{\mathcal{P}}_{l_{C\tilde{\mathcal{H}}_l}}\right),\ \breve{\mathcal{P}}_{l_{F\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}} = \left(\breve{\mathcal{P}}_{l_{F\tilde{\mathcal{R}}_l}} \cap \breve{\mathcal{P}}_{l_{F\tilde{\mathcal{H}}_l}}\right)$$

$$T^L_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \frac{T^L_{\tilde{\mathcal{R}}_l} + T^L_{\tilde{\mathcal{H}}_l}}{2},\ T^U_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \frac{T^U_{\tilde{\mathcal{R}}_l} + T^U_{\tilde{\mathcal{H}}_l}}{2},\ U^L_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \frac{U^L_{\tilde{\mathcal{R}}_l} + U^L_{\tilde{\mathcal{H}}_l}}{2},\ U^U_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \frac{U^U_{\tilde{\mathcal{R}}_l} + U^U_{\tilde{\mathcal{H}}_l}}{2}$$

$$C^L_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \frac{C^L_{\tilde{\mathcal{R}}_l} + C^L_{\tilde{\mathcal{H}}_l}}{2},\ C^U_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \frac{C^U_{\tilde{\mathcal{R}}_l} + C^U_{\tilde{\mathcal{H}}_l}}{2},\ F^L_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \frac{F^L_{\tilde{\mathcal{R}}_l} + F^L_{\tilde{\mathcal{H}}_l}}{2},\ F^U_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \frac{F^U_{\tilde{\mathcal{R}}_l} + F^U_{\tilde{\mathcal{H}}_l}}{2}$$

Example 3.2.2: From Example 3.1.2, for $\hat{\mathcal{X}}$ and $\hat{\mathcal{Z}}$, we obtain that

i. $\hat{\mathcal{X}} \cup'_M \hat{\mathcal{Z}} = \{< \{\acute{n}, \acute{t}, \acute{r}, \acute{s}, \acute{l}, \acute{k}\}, \{\acute{n}, \acute{r}, \acute{f}, \acute{b}, \acute{l}\}[0.05,0.7], \{\acute{n}, \acute{s}, \acute{l}\}[0.25,0.65], \{\acute{n}, \acute{r}, \acute{s}, \acute{l}, \acute{f}, \acute{k}\}[0.4,0.65],$

$\{\acute{t}, \acute{f}\}[0.45,0.8]; \{\acute{f}, \acute{k}, \acute{l}\}, \{\acute{t}, \acute{r}, \acute{s}, \acute{f}, \acute{l}\}[0.3,0.6], \{\acute{t}, \acute{r}, \acute{b}\}[0.4,0.65], \{\acute{n}, \acute{t}, \acute{r}, \acute{s}, \acute{l}\}[0.25,0.7], \{\acute{l}, \acute{k}\}[0.2,0.55] >\}$

ii. $\hat{\mathcal{X}} \cap'_M \hat{\mathcal{Z}} = \{< \{\acute{n}, \acute{t}, \acute{l}\}, \{\acute{f}, \acute{l}\}[0.05,0.7], \{\acute{l}\}[0.25,0.65], \{\acute{r}\}[0.4,0.65], \{\acute{f}\}[0.45,0.8];$

$\{\acute{k}\}, \{\acute{f}, \acute{l}\}[0.3,0.6], \{\acute{b}\}[0.4,0.65], \{\acute{n}, \acute{s}\}[0.25,0.7], \{\acute{l}\}[0.2,0.55] >\}$

Definition 3.2.3: Let

$\tilde{\mathcal{R}} = \{< \breve{\mathcal{P}}_{1_{\tilde{\mathcal{R}}_1}}, \breve{\mathcal{P}}_{1_{T_{\tilde{\mathcal{R}}_1}}}[T^L_{\tilde{\mathcal{R}}_1}, T^U_{\tilde{\mathcal{R}}_1}], \breve{\mathcal{P}}_{1_{U_{\tilde{\mathcal{R}}_1}}}[U^L_{\tilde{\mathcal{R}}_1}, U^U_{\tilde{\mathcal{R}}_1}], \breve{\mathcal{P}}_{1_{C_{\tilde{\mathcal{R}}_1}}}[C^L_{\tilde{\mathcal{R}}_1}, C^U_{\tilde{\mathcal{R}}_1}], \breve{\mathcal{P}}_{1_{F_{\tilde{\mathcal{R}}_1}}}[F^L_{\tilde{\mathcal{R}}_1}, F^U_{\tilde{\mathcal{R}}_1}];$

$\breve{\mathcal{P}}_{2_{\tilde{\mathcal{R}}_2}}, \breve{\mathcal{P}}_{2_{T_{\tilde{\mathcal{R}}_2}}}[T^L_{\tilde{\mathcal{R}}_2}, T^U_{\tilde{\mathcal{R}}_2}], \breve{\mathcal{P}}_{2_{U_{\tilde{\mathcal{R}}_2}}}[U^L_{\tilde{\mathcal{R}}_2}, U^U_{\tilde{\mathcal{R}}_2}], \breve{\mathcal{P}}_{2_{C_{\tilde{\mathcal{R}}_2}}}[C^L_{\tilde{\mathcal{R}}_2}, C^U_{\tilde{\mathcal{R}}_2}], \breve{\mathcal{P}}_{2_{F_{\tilde{\mathcal{R}}_2}}}[F^L_{\tilde{\mathcal{R}}_2}, F^U_{\tilde{\mathcal{R}}_2}];$

…

$\breve{\mathcal{P}}_{s_{\tilde{\mathcal{R}}_s}}, \breve{\mathcal{P}}_{s_{T_{\tilde{\mathcal{R}}_s}}}[T^L_{\tilde{\mathcal{R}}_s}, T^U_{\tilde{\mathcal{R}}_s}], \breve{\mathcal{P}}_{s_{U_{\tilde{\mathcal{R}}_s}}}[U^L_{\tilde{\mathcal{R}}_s}, U^U_{\tilde{\mathcal{R}}_s}], \breve{\mathcal{P}}_{s_{C_{\tilde{\mathcal{R}}_s}}}[C^L_{\tilde{\mathcal{R}}_s}, C^U_{\tilde{\mathcal{R}}_s}], \breve{\mathcal{P}}_{s_{F_{\tilde{\mathcal{R}}_s}}}[F^L_{\tilde{\mathcal{R}}_s}, F^U_{\tilde{\mathcal{R}}_s}] >,$

$\breve{\mathcal{P}}_{l_{\tilde{\mathcal{R}}_l}}, \breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{R}}_l}}}, \breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{R}}_l}}}, \breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{R}}_l}}}, \breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{R}}_l}}} \in P(\mathcal{N}); l = 1,2,3, \dots, s\}$

and

$\tilde{\mathcal{H}} = \{< \breve{\mathcal{P}}_{1_{\tilde{\mathcal{H}}_1}}, \breve{\mathcal{P}}_{1_{T_{\tilde{\mathcal{H}}_1}}}[T^L_{\tilde{\mathcal{H}}_1}, T^U_{\tilde{\mathcal{H}}_1}], \breve{\mathcal{P}}_{1_{U_{\tilde{\mathcal{H}}_1}}}[U^L_{\tilde{\mathcal{H}}_1}, U^U_{\tilde{\mathcal{H}}_1}], \breve{\mathcal{P}}_{1_{C_{\tilde{\mathcal{H}}_1}}}[C^L_{\tilde{\mathcal{H}}_1}, C^U_{\tilde{\mathcal{H}}_1}], \breve{\mathcal{P}}_{1_{F_{\tilde{\mathcal{H}}_1}}}[F^L_{\tilde{\mathcal{H}}_1}, F^U_{\tilde{\mathcal{H}}_1}];$

$\breve{\mathcal{P}}_{2_{\tilde{\mathcal{H}}_2}}, \breve{\mathcal{P}}_{2_{T_{\tilde{\mathcal{H}}_2}}}[T^L_{\tilde{\mathcal{H}}_2}, T^U_{\tilde{\mathcal{H}}_2}], \breve{\mathcal{P}}_{2_{U_{\tilde{\mathcal{H}}_2}}}[U^L_{\tilde{\mathcal{H}}_2}, U^U_{\tilde{\mathcal{H}}_2}], \breve{\mathcal{P}}_{2_{C_{\tilde{\mathcal{H}}_2}}}[C^L_{\tilde{\mathcal{H}}_2}, C^U_{\tilde{\mathcal{H}}_2}], \breve{\mathcal{P}}_{2_{F_{\tilde{\mathcal{H}}_2}}}[F^L_{\tilde{\mathcal{H}}_2}, F^U_{\tilde{\mathcal{H}}_2}];$

…

$\breve{\mathcal{P}}_{s_{\tilde{\mathcal{H}}_s}}, \breve{\mathcal{P}}_{s_{T_{\tilde{\mathcal{H}}_s}}}[T^L_{\tilde{\mathcal{H}}_s}, T^U_{\tilde{\mathcal{H}}_s}], \breve{\mathcal{P}}_{s_{U_{\tilde{\mathcal{H}}_s}}}[U^L_{\tilde{\mathcal{H}}_s}, U^U_{\tilde{\mathcal{H}}_s}], \breve{\mathcal{P}}_{s_{C_{\tilde{\mathcal{H}}_s}}}[C^L_{\tilde{\mathcal{H}}_s}, C^U_{\tilde{\mathcal{H}}_s}], \breve{\mathcal{P}}_{s_{F_{\tilde{\mathcal{H}}_s}}}[F^L_{\tilde{\mathcal{H}}_s}, F^U_{\tilde{\mathcal{H}}_s}] >,$

$\breve{\mathcal{P}}_{l_{\tilde{\mathcal{H}}_l}}, \breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{H}}_l}}}, \breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{H}}_l}}}, \breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{H}}_l}}}, \breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{H}}_l}}} \in P(\mathcal{N}); l = 1,2,3, \dots, s\}$

be two IGSVNQSs.

i. For $\tilde{\mathcal{R}}$ and $\tilde{\mathcal{H}}$, the operation " optimistic $\cup$'' is defined as

$\tilde{\mathcal{R}} \cup'_O \tilde{\mathcal{H}} = \{< \breve{\mathcal{P}}_{1_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}}, \breve{\mathcal{P}}_{1_{T_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}}}[T^L_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}, T^U_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}], \breve{\mathcal{P}}_{1_{U_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}}}[U^L_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}, U^U_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}],$

$\breve{\mathcal{P}}_{1_{C_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}}}[C^L_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}, C^U_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}], \breve{\mathcal{P}}_{1_{F_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}}}[F^L_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}, F^U_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}];$

$\breve{\mathcal{P}}_{2_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}}, \breve{\mathcal{P}}_{2_{T_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}}}[T^L_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}, T^U_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}], \breve{\mathcal{P}}_{2_{U_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}}}[U^L_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}, U^U_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}],$

$\breve{\mathcal{P}}_{2_{C_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}}}[C^L_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}, C^U_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}], \breve{\mathcal{P}}_{2_{F_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}}}[F^L_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}, F^U_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}];$

…

$\breve{\mathcal{P}}_{s_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}}, \breve{\mathcal{P}}_{s_{T_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}}}[T^L_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}, T^U_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}], \breve{\mathcal{P}}_{s_{U_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}}}[U^L_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}, U^U_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}],$

$\breve{\mathcal{P}}_{s_{C_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}}}[C^L_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}, C^U_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}], \breve{\mathcal{P}}_{s_{F_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}}}[F^L_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}, F^U_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}] >,$

$$\breve{\mathcal{P}}_{l_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}, \breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}}, \breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}}, \breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}}, \breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}} \in P(\mathcal{N}); l = 1,2,3,\dots,s\}$$

Here, for $l = 1,2,3,\dots,s$;

$$\breve{\mathcal{P}}_{l_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}} = \left(\breve{\mathcal{P}}_{l_{\tilde{\mathcal{R}}_l}} \cup \breve{\mathcal{P}}_{l_{\tilde{\mathcal{H}}_l}}\right),\ \breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}} = \left(\breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{R}}_l}}} \cup \breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{H}}_l}}}\right),\ \breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}} = \left(\breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{R}}_l}}} \cup \breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{H}}_l}}}\right)$$

$$\breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}} = \left(\breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{R}}_l}}} \cup \breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{H}}_l}}}\right),\ \breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}} = \left(\breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{R}}_l}}} \cup \breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{H}}_l}}}\right)$$

$$T^L_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \max\{T^L_{\tilde{\mathcal{R}}_l}, T^L_{\tilde{\mathcal{H}}_l}\},\ T^U_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \max\{T^U_{\tilde{\mathcal{R}}_l}, T^U_{\tilde{\mathcal{H}}_l}\}$$

$$U^L_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \min\{U^L_{\tilde{\mathcal{R}}_l}, U^L_{\tilde{\mathcal{H}}_l}\},\ U^U{}_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \min\{U^U_{\tilde{\mathcal{R}}_l}, U^U_{\tilde{\mathcal{H}}_l}\}$$

$$C^L_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \min\{C^L_{\tilde{\mathcal{R}}_l}, C^L_{\tilde{\mathcal{H}}_l}\},\ C^U_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \min\{C^U_{\tilde{\mathcal{R}}_l}, C^U_{\tilde{\mathcal{H}}_l}\}$$

$$F^L_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \min\{F^L_{\tilde{\mathcal{R}}_l}, F^L_{\mathcal{H}_l}\},\ F^U_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \min\{F^U_{\mathcal{R}_l}, F^U_{\tilde{\mathcal{H}}_l}\}$$

ii. For $\tilde{\mathcal{R}}$ and $\tilde{\mathcal{H}}$, the " optimistic $\cap$''operation is defined as

$$\tilde{\mathcal{R}} \cap'_O \tilde{\mathcal{H}} = \{< \breve{\mathcal{P}}_{1_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}},\ \breve{\mathcal{P}}_{1_{T_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}}}[\ T^L_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}, T^U_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}],\ \breve{\mathcal{P}}_{1_{U_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}}}[U^L_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1},\ U^U_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}],$$

$$\breve{\mathcal{P}}_{1_{C_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}}}[C^L_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}, C^U_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}], \breve{\mathcal{P}}_{1_{F_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}}}[F^L_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}, F^U_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}];$$

$$\breve{\mathcal{P}}_{2_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}},\ \breve{\mathcal{P}}_{2_{T_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}}}[\ T^L_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}, T^U_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}],\ \breve{\mathcal{P}}_{2_{U_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}}}[U^L_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2},\ U^U_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}],$$

$$\breve{\mathcal{P}}_{2_{C_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}}}[C^L_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}, C^U_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}], \breve{\mathcal{P}}_{2_{F_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}}}[F^L_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}, F^U_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}];$$

...

$$\breve{\mathcal{P}}_{s_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}},\ \breve{\mathcal{P}}_{s_{T_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}}}[\ T^L_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}, T^U_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}],\ \breve{\mathcal{P}}_{s_{U_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}}}[U^L_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s},\ U^U_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}],$$

$$\breve{\mathcal{P}}_{s_{C_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}}}[C^L_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}, C^U_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}], \breve{\mathcal{P}}_{s_{F_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}}}[F^L_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}, F^U_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}] >,$$

$$\breve{\mathcal{P}}_{l_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}, \breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}}, \breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}}, \breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}}, \breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}} \in P(\mathcal{N}); l = 1,2,3,\dots,s\}$$

Here, for $l = 1,2,3,\dots,s$;

$$\breve{\mathcal{P}}_{l_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}} = \left(\breve{\mathcal{P}}_{l_{\tilde{\mathcal{R}}_l}} \cap \breve{\mathcal{P}}_{l_{\tilde{\mathcal{H}}_l}}\right),\ \breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}} = \left(\breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{R}}_l}}} \cap \breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{H}}_l}}}\right),\ \breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}} = \left(\breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{R}}_l}}} \cap \breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{H}}_l}}}\right)$$

$$\breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}} = \left(\breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{R}}_l}}} \cap \breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{H}}_l}}}\right),\ \breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}} = \left(\breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{R}}_l}}} \cap \breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{H}}_l}}}\right)$$

$$T^L_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \max\{T^L_{\tilde{\mathcal{R}}_l}, T^L_{\tilde{\mathcal{H}}_l}\},\ T^U_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \max\{T^U_{\tilde{\mathcal{R}}_l}, T^U_{\tilde{\mathcal{H}}_l}\}$$

$$U^L_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \min\{U^L_{\tilde{\mathcal{R}}_l}, U^L_{\tilde{\mathcal{H}}_l}\},\ U^U{}_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \min\{U^U_{\tilde{\mathcal{R}}_l}, U^U_{\tilde{\mathcal{H}}_l}\}$$

$$C^L_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \min\{C^L_{\tilde{\mathcal{R}}_l}, C^L_{\tilde{\mathcal{H}}_l}\},\ C^U_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \min\{C^U_{\tilde{\mathcal{R}}_l}, C^U_{\tilde{\mathcal{H}}_l}\}$$

$$F^L_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \min\{F^L_{\tilde{\mathcal{R}}_l}, F^L_{\mathcal{H}_l}\},\ F^U_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \min\{F^U_{\mathcal{R}_l}, F^U_{\tilde{\mathcal{H}}_l}\}$$

Example 3.2.4: From Example 3.1.2, for $\hat{\mathcal{X}}$ and $\hat{\mathcal{Z}}$, we obtain that

i. $\hat{\mathcal{X}} \cup_O' \hat{\mathcal{Z}} = \{< \{\acute{n}, \acute{t}, \acute{r}, \acute{s}, \acute{l}, \acute{k}\}, \{\acute{n}, \acute{r}, \acute{f}, \acute{b}, \acute{l}\}[0.1,0.8], \{\acute{n}, \acute{s}, \acute{l}\}[0.2,0.5], \{\acute{n}, \acute{r}, \acute{s}, \acute{l}, \acute{f}, \acute{k}\}[0.3,0.6],$

$\{\acute{t}, \acute{f}\}[0.3,0.7]; \{\acute{f}, \acute{k}, \acute{l}\}, \{\acute{t}, \acute{r}, \acute{s}, \acute{f}, \acute{l}\}[0.5,0.6], \{\acute{t}, \acute{r}, \acute{b}\}[0.3,0.4], \{\acute{n}, \acute{t}, \acute{r}, \acute{s}, \acute{l}\}[0.1,0.6], \{\acute{l}, \acute{k}\}[0.1,0.3] >\}$

ii. $\hat{\mathcal{X}} \cap_O' \hat{\mathcal{Z}} = \{< \{\acute{n}, \acute{t}, \acute{l}\}, \{\acute{f}, \acute{l}\}[0.1,0.8], \{\acute{l}\}[0.2,0.5], \{\acute{r}\}[0.3,0.6], \{\acute{f}\}[0.3,0.7];$

$\{\acute{k}\}, \{\acute{f}, \acute{l}\}[0.5,0.6], \{\acute{b}\}[0.3,0.4], \{\acute{n}, \acute{s}\}[0.1,0.6], \{\acute{l}\}[0.1,0.3] >\}$

Definition 3.2.5: Let

$\tilde{\mathcal{R}} = \{< \breve{\mathcal{P}}_{1\tilde{\mathcal{R}}_1}, \breve{\mathcal{P}}_{1T\tilde{\mathcal{R}}_1}[T^L_{\tilde{\mathcal{R}}_1}, T^U_{\tilde{\mathcal{R}}_1}], \breve{\mathcal{P}}_{1U\tilde{\mathcal{R}}_1}[U^L_{\tilde{\mathcal{R}}_1}, U^U_{\tilde{\mathcal{R}}_1}], \breve{\mathcal{P}}_{1C\tilde{\mathcal{R}}_1}[C^L_{\tilde{\mathcal{R}}_1}, C^U_{\tilde{\mathcal{R}}_1}], \breve{\mathcal{P}}_{1F\tilde{\mathcal{R}}_1}[F^L_{\tilde{\mathcal{R}}_1}, F^U_{\tilde{\mathcal{R}}_1}];$

$\breve{\mathcal{P}}_{2\tilde{\mathcal{R}}_2}, \breve{\mathcal{P}}_{2T\tilde{\mathcal{R}}_2}[T^L_{\tilde{\mathcal{R}}_2}, T^U_{\tilde{\mathcal{R}}_2}], \breve{\mathcal{P}}_{2U\tilde{\mathcal{R}}_2}[U^L_{\tilde{\mathcal{R}}_2}, U^U_{\tilde{\mathcal{R}}_2}], \breve{\mathcal{P}}_{2C\tilde{\mathcal{R}}_2}[C^L_{\tilde{\mathcal{R}}_2}, C^U_{\tilde{\mathcal{R}}_2}], \breve{\mathcal{P}}_{2F\tilde{\mathcal{R}}_2}[F^L_{\tilde{\mathcal{R}}_2}, F^U_{\tilde{\mathcal{R}}_2}];$

...

$\breve{\mathcal{P}}_{s\tilde{\mathcal{R}}_s}, \breve{\mathcal{P}}_{sT\tilde{\mathcal{R}}_s}[T^L_{\tilde{\mathcal{R}}_s}, T^U_{\tilde{\mathcal{R}}_s}], \breve{\mathcal{P}}_{sU\tilde{\mathcal{R}}_s}[U^L_{\tilde{\mathcal{R}}_s}, U^U_{\tilde{\mathcal{R}}_s}], \breve{\mathcal{P}}_{sC\tilde{\mathcal{R}}_s}[C^L_{\tilde{\mathcal{R}}_s}, C^U_{\tilde{\mathcal{R}}_s}], \breve{\mathcal{P}}_{sF\tilde{\mathcal{R}}_s}[F^L_{\tilde{\mathcal{R}}_s}, F^U_{\tilde{\mathcal{R}}_s}] >,$

$\breve{\mathcal{P}}_{l\tilde{\mathcal{R}}_l}, \breve{\mathcal{P}}_{lT\tilde{\mathcal{R}}_l}, \breve{\mathcal{P}}_{lU\tilde{\mathcal{R}}_l}, \breve{\mathcal{P}}_{lC\tilde{\mathcal{R}}_l}, \breve{\mathcal{P}}_{lF\tilde{\mathcal{R}}_l} \in P(\mathcal{N}); l = 1,2,3, \dots, s\}$

and

$\tilde{\mathcal{H}} = \{< \breve{\mathcal{P}}_{1\tilde{\mathcal{H}}_1}, \breve{\mathcal{P}}_{1T\tilde{\mathcal{H}}_1}[T^L_{\tilde{\mathcal{H}}_1}, T^U_{\tilde{\mathcal{H}}_1}], \breve{\mathcal{P}}_{1U\tilde{\mathcal{H}}_1}[U^L_{\tilde{\mathcal{H}}_1}, U^U_{\tilde{\mathcal{H}}_1}], \breve{\mathcal{P}}_{1C\tilde{\mathcal{H}}_1}[C^L_{\tilde{\mathcal{H}}_1}, C^U_{\tilde{\mathcal{H}}_1}], \breve{\mathcal{P}}_{1F\tilde{\mathcal{H}}_1}[F^L_{\tilde{\mathcal{H}}_1}, F^U_{\tilde{\mathcal{H}}_1}];$

$\breve{\mathcal{P}}_{2\tilde{\mathcal{H}}_2}, \breve{\mathcal{P}}_{2T\tilde{\mathcal{H}}_2}[T^L_{\tilde{\mathcal{H}}_2}, T^U_{\tilde{\mathcal{H}}_2}], \breve{\mathcal{P}}_{2U\tilde{\mathcal{H}}_2}[U^L_{\tilde{\mathcal{H}}_2}, U^U_{\tilde{\mathcal{H}}_2}], \breve{\mathcal{P}}_{2C\tilde{\mathcal{H}}_2}[C^L_{\tilde{\mathcal{H}}_2}, C^U_{\tilde{\mathcal{H}}_2}], \breve{\mathcal{P}}_{2F\tilde{\mathcal{H}}_2}[F^L_{\tilde{\mathcal{H}}_2}, F^U_{\tilde{\mathcal{H}}_2}];$

...

$\breve{\mathcal{P}}_{s\tilde{\mathcal{H}}_s}, \breve{\mathcal{P}}_{sT\tilde{\mathcal{H}}_s}[T^L_{\tilde{\mathcal{H}}_s}, T^U_{\tilde{\mathcal{H}}_s}], \breve{\mathcal{P}}_{sU\tilde{\mathcal{H}}_s}[U^L_{\tilde{\mathcal{H}}_s}, U^U_{\tilde{\mathcal{H}}_s}], \breve{\mathcal{P}}_{sC\tilde{\mathcal{H}}_s}[C^L_{\tilde{\mathcal{H}}_s}, C^U_{\tilde{\mathcal{H}}_s}], \breve{\mathcal{P}}_{sF\tilde{\mathcal{H}}_s}[F^L_{\tilde{\mathcal{H}}_s}, F^U_{\tilde{\mathcal{H}}_s}] >,$

$\breve{\mathcal{P}}_{l\tilde{\mathcal{H}}_l}, \breve{\mathcal{P}}_{lT\tilde{\mathcal{H}}_l}, \breve{\mathcal{P}}_{lU\tilde{\mathcal{H}}_l}, \breve{\mathcal{P}}_{lC\tilde{\mathcal{H}}_l}, \breve{\mathcal{P}}_{lF\tilde{\mathcal{H}}_l} \in P(\mathcal{N}); l = 1,2,3, \dots, s\}$

be two IGSVNQSs.

i. The "pessimistic $\cup$'' operation for $\tilde{\mathcal{R}}$ and $\tilde{\mathcal{H}}$ is defined as

$\tilde{\mathcal{R}} \cup_P' \tilde{\mathcal{H}} = \{< \breve{\mathcal{P}}_{1\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}, \breve{\mathcal{P}}_{1T\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}[T^L_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}, T^U_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}], \breve{\mathcal{P}}_{1U\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}[U^L_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}, U^U_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}],$

$\breve{\mathcal{P}}_{1C\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}[C^L_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}, C^U_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}], \breve{\mathcal{P}}_{1F\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}[F^L_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}, F^U_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}];$

$\breve{\mathcal{P}}_{2\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}, \breve{\mathcal{P}}_{2T\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}[T^L_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}, T^U_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}], \breve{\mathcal{P}}_{2U\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}[U^L_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}, U^U_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}],$

$\breve{\mathcal{P}}_{2C\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}[C^L_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}, C^U_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}], \breve{\mathcal{P}}_{2F\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}[F^L_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}, F^U_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}];$

...

$\breve{\mathcal{P}}_{s\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}, \breve{\mathcal{P}}_{sT\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}[T^L_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}, T^U_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}], \breve{\mathcal{P}}_{sU\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}[U^L_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}, U^U_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}],$

$$\breve{\mathcal{P}}_{s_{C\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}}[C^{L}{}_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}, C^{U}{}_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}], \breve{\mathcal{P}}_{s_{F\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}}[F^{L}{}_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}, F^{U}{}_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}] >,$$

$$\breve{\mathcal{P}}_{l_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}, \breve{\mathcal{P}}_{l_{T\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}, \breve{\mathcal{P}}_{l_{U\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}, \breve{\mathcal{P}}_{l_{C\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}, \breve{\mathcal{P}}_{l_{F\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}} \in P(\mathcal{N}); l = 1,2,3,\dots,s\}$$

Here, for $l = 1,2,3,\dots,s$;

$$\breve{\mathcal{P}}_{l_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}} = \left(\breve{\mathcal{P}}_{l_{\tilde{\mathcal{R}}_l}} \cup \breve{\mathcal{P}}_{l_{\tilde{\mathcal{H}}_l}}\right),\ \breve{\mathcal{P}}_{l_{T\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}} = \left(\breve{\mathcal{P}}_{l_{T\tilde{\mathcal{R}}_l}} \cup \breve{\mathcal{P}}_{l_{T\tilde{\mathcal{H}}_l}}\right),\ \breve{\mathcal{P}}_{l_{U\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}} = \left(\breve{\mathcal{P}}_{l_{U\tilde{\mathcal{R}}_l}} \cup \breve{\mathcal{P}}_{l_{U\tilde{\mathcal{H}}_l}}\right)$$

$$\breve{\mathcal{P}}_{l_{C\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}} = \left(\breve{\mathcal{P}}_{l_{C\tilde{\mathcal{R}}_l}} \cup \breve{\mathcal{P}}_{l_{C\tilde{\mathcal{H}}_l}}\right),\ \breve{\mathcal{P}}_{l_{F\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}} = \left(\breve{\mathcal{P}}_{l_{F\tilde{\mathcal{R}}_l}} \cup \breve{\mathcal{P}}_{l_{F\tilde{\mathcal{H}}_l}}\right)$$

$$T^{L}{}_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \min\{T^{L}{}_{\tilde{\mathcal{R}}_l}, T^{L}{}_{\tilde{\mathcal{H}}_l}\},\ T^{U}{}_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \min\{T^{U}{}_{\tilde{\mathcal{R}}_l}, T^{U}{}_{\tilde{\mathcal{H}}_l}\}$$

$$U^{L}{}_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \max\{U^{L}{}_{\tilde{\mathcal{R}}_l}, U^{L}{}_{\tilde{\mathcal{H}}_l}\},\ U^{U}{}_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \max\{U^{U}{}_{\tilde{\mathcal{R}}_l}, U^{U}{}_{\tilde{\mathcal{H}}_l}\}$$

$$C^{L}{}_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \max\ \{C^{L}{}_{\tilde{\mathcal{R}}_l}, C^{L}{}_{\tilde{\mathcal{H}}_l}\},\ C^{U}{}_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \max\{C^{U}{}_{\tilde{\mathcal{R}}_l}, C^{U}{}_{\tilde{\mathcal{H}}_l}\}$$

$$F^{L}{}_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \max\{F^{L}{}_{\tilde{\mathcal{R}}_l}, F^{L}{}_{\tilde{\mathcal{H}}_l}\},\ F^{U}{}_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \max\{F^{U}{}_{\tilde{\mathcal{R}}_l}, F^{U}{}_{\tilde{\mathcal{H}}_l}\}$$

ii. For $\tilde{\mathcal{R}}$ and $\tilde{\mathcal{H}}$, the " pessimistic $\cap'$'operation is defined as

$$\tilde{\mathcal{R}} \cap'_P \tilde{\mathcal{H}} = \{< \breve{\mathcal{P}}_{1_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}},\ \breve{\mathcal{P}}_{1_{T\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}}[\ T^{L}{}_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}, T^{U}{}_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}],\ \breve{\mathcal{P}}_{1_{U\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}}[U^{L}{}_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1},\ U^{U}{}_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}],$$

$$\breve{\mathcal{P}}_{1_{C\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}}[C^{L}{}_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}, C^{U}{}_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}], \breve{\mathcal{P}}_{1_{F\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}}[F^{L}{}_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}, F^{U}{}_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}];$$

$$\breve{\mathcal{P}}_{2_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}},\ \breve{\mathcal{P}}_{2_{T\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}}[\ T^{L}{}_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}, T^{U}{}_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}],\ \breve{\mathcal{P}}_{2_{U\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}}[U^{L}{}_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2},\ U^{U}{}_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}],$$

$$\breve{\mathcal{P}}_{2_{C\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}}[C^{L}{}_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}, C^{U}{}_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}], \breve{\mathcal{P}}_{2_{F\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}}[F^{L}{}_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}, F^{U}{}_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}];$$

$$\dots$$

$$\breve{\mathcal{P}}_{s_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}},\ \breve{\mathcal{P}}_{s_{T\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}}[\ T^{L}{}_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}, T^{U}{}_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}],\ \breve{\mathcal{P}}_{s_{U\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}}[U^{L}{}_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s},\ U^{U}{}_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}],$$

$$\breve{\mathcal{P}}_{s_{C\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}}[C^{L}{}_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}, C^{U}{}_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}], \breve{\mathcal{P}}_{s_{F\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}}[F^{L}{}_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}, F^{U}{}_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}] >,$$

$$\breve{\mathcal{P}}_{l_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}, \breve{\mathcal{P}}_{l_{T\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}, \breve{\mathcal{P}}_{l_{U\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}, \breve{\mathcal{P}}_{l_{C\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}, \breve{\mathcal{P}}_{l_{F\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}} \in P(\mathcal{N}); l = 1,2,3,\dots,s\}$$

Here, for $l = 1,2,3,\dots,s$;

$$\breve{\mathcal{P}}_{l_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}} = \left(\breve{\mathcal{P}}_{l_{\tilde{\mathcal{R}}_l}} \cap \breve{\mathcal{P}}_{l_{\tilde{\mathcal{H}}_l}}\right),\ \breve{\mathcal{P}}_{l_{T\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}} = \left(\breve{\mathcal{P}}_{l_{T\tilde{\mathcal{R}}_l}} \cap \breve{\mathcal{P}}_{l_{T\tilde{\mathcal{H}}_l}}\right),\ \breve{\mathcal{P}}_{l_{U\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}} = \left(\breve{\mathcal{P}}_{l_{U\tilde{\mathcal{R}}_l}} \cap \breve{\mathcal{P}}_{l_{U\tilde{\mathcal{H}}_l}}\right)$$

$$\breve{\mathcal{P}}_{l_{C\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}} = \left(\breve{\mathcal{P}}_{l_{C\tilde{\mathcal{R}}_l}} \cap \breve{\mathcal{P}}_{l_{C\tilde{\mathcal{H}}_l}}\right),\ \breve{\mathcal{P}}_{l_{F\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}} = \left(\breve{\mathcal{P}}_{l_{F\tilde{\mathcal{R}}_l}} \cap \breve{\mathcal{P}}_{l_{F\tilde{\mathcal{H}}_l}}\right)$$

$$T^{L}{}_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \min\{T^{L}{}_{\tilde{\mathcal{R}}_l}, T^{L}{}_{\tilde{\mathcal{H}}_l}\},\ T^{U}{}_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \min\{T^{U}{}_{\tilde{\mathcal{R}}_l}, T^{U}{}_{\tilde{\mathcal{H}}_l}\}$$

$$U^{L}{}_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \max\{U^{L}{}_{\tilde{\mathcal{R}}_l}, U^{L}{}_{\tilde{\mathcal{H}}_l}\},\ U^{U}{}_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \max\{U^{U}{}_{\tilde{\mathcal{R}}_l}, U^{U}{}_{\tilde{\mathcal{H}}_l}\}$$

$$C^{L}{}_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \max\ \{C^{L}{}_{\tilde{\mathcal{R}}_l}, C^{L}{}_{\tilde{\mathcal{H}}_l}\},\ C^{U}{}_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \max\{C^{U}{}_{\tilde{\mathcal{R}}_l}, C^{U}{}_{\tilde{\mathcal{H}}_l}\}$$

$F^{L}_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \max\{F^{L}_{\tilde{\mathcal{R}}_l}, F^{L}_{\tilde{\mathcal{H}}_l}\},\ F^{U}_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \max\{F^{U}_{\tilde{\mathcal{R}}_l}, F^{U}_{\tilde{\mathcal{H}}_l}\}$

Example 3.2.6: From Example 3.1.2, for $\hat{\mathcal{X}}$ and $\hat{\mathcal{Z}}$, we obtain that

i. $\hat{\mathcal{X}} \cup'_P \hat{\mathcal{Z}} = \{< \{\acute{n}, \acute{t}, \acute{r}, \acute{s}, \acute{l}, \acute{k}\}, \{\acute{n}, \acute{r}, \acute{f}, \acute{b}, \acute{l}\}[0,0.6], \{\acute{n}, \acute{s}, \acute{l}\}[0.3,0.8], \{\acute{n}, \acute{r}, \acute{s}, \acute{l}, \acute{f}, \acute{k}\}[0.5,0.7], \{\acute{t}, \acute{f}\}[0.6,0.9];$

$\{\acute{f}, \acute{k}, \acute{l}\}, \{\acute{t}, \acute{r}, \acute{s}, \acute{f}, \acute{l}\}[0.1,0.6], \{\acute{t}, \acute{r}, \acute{b}\}[0.5,0.9], \{\acute{n}, \acute{t}, \acute{r}, \acute{s}, \acute{l}\}[0.4,0.8], \{\acute{l}, \acute{k}\}[0.3,0.8] >\}$

ii. $\hat{\mathcal{X}} \cap'_P \hat{\mathcal{Z}} = \{< \{\acute{n}, \acute{t}, \acute{l}\}, \{\acute{f}, \acute{l}\}[0,0.6], \{\acute{l}\}[0.3,0.8], \{\acute{r}\}[0.5,0.7], \{\acute{f}\}[0.6,0.9];$

$\{\acute{k}\}, \{\acute{f}, \acute{l}\}[0.1,0.6], \{\acute{b}\}[0.5,0.9], \{\acute{n}, \acute{s}\}[0.4,0.8], \{\acute{l}\}[0.3,0.8] >\}$

Properties 3.2.9: Let

$\tilde{\mathcal{R}} = \{< \breve{\mathcal{P}}_{1\tilde{\mathcal{R}}_1},\ \breve{\mathcal{P}}_{1_{T\tilde{\mathcal{R}}_1}}[\ T^{L}_{\tilde{\mathcal{R}}_1}, T^{U}_{\tilde{\mathcal{R}}_1}],\ \breve{\mathcal{P}}_{1_{U\tilde{\mathcal{R}}_1}}[U^{L}_{\tilde{\mathcal{R}}_1},\ U^{U}_{\tilde{\mathcal{R}}_1}],\ \breve{\mathcal{P}}_{1_{C\tilde{\mathcal{R}}_1}}[C^{L}_{\tilde{\mathcal{R}}_1},\ C^{U}_{\tilde{\mathcal{R}}_1}],\ \breve{\mathcal{P}}_{1_{F\tilde{\mathcal{R}}_1}}[F^{L}_{\tilde{\mathcal{R}}_1}, F^{U}_{\tilde{\mathcal{R}}_1}];$

$\breve{\mathcal{P}}_{2\tilde{\mathcal{R}}_2},\ \breve{\mathcal{P}}_{2_{T\tilde{\mathcal{R}}_2}}[\ T^{L}_{\tilde{\mathcal{R}}_2}, T^{U}_{\tilde{\mathcal{R}}_2}],\ \breve{\mathcal{P}}_{2_{U\tilde{\mathcal{R}}_2}}[U^{L}_{\tilde{\mathcal{R}}_2},\ U^{U}_{\tilde{\mathcal{R}}_2}],\ \breve{\mathcal{P}}_{2_{C\tilde{\mathcal{R}}_2}}[C^{L}_{\tilde{\mathcal{R}}_2},\ C^{U}_{\tilde{\mathcal{R}}_2}],\ \breve{\mathcal{P}}_{2_{F\tilde{\mathcal{R}}_2}}[F^{L}_{\tilde{\mathcal{R}}_2}, F^{U}_{\tilde{\mathcal{R}}_2}];$

…

$\breve{\mathcal{P}}_{s\tilde{\mathcal{R}}_s},\ \breve{\mathcal{P}}_{s_{T\tilde{\mathcal{R}}_s}}[\ T^{L}_{\tilde{\mathcal{R}}_s}, T^{U}_{\tilde{\mathcal{R}}_s}],\ \breve{\mathcal{P}}_{s_{U\tilde{\mathcal{R}}_s}}[U^{L}_{\tilde{\mathcal{R}}_s},\ U^{U}_{\tilde{\mathcal{R}}_s}],\ \breve{\mathcal{P}}_{s_{C\tilde{\mathcal{R}}_s}}[C^{L}_{\tilde{\mathcal{R}}_s},\ C^{U}_{\tilde{\mathcal{R}}_s}],\ \breve{\mathcal{P}}_{s_{F\tilde{\mathcal{R}}_s}}[F^{L}_{\tilde{\mathcal{R}}_s}, F^{U}_{\tilde{\mathcal{R}}_s}] >,$

$\breve{\mathcal{P}}_{l\tilde{\mathcal{R}}_l}, \breve{\mathcal{P}}_{l_{T\tilde{\mathcal{R}}_l}}, \breve{\mathcal{P}}_{l_{U\tilde{\mathcal{R}}_l}}, \breve{\mathcal{P}}_{l_{C\tilde{\mathcal{R}}_l}}, \breve{\mathcal{P}}_{l_{F\tilde{\mathcal{R}}_l}} \in P(\mathcal{N}); l = 1,2,3, \dots, s\}$

$\tilde{\mathcal{H}} = \{< \breve{\mathcal{P}}_{1\tilde{\mathcal{H}}_1},\ \breve{\mathcal{P}}_{1_{T\tilde{\mathcal{H}}_1}}[\ T^{L}_{\tilde{\mathcal{H}}_1}, T^{U}_{\tilde{\mathcal{H}}_1}],\ \breve{\mathcal{P}}_{1_{U\tilde{\mathcal{H}}_1}}[U^{L}_{\tilde{\mathcal{H}}_1},\ U^{U}_{\tilde{\mathcal{H}}_1}],\ \breve{\mathcal{P}}_{1_{C\tilde{\mathcal{H}}_1}}[C^{L}_{\tilde{\mathcal{H}}_1},\ C^{U}_{\tilde{\mathcal{H}}_1}],\ \breve{\mathcal{P}}_{1_{F\tilde{\mathcal{H}}_1}}[F^{L}_{\tilde{\mathcal{H}}_1}, F^{U}_{\tilde{\mathcal{H}}_1}];$

$\breve{\mathcal{P}}_{2\tilde{\mathcal{H}}_2},\ \breve{\mathcal{P}}_{2_{T\tilde{\mathcal{H}}_2}}[\ T^{L}_{\tilde{\mathcal{H}}_2}, T^{U}_{\tilde{\mathcal{H}}_2}],\ \breve{\mathcal{P}}_{2_{U\tilde{\mathcal{H}}_2}}[U^{L}_{\tilde{\mathcal{H}}_2},\ U^{U}_{\tilde{\mathcal{H}}_2}],\ \breve{\mathcal{P}}_{2_{C\tilde{\mathcal{H}}_2}}[C^{L}_{\tilde{\mathcal{H}}_2},\ C^{U}_{\tilde{\mathcal{H}}_2}],\ \breve{\mathcal{P}}_{2_{F\tilde{\mathcal{H}}_2}}[F^{L}_{\tilde{\mathcal{H}}_2}, F^{U}_{\tilde{\mathcal{H}}_2}];$

…

$\breve{\mathcal{P}}_{s\tilde{\mathcal{H}}_s},\ \breve{\mathcal{P}}_{s_{T\tilde{\mathcal{H}}_s}}[\ T^{L}_{\tilde{\mathcal{H}}_s}, T^{U}_{\tilde{\mathcal{H}}_s}],\ \breve{\mathcal{P}}_{s_{U\tilde{\mathcal{H}}_s}}[U^{L}_{\tilde{\mathcal{H}}_s},\ U^{U}_{\tilde{\mathcal{H}}_s}],\ \breve{\mathcal{P}}_{s_{C\tilde{\mathcal{H}}_s}}[C^{L}_{\tilde{\mathcal{H}}_s},\ C^{U}_{\tilde{\mathcal{H}}_s}],\ \breve{\mathcal{P}}_{s_{F\tilde{\mathcal{H}}_s}}[F^{L}_{\tilde{\mathcal{H}}_s}, F^{U}_{\tilde{\mathcal{H}}_s}] >,$

$\breve{\mathcal{P}}_{l\tilde{\mathcal{H}}_l}, \breve{\mathcal{P}}_{l_{T\tilde{\mathcal{H}}_l}}, \breve{\mathcal{P}}_{l_{U\tilde{\mathcal{H}}_l}}, \breve{\mathcal{P}}_{l_{C\tilde{\mathcal{H}}_l}}, \breve{\mathcal{P}}_{l_{F\tilde{\mathcal{H}}_l}} \in P(\mathcal{N}); l = 1,2,3, \dots, s\}$

and

$\tilde{\mathcal{G}} = \{< \breve{\mathcal{P}}_{1\tilde{\mathcal{G}}_1},\ \breve{\mathcal{P}}_{1_{T\tilde{\mathcal{G}}_1}}[\ T^{L}_{\tilde{\mathcal{G}}_1}, T^{U}_{\tilde{\mathcal{G}}_1}],\ \breve{\mathcal{P}}_{1_{U\tilde{\mathcal{G}}_1}}[U^{L}_{\tilde{\mathcal{G}}_1},\ U^{U}_{\tilde{\mathcal{G}}_1}],\ \breve{\mathcal{P}}_{1_{C\tilde{\mathcal{G}}_1}}[C^{L}_{\tilde{\mathcal{G}}_1},\ C^{U}_{\tilde{\mathcal{G}}_1}],\ \breve{\mathcal{P}}_{1_{F\tilde{\mathcal{G}}_1}}[F^{L}_{\tilde{\mathcal{G}}_1}, F^{U}_{\tilde{\mathcal{G}}_1}];$

$\breve{\mathcal{P}}_{2\tilde{\mathcal{G}}_2},\ \breve{\mathcal{P}}_{2_{T\tilde{\mathcal{G}}_2}}[\ T^{L}_{\tilde{\mathcal{G}}_2}, T^{U}_{\tilde{\mathcal{G}}_2}],\ \breve{\mathcal{P}}_{2_{U\tilde{\mathcal{G}}_2}}[U^{L}_{\tilde{\mathcal{G}}_2},\ U^{U}_{\tilde{\mathcal{G}}_2}],\ \breve{\mathcal{P}}_{2_{C\tilde{\mathcal{G}}_2}}[C^{L}_{\tilde{\mathcal{G}}_2},\ C^{U}_{\tilde{\mathcal{G}}_2}],\ \breve{\mathcal{P}}_{2_{F\tilde{\mathcal{G}}_2}}[F^{L}_{\tilde{\mathcal{G}}_2}, F^{U}_{\tilde{\mathcal{G}}_2}];$

…

$\breve{\mathcal{P}}_{s\tilde{\mathcal{G}}_s},\ \breve{\mathcal{P}}_{s_{T\tilde{\mathcal{G}}_s}}[\ T^{L}_{\tilde{\mathcal{G}}_s}, T^{U}_{\tilde{\mathcal{G}}_s}],\ \breve{\mathcal{P}}_{s_{U\tilde{\mathcal{G}}_s}}[U^{L}_{\tilde{\mathcal{G}}_s},\ U^{U}_{\tilde{\mathcal{G}}_s}],\ \breve{\mathcal{P}}_{s_{C\tilde{\mathcal{G}}_s}}[C^{L}_{\tilde{\mathcal{G}}_s},\ C^{U}_{\tilde{\mathcal{G}}_s}],\ \breve{\mathcal{P}}_{s_{F\tilde{\mathcal{G}}_s}}[F^{L}_{\tilde{\mathcal{G}}_s}, F^{U}_{\tilde{\mathcal{G}}_s}] >,$

$\breve{\mathcal{P}}_{l\tilde{\mathcal{G}}_l}, \breve{\mathcal{P}}_{l_{T\tilde{\mathcal{G}}_l}}, \breve{\mathcal{P}}_{l_{U\tilde{\mathcal{G}}_l}}, \breve{\mathcal{P}}_{l_{C\tilde{\mathcal{G}}_l}}, \breve{\mathcal{P}}_{l_{F\tilde{\mathcal{G}}_l}} \in P(\mathcal{N}); l = 1,2,3, \dots, s\}$

be three IGSVNQS. Then, following conditions are satisfied.

i. $\left(\tilde{\mathcal{R}} \cup'_O \tilde{\mathcal{H}}\right) =' \left(\tilde{\mathcal{H}} \cup'_O \tilde{\mathcal{R}}\right)$

ii. $\left(\tilde{\mathcal{R}} \cup'_P \tilde{\mathcal{H}}\right) =' \left(\mathcal{H} \cup'_P \tilde{\mathcal{R}}\right)$

iii. $(\mathcal{R} \cap'_M \mathcal{H}) =' (\mathcal{H} \cap'_M \mathcal{R})$

iv. $(\mathcal{R} \cap'_O \mathcal{H}) =' (\mathcal{H} \cap'_O \mathcal{R})$

v. $(\mathcal{R} \cap'_P \mathcal{H}) =' (\mathcal{H} \cap'_P \mathcal{R})$

vi. If

$$T^L_{\tilde{\mathcal{R}}_l} = T^L_{\tilde{\mathcal{G}}_l} \ , \ T^U_{\tilde{\mathcal{R}}_l} = T^U_{\tilde{\mathcal{G}}_l} \ , \ U^L_{\tilde{\mathcal{R}}_l} = U^L_{\tilde{\mathcal{G}}_l} \ , \ U^U_{\tilde{\mathcal{R}}_l} = U^U_{\tilde{\mathcal{G}}_l}$$

$$C^L_{\tilde{\mathcal{R}}_l} = C^L_{\tilde{\mathcal{G}}_l} \ , \ C^U_{\tilde{\mathcal{R}}_l} = C^U_{\tilde{\mathcal{G}}_l} \ , \ F^L_{\tilde{\mathcal{R}}_l} = F^L_{\tilde{\mathcal{G}}_l} \ , \ F^U_{\tilde{\mathcal{R}}_l} = F^U_{\tilde{\mathcal{G}}_l}$$

then

$$\tilde{\mathcal{R}} \cup'_M \left(\tilde{\mathcal{H}} \cup'_M \tilde{\mathcal{G}}\right) =' \left(\tilde{\mathcal{R}} \cup'_M \tilde{\mathcal{H}}\right) \cup'_M \tilde{\mathcal{G}}$$

vii. $\tilde{\mathcal{R}} \cup'_O (\tilde{\mathcal{H}} \cup'_O \tilde{\mathcal{G}}) =' (\tilde{\mathcal{R}} \cup'_O \tilde{\mathcal{H}}) \cup'_O \tilde{\mathcal{G}}$

viii. $\tilde{\mathcal{R}} \cup'_P (\tilde{\mathcal{H}} \cup'_P \tilde{\mathcal{G}}) =' (\tilde{\mathcal{R}} \cup'_P \tilde{\mathcal{H}}) \cup'_P \tilde{\mathcal{G}}$

ix. If

$$T^L_{\tilde{\mathcal{R}}_l} = T^L_{\tilde{\mathcal{G}}_l} \ , \ T^U_{\tilde{\mathcal{R}}_l} = T^U_{\tilde{\mathcal{G}}_l} \ , \ U^L_{\tilde{\mathcal{R}}_l} = U^L_{\tilde{\mathcal{G}}_l} \ , \ U^U_{\tilde{\mathcal{R}}_l} = U^U_{\tilde{\mathcal{G}}_l}$$

$$C^L_{\tilde{\mathcal{R}}_l} = C^L_{\tilde{\mathcal{G}}_l} \ , \ C^U_{\tilde{\mathcal{R}}_l} = C^U_{\tilde{\mathcal{G}}_l} \ , \ F^L_{\tilde{\mathcal{R}}_l} = F^L_{\tilde{\mathcal{G}}_l} \ , \ F^U_{\tilde{\mathcal{R}}_l} = F^U_{\tilde{\mathcal{G}}_l}$$

then

$$\tilde{\mathcal{R}} \cap'_M (\tilde{\mathcal{H}} \cap'_M \tilde{\mathcal{G}}) =' (\tilde{\mathcal{R}} \cap'_M \tilde{\mathcal{H}}) \cap'_M \tilde{\mathcal{G}}.$$

x. $\tilde{\mathcal{R}} \cap'_O (\tilde{\mathcal{H}} \cap'_O \tilde{\mathcal{G}}) =' (\tilde{\mathcal{R}} \cap'_O \tilde{\mathcal{H}}) \cap'_O \tilde{\mathcal{G}}$

xi. $\tilde{\mathcal{R}} \cap'_P (\tilde{\mathcal{H}} \cap'_P \tilde{\mathcal{G}}) =' (\tilde{\mathcal{R}} \cap'_P \tilde{\mathcal{H}}) \cap'_P \tilde{\mathcal{G}}$

xii. $\tilde{\mathcal{R}} \cap'_M (\tilde{\mathcal{H}} \cup'_M \tilde{\mathcal{G}}) =' (\tilde{\mathcal{R}} \cap'_M \tilde{\mathcal{H}}) \cup'_M (\tilde{\mathcal{R}} \cap'_M \tilde{\mathcal{G}})$

xiii. $\tilde{\mathcal{R}} \cap'_O (\tilde{\mathcal{H}} \cup'_O \tilde{\mathcal{G}}) =' (\tilde{\mathcal{R}} \cap'_O \tilde{\mathcal{H}}) \cup'_O (\tilde{\mathcal{R}} \cap'_O \tilde{\mathcal{G}})$

xiv. $\tilde{\mathcal{R}} \cap'_P (\tilde{\mathcal{H}} \cup'_P \tilde{\mathcal{G}}) =' (\tilde{\mathcal{R}} \cap'_P \tilde{\mathcal{H}}) \cup'_P (\tilde{\mathcal{R}} \cap'_P \tilde{\mathcal{G}})$

xv. $\tilde{\mathcal{R}} \cup'_M (\tilde{\mathcal{H}} \cap'_M \tilde{\mathcal{G}}) =' (\tilde{\mathcal{R}} \cup'_M \tilde{\mathcal{H}}) \cap'_M (\tilde{\mathcal{R}} \cup'_M \tilde{\mathcal{G}})$

xvi. $\tilde{\mathcal{R}} \cup'_O (\tilde{\mathcal{H}} \cap'_O \tilde{\mathcal{G}}) =' (\tilde{\mathcal{R}} \cup'_O \tilde{\mathcal{H}}) \cap'_O (\tilde{\mathcal{R}} \cup'_O \tilde{\mathcal{G}})$

xvii. $\tilde{\mathcal{R}} \cup'_P (\tilde{\mathcal{H}} \cap'_P \tilde{\mathcal{G}}) =' (\tilde{\mathcal{R}} \cup'_P \tilde{\mathcal{H}}) \cap'_P (\tilde{\mathcal{R}} \cup'_P \tilde{\mathcal{G}})$

xviii. İf $\tilde{\mathcal{R}} =' \tilde{\mathcal{H}}$ then $\left(\tilde{\mathcal{R}} \cup'_M \tilde{\mathcal{H}}\right) =' \left(\tilde{\mathcal{R}} \cup'_O \tilde{\mathcal{H}}\right) =' \left(\tilde{\mathcal{R}} \cup'_P \tilde{\mathcal{H}}\right)$

xix. İf $\tilde{\mathcal{R}} =' \tilde{\mathcal{H}}$ then $\left(\tilde{\mathcal{R}} \cap'_M \tilde{\mathcal{H}}\right) =' \left(\tilde{\mathcal{R}} \cap'_O \tilde{\mathcal{H}}\right) =' \left(\tilde{\mathcal{R}} \cap'_P \tilde{\mathcal{H}}\right)$

Proof:

i. From Definition 3.2.1 for $(\tilde{\mathcal{R}} \cup'_M \tilde{\mathcal{H}})$, we get

$$\breve{\mathcal{P}}_{l_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}} = \left(\breve{\mathcal{P}}_{l_{\tilde{\mathcal{R}}_l}} \cup \breve{\mathcal{P}}_{l_{\tilde{\mathcal{H}}_l}}\right),\ \breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}} = \left(\breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{R}}_l}}} \cup \breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{H}}_l}}}\right),\ \breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}} = \left(\breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{R}}_l}}} \cup \breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{H}}_l}}}\right)$$

$$\breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}} = \left(\breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{R}}_l}}} \cup \breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{H}}_l}}}\right),\ \breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}} = \left(\breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{R}}_l}}} \cup \breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{H}}_l}}}\right)$$

$$T^L_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \frac{T^L_{\tilde{\mathcal{R}}_l}+T^L_{\tilde{\mathcal{H}}_l}}{2},\ T^U_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \frac{T^U_{\tilde{\mathcal{R}}_l}+T^U_{\tilde{\mathcal{H}}_l}}{2},\ U^L_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \frac{U^L_{\tilde{\mathcal{R}}_l}+U^L_{\tilde{\mathcal{H}}_l}}{2},\ U^U_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \frac{U^U_{\tilde{\mathcal{R}}_l}+U^U_{\tilde{\mathcal{H}}_l}}{2}$$

$$C^L_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \frac{C^L_{\tilde{\mathcal{R}}_l}+C^L_{\tilde{\mathcal{H}}_l}}{2},\ C^U_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \frac{C^U_{\tilde{\mathcal{R}}_l}+C^U_{\tilde{\mathcal{H}}_l}}{2},\ F^L_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \frac{F^L_{\tilde{\mathcal{R}}_l}+F^L_{\tilde{\mathcal{H}}_l}}{2},\ F^U_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \frac{F^U_{\tilde{\mathcal{R}}_l}+F^U_{\tilde{\mathcal{H}}_l}}{2} \quad (1)$$

From Definition 3.2.1 for $(\tilde{\mathcal{H}} \cup'_M \tilde{\mathcal{R}})$, we get

$$\breve{\mathcal{P}}_{l_{\tilde{\mathcal{H}}_l\tilde{\mathcal{R}}_l}} = \left(\breve{\mathcal{P}}_{l_{\tilde{\mathcal{H}}_l}} \cup \breve{\mathcal{P}}_{l_{\tilde{\mathcal{R}}_l}}\right),\ \breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{H}}_l\tilde{\mathcal{R}}_l}}} = \left(\breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{H}}_l}}} \cup \breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{R}}_l}}}\right),\ \breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{H}}_l\tilde{\mathcal{R}}_l}}} = \left(\breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{H}}_l}}} \cup \breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{R}}_l}}}\right)$$

$$\breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{H}}_l\tilde{\mathcal{R}}_l}}} = \left(\breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{H}}_l}}} \cup \breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{R}}_l}}}\right),\ \breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{H}}_l\tilde{\mathcal{R}}_l}}} = \left(\breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{H}}_l}}} \cup \breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{R}}_l}}}\right)$$

$$T^L_{\tilde{\mathcal{H}}_l\tilde{\mathcal{R}}_l} = \frac{T^L_{\tilde{\mathcal{H}}_l}+T^L_{\tilde{\mathcal{R}}_l}}{2}, T^U_{\tilde{\mathcal{H}}_l\tilde{\mathcal{R}}_l} = \frac{T^U_{\tilde{\mathcal{H}}_l}+T^U_{\tilde{\mathcal{R}}_l}}{2}, U^L_{\tilde{\mathcal{H}}_l\tilde{\mathcal{R}}_l} = \frac{U^L_{\tilde{\mathcal{H}}_l}+U^L_{\tilde{\mathcal{R}}_l}}{2}, U^U_{\tilde{\mathcal{H}}_l\tilde{\mathcal{R}}_l} = \frac{U^U_{\tilde{\mathcal{H}}_l}+U^U_{\tilde{\mathcal{R}}_l}}{2}$$

$$C^L_{\tilde{\mathcal{H}}_l\tilde{\mathcal{R}}_l} = \frac{C^L_{\tilde{\mathcal{H}}_l}+C^L_{\tilde{\mathcal{R}}_l}}{2}, C^U_{\tilde{\mathcal{H}}_l\tilde{\mathcal{R}}_l} = \frac{C^U_{\tilde{\mathcal{H}}_l}+C^U_{\tilde{\mathcal{R}}_l}}{2}, F^L_{\tilde{\mathcal{H}}_l\tilde{\mathcal{R}}_l} = \frac{F^L_{\tilde{\mathcal{H}}_l}+F^L_{\tilde{\mathcal{R}}_l}}{2}, F^U_{\tilde{\mathcal{H}}_l\tilde{\mathcal{R}}_l} = \frac{F^U_{\tilde{\mathcal{H}}_l}+F^U_{\tilde{\mathcal{R}}_l}}{2} \quad (2)$$

and from Definition 3.1.1, (1) and (2), we get

$$(\mathcal{R} \cup'_M \mathcal{H}) =' (\mathcal{H} \cup'_M \mathcal{R}).$$

Proof of {ii, iii} are obtained similar to i.

iv. From Definition 3.2.1 for $(\mathcal{R} \cap'_M \mathcal{H})$, we get

$$\breve{\mathcal{P}}_{l_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}} = \left(\breve{\mathcal{P}}_{l_{\tilde{\mathcal{R}}_l}} \cap \breve{\mathcal{P}}_{l_{\tilde{\mathcal{H}}_l}}\right),\ \breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}} = \left(\breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{R}}_l}}} \cap \breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{H}}_l}}}\right),\ \breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}} = \left(\breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{R}}_l}}} \cap \breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{H}}_l}}}\right)$$

$$\breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}} = \left(\breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{R}}_l}}} \cap \breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{H}}_l}}}\right),\ \breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}} = \left(\breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{R}}_l}}} \cap \breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{H}}_l}}}\right)$$

$$T^L_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \frac{T^L_{\tilde{\mathcal{R}}_l}+T^L_{\tilde{\mathcal{H}}_l}}{2},\ T^U_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \frac{T^U_{\tilde{\mathcal{R}}_l}+T^U_{\tilde{\mathcal{H}}_l}}{2},\ U^L_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \frac{U^L_{\tilde{\mathcal{R}}_l}+U^L_{\tilde{\mathcal{H}}_l}}{2},\ U^U_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \frac{U^U_{\tilde{\mathcal{R}}_l}+U^U_{\tilde{\mathcal{H}}_l}}{2}$$

$$C^L_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \frac{C^L_{\tilde{\mathcal{R}}_l}+C^L_{\tilde{\mathcal{H}}_l}}{2},\ C^U_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \frac{C^U_{\tilde{\mathcal{R}}_l}+C^U_{\tilde{\mathcal{H}}_l}}{2},\ F^L_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \frac{F^L_{\tilde{\mathcal{R}}_l}+F^L_{\tilde{\mathcal{H}}_l}}{2},\ F^U_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \frac{F^U_{\tilde{\mathcal{R}}_l}+F^U_{\tilde{\mathcal{H}}_l}}{2} \quad (3)$$

Definition 3.2.1 for $(\tilde{\mathcal{H}} \cap'_M \tilde{\mathcal{R}})$

$\breve{\mathcal{P}}_{l_{\tilde{\mathcal{H}}_l\tilde{\mathcal{R}}_l}} = \left(\breve{\mathcal{P}}_{l_{\tilde{\mathcal{H}}_l}} \cap \breve{\mathcal{P}}_{l_{\tilde{\mathcal{R}}_l}}\right),\ \breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{H}}_l\tilde{\mathcal{R}}_l}}} = \left(\breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{H}}_l}}} \cap \breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{R}}_l}}}\right),\ \breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{H}}_l\tilde{\mathcal{R}}_l}}} = \left(\breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{H}}_l}}} \cap \breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{R}}_l}}}\right)$

$\breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{H}}_l\tilde{\mathcal{R}}_l}}} = \left(\breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{H}}_l}}} \cap \breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{R}}_l}}}\right),\ \breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{H}}_l\tilde{\mathcal{R}}_l}}} = \left(\breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{H}}_l}}} \cap \breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{R}}_l}}}\right)$

$T^L_{\tilde{\mathcal{H}}_l\tilde{\mathcal{R}}_l} = \frac{T^L_{\tilde{\mathcal{H}}_l} + T^L_{\tilde{\mathcal{R}}_l}}{2},\ T^U_{\tilde{\mathcal{H}}_l\tilde{\mathcal{R}}_l} = \frac{T^U_{\tilde{\mathcal{H}}_l} + T^U_{\tilde{\mathcal{R}}_l}}{2},\ U^L_{\tilde{\mathcal{H}}_l\tilde{\mathcal{R}}_l} = \frac{U^L_{\tilde{\mathcal{H}}_l} + U^L_{\tilde{\mathcal{R}}_l}}{2},\ U^U_{\tilde{\mathcal{H}}_l\tilde{\mathcal{R}}_l} = \frac{U^U_{\tilde{\mathcal{H}}_l} + U^U_{\tilde{\mathcal{R}}_l}}{2}$

$C^L_{\tilde{\mathcal{H}}_l\tilde{\mathcal{R}}_l} = \frac{C^L_{\tilde{\mathcal{H}}_l} + C^L_{\tilde{\mathcal{R}}_l}}{2},\ C^U_{\tilde{\mathcal{H}}_l\tilde{\mathcal{R}}_l} = \frac{C^U_{\tilde{\mathcal{H}}_l} + C^U_{\tilde{\mathcal{R}}_l}}{2},\ F^L_{\tilde{\mathcal{H}}_l\tilde{\mathcal{R}}_l} = \frac{F^L_{\tilde{\mathcal{H}}_l} + F^L_{\tilde{\mathcal{R}}_l}}{2},\ F^U_{\tilde{\mathcal{H}}_l\tilde{\mathcal{R}}_l} = \frac{F^U_{\tilde{\mathcal{H}}_l} + F^U_{\tilde{\mathcal{R}}_l}}{2}$ (4)

From Definition 3.1.1, (3) and (4) we get

$$\left(\tilde{\mathcal{R}} \cap'_M \tilde{\mathcal{H}}\right) =' \left(\tilde{\mathcal{H}} \cap'_M \tilde{\mathcal{R}}\right)$$

Proof of {v, vi} are obtained similar to iv.

vii. We assume that

$$T^L_{\tilde{\mathcal{R}}_l} = T^L_{\tilde{\mathcal{G}}_l}\ ,\ T^U_{\tilde{\mathcal{R}}_l} = T^U_{\tilde{\mathcal{G}}_l}\ ,\ U^L_{\tilde{\mathcal{R}}_l} = U^L_{\tilde{\mathcal{G}}_l}\ ,\ U^U_{\tilde{\mathcal{R}}_l} = U^U_{\tilde{\mathcal{G}}_l}$$

$$C^L_{\tilde{\mathcal{R}}_l} = C^L_{\tilde{\mathcal{G}}_l}\ ,\ C^U_{\tilde{\mathcal{R}}_l} = C^U_{\tilde{\mathcal{G}}_l}\ ,\ F^L_{\tilde{\mathcal{R}}_l} = F^L_{\tilde{\mathcal{G}}_l}\ ,\ F^U_{\tilde{\mathcal{R}}_l} = F^U_{\tilde{\mathcal{G}}_l}.$$

For $\tilde{\mathcal{R}} \cup'_M \left(\tilde{\mathcal{H}} \cup'_M \tilde{\mathcal{G}}\right)$ from Definition 3.2.1, we get

$\breve{\mathcal{P}}_{l_{\tilde{\mathcal{R}}_l(\tilde{\mathcal{H}}_l\tilde{\mathcal{G}}_l)}} = \left(\breve{\mathcal{P}}_{l_{\tilde{\mathcal{R}}_l}} \cup (\breve{\mathcal{P}}_{l_{\tilde{\mathcal{H}}_l}} \cup \breve{\mathcal{P}}_{l_{\tilde{\mathcal{G}}_l}})\right),\ \breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{R}}_l(\tilde{\mathcal{H}}_l\tilde{\mathcal{G}}_l)}}} = \left(\breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{R}}_l}}} \cup (\breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{H}}_l}}} \cup \breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{G}}_l}}})\right),$

$\breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{R}}_l(\tilde{\mathcal{H}}_l\tilde{\mathcal{G}}_l)}}} = \left(\breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{R}}_l}}} \cup (\breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{H}}_l}}} \cup \breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{G}}_l}}})\right),\ \breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{R}}_l(\tilde{\mathcal{H}}_l\tilde{\mathcal{G}}_l)}}} = \left(\breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{R}}_l}}} \cup (\breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{H}}_l}}} \cup \breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{G}}_l}}})\right)$

$\breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{R}}_l(\tilde{\mathcal{H}}_l\tilde{\mathcal{G}}_l)}}} = \left(\breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{R}}_l}}} \cup (\breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{H}}_l}}} \cup \breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{G}}_l}}})\right)$

$$T^L_{\tilde{\mathcal{R}}_l(\tilde{\mathcal{H}}_l\tilde{\mathcal{G}}_l)} = \frac{T^L_{\tilde{\mathcal{R}}_l} + \frac{T^L_{\tilde{\mathcal{H}}_l} + T^L_{\tilde{\mathcal{G}}_l}}{2}}{2}, \qquad T^U_{\tilde{\mathcal{R}}_l(\tilde{\mathcal{H}}_l\tilde{\mathcal{G}}_l)} = \frac{T^U_{\tilde{\mathcal{R}}_l} + \frac{T^U_{\tilde{\mathcal{H}}_l} + T^U_{\tilde{\mathcal{G}}_l}}{2}}{2}$$

$$U^L_{\tilde{\mathcal{R}}_l(\tilde{\mathcal{H}}_l\tilde{\mathcal{G}}_l)} = \frac{U^L_{\tilde{\mathcal{R}}_l} + \frac{U^L_{\tilde{\mathcal{H}}_l} + U^L_{\tilde{\mathcal{G}}_l}}{2}}{2}, \qquad U^U_{\tilde{\mathcal{R}}_l(\tilde{\mathcal{H}}_l\tilde{\mathcal{G}}_l)} = \frac{U^U_{\tilde{\mathcal{R}}_l} + \frac{U^U_{\tilde{\mathcal{H}}_l} + U^U_{\tilde{\mathcal{G}}_l}}{2}}{2}$$

$$C^L_{\tilde{\mathcal{R}}_l(\tilde{\mathcal{H}}_l\tilde{\mathcal{G}}_l)} = \frac{C^L_{\tilde{\mathcal{R}}_l} + \frac{C^L_{\tilde{\mathcal{H}}_l} + C^L_{\tilde{\mathcal{G}}_l}}{2}}{2}, \qquad C^U_{\tilde{\mathcal{R}}_l(\tilde{\mathcal{H}}_l\tilde{\mathcal{G}}_l)} = \frac{C^U_{\tilde{\mathcal{R}}_l} + \frac{C^U_{\tilde{\mathcal{H}}_l} + C^U_{\tilde{\mathcal{G}}_l}}{2}}{2}$$

$$F^L_{\tilde{\mathcal{R}}_l(\tilde{\mathcal{H}}_l\tilde{\mathcal{G}}_l)} = \frac{F^L_{\tilde{\mathcal{R}}_l} + \frac{F^L_{\tilde{\mathcal{H}}_l} + F^L_{\tilde{\mathcal{G}}_l}}{2}}{2},$$

$$F^U_{\tilde{\mathcal{R}}_l(\tilde{\mathcal{H}}_l\tilde{\mathcal{G}}_l)} = \frac{F^U_{\tilde{\mathcal{R}}_l} + \frac{F^U_{\tilde{\mathcal{H}}_l} + F^U_{\tilde{\mathcal{G}}_l}}{2}}{2}.$$ (5)

From Definition 3.2.1 for $(\tilde{\mathcal{R}} \cup'_M \tilde{\mathcal{H}}) \cup'_M \tilde{\mathcal{G}}$,

$$\breve{\mathcal{P}}_{l_{(\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l)\tilde{\mathcal{G}}_l}} = \left((\breve{\mathcal{P}}_{l_{\tilde{\mathcal{R}}_l}} \cup \breve{\mathcal{P}}_{l_{\tilde{\mathcal{H}}_l}}) \cup \breve{\mathcal{P}}_{l_{\tilde{\mathcal{G}}_l}}\right),\ \breve{\mathcal{P}}_{l_{T_{(\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l)\tilde{\mathcal{G}}_l}}} = \left((\breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{R}}_l}}} \cup \breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{H}}_l}}}) \cup \breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{G}}_l}}}\right)$$

$$\breve{\mathcal{P}}_{l_{U_{(\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l)\tilde{\mathcal{G}}_l}}} = \left((\breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{R}}_l}}} \cup \breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{H}}_l}}}) \cup \breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{G}}_l}}}\right),\ \breve{\mathcal{P}}_{l_{C_{(\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l)\tilde{\mathcal{G}}_l}}} = \left((\breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{R}}_l}}} \cup \breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{H}}_l}}}) \cup \breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{G}}_l}}}\right)$$

$$\breve{\mathcal{P}}_{l_{F_{(\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l)\tilde{\mathcal{G}}_l}}} = \left((\breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{R}}_l}}} \cup \breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{H}}_l}}}) \cup \breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{G}}_l}}}\right)$$

$$T^L_{(\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l)\tilde{\mathcal{G}}_l} = \frac{\frac{T^L_{\tilde{\mathcal{R}}_l} + T^L_{\tilde{\mathcal{H}}_l}}{2} + T^L_{\tilde{\mathcal{G}}_l}}{2}, \qquad T^U_{(\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l)\tilde{\mathcal{G}}_l} = \frac{\frac{T^U_{\tilde{\mathcal{R}}_l} + T^U_{\tilde{\mathcal{H}}_l}}{2} + T^U_{\tilde{\mathcal{G}}_l}}{2}$$

$$U^L_{(\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l)\tilde{\mathcal{G}}_l} = \frac{\frac{U^L_{\tilde{\mathcal{R}}_l} + U^L_{\tilde{\mathcal{H}}_l}}{2} + U^L_{\tilde{\mathcal{G}}_l}}{2}, \qquad U^U_{(\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l)\tilde{\mathcal{G}}_l} = \frac{\frac{U^U_{\tilde{\mathcal{R}}_l} + U^U_{\tilde{\mathcal{H}}_l}}{2} + U^U_{\tilde{\mathcal{G}}_l}}{2}$$

$$C^L_{(\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l)\tilde{\mathcal{G}}_l} = \frac{\frac{C^L_{\tilde{\mathcal{R}}_l} + C^L_{\tilde{\mathcal{H}}_l}}{2} + C^L_{\tilde{\mathcal{G}}_l}}{2}, \qquad C^U_{(\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l)\tilde{\mathcal{G}}_l} = \frac{\frac{C^U_{\tilde{\mathcal{R}}_l} + C^U_{\tilde{\mathcal{H}}_l}}{2} + C^U_{\tilde{\mathcal{G}}_l}}{2}$$

$$F^L_{(\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l)\tilde{\mathcal{G}}_l} = \frac{\frac{F^L_{\tilde{\mathcal{R}}_l} + F^L_{\tilde{\mathcal{H}}_l}}{2} + F^L_{\tilde{\mathcal{G}}_l}}{2},$$

$$F^U_{(\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l)\tilde{\mathcal{G}}_l} = \frac{\frac{F^U_{\tilde{\mathcal{R}}_l} + F^U_{\tilde{\mathcal{H}}_l}}{2} + F^U_{\tilde{\mathcal{G}}_l}}{2}. \tag{6}$$

From Definition 3.1.1, (5) and (6)

we get

$$\tilde{\mathcal{R}} \cup'_M (\tilde{\mathcal{H}} \cup'_M \tilde{\mathcal{G}}) =' (\tilde{\mathcal{R}} \cup'_M \tilde{\mathcal{H}}) \cup'_M \tilde{\mathcal{G}}$$

Proof of {viii, ix} are obtained similar to vii.

x. We assume that

$$T^L_{\tilde{\mathcal{R}}_l} = T^L_{\tilde{\mathcal{G}}_l}\ ,\ T^U_{\tilde{\mathcal{R}}_l} = T^U_{\tilde{\mathcal{G}}_l}\ ,\ U^L_{\tilde{\mathcal{R}}_l} = U^L_{\tilde{\mathcal{G}}_l}\ ,\ U^U_{\tilde{\mathcal{R}}_l} = U^U_{\tilde{\mathcal{G}}_l}$$

$$C^L_{\tilde{\mathcal{R}}_l} = C^L_{\tilde{\mathcal{G}}_l}\ ,\ C^U_{\tilde{\mathcal{R}}_l} = C^U_{\tilde{\mathcal{G}}_l}\ ,\ F^L_{\tilde{\mathcal{R}}_l} = F^L_{\tilde{\mathcal{G}}_l}\ ,\ F^U_{\tilde{\mathcal{R}}_l} = F^U_{\tilde{\mathcal{G}}_l}.$$

From Definition 3.2.1 for $\mathcal{R} \cap'_M (\mathcal{H} \cap'_M \mathcal{G})$, we get

$$\breve{\mathcal{P}}_{l_{\tilde{\mathcal{R}}_l(\tilde{\mathcal{H}}_l\tilde{\mathcal{G}}_l)}} = \left(\breve{\mathcal{P}}_{l_{\tilde{\mathcal{R}}_l}} \cap (\breve{\mathcal{P}}_{l_{\tilde{\mathcal{H}}_l}} \cap \breve{\mathcal{P}}_{l_{\tilde{\mathcal{G}}_l}})\right),\ \breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{R}}_l(\tilde{\mathcal{H}}_l\tilde{\mathcal{G}}_l)}}} = \left(\breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{R}}_l}}} \cap (\breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{H}}_l}}} \cap \breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{G}}_l}}})\right)$$

$$\breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{R}}_l(\tilde{\mathcal{H}}_l\tilde{\mathcal{G}}_l)}}} = \left(\breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{R}}_l}}} \cap (\breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{H}}_l}}} \cap \breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{G}}_l}}})\right),\ \breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{R}}_l(\tilde{\mathcal{H}}_l\tilde{\mathcal{G}}_l)}}} = \left(\breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{R}}_l}}} \cap (\breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{H}}_l}}} \cap \breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{G}}_l}}})\right)$$

$$\breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{R}}_l(\tilde{\mathcal{H}}_l\tilde{\mathcal{G}}_l)}}} = \left(\breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{R}}_l}}} \cap (\breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{H}}_l}}} \cap \breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{G}}_l}}})\right)$$

$$T^{L}_{\tilde{\mathcal{R}}_l(\tilde{\mathcal{H}}_l\tilde{\mathcal{G}}_l)} = \frac{T^{L}_{\tilde{\mathcal{R}}_l} + \frac{T^{L}_{\tilde{\mathcal{H}}_l} + T^{L}_{\tilde{\mathcal{G}}_l}}{2}}{2}, \qquad T^{U}_{\tilde{\mathcal{R}}_l(\tilde{\mathcal{H}}_l\tilde{\mathcal{G}}_l)} = \frac{T^{U}_{\tilde{\mathcal{R}}_l} + \frac{T^{U}_{\tilde{\mathcal{H}}_l} + T^{U}_{\tilde{\mathcal{G}}_l}}{2}}{2}$$

$$U^{L}_{\tilde{\mathcal{R}}_l(\tilde{\mathcal{H}}_l\tilde{\mathcal{G}}_l)} = \frac{U^{L}_{\tilde{\mathcal{R}}_l} + \frac{U^{L}_{\tilde{\mathcal{H}}_l} + U^{L}_{\tilde{\mathcal{G}}_l}}{2}}{2}, \qquad U^{U}_{\tilde{\mathcal{R}}_l(\tilde{\mathcal{H}}_l\tilde{\mathcal{G}}_l)} = \frac{U^{U}_{\tilde{\mathcal{R}}_l} + \frac{U^{U}_{\tilde{\mathcal{H}}_l} + U^{U}_{\tilde{\mathcal{G}}_l}}{2}}{2}$$

$$C^{L}_{\tilde{\mathcal{R}}_l(\tilde{\mathcal{H}}_l\tilde{\mathcal{G}}_l)} = \frac{C^{L}_{\tilde{\mathcal{R}}_l} + \frac{C^{L}_{\tilde{\mathcal{H}}_l} + C^{L}_{\tilde{\mathcal{G}}_l}}{2}}{2}, \qquad C^{U}_{\tilde{\mathcal{R}}_l(\tilde{\mathcal{H}}_l\tilde{\mathcal{G}}_l)} = \frac{C^{U}_{\tilde{\mathcal{R}}_l} + \frac{C^{U}_{\tilde{\mathcal{H}}_l} + C^{U}_{\tilde{\mathcal{G}}_l}}{2}}{2}$$

$$F^{L}_{\tilde{\mathcal{R}}_l(\tilde{\mathcal{H}}_l\tilde{\mathcal{G}}_l)} = \frac{F^{L}_{\tilde{\mathcal{R}}_l} + \frac{F^{L}_{\tilde{\mathcal{H}}_l} + F^{L}_{\tilde{\mathcal{G}}_l}}{2}}{2},$$

$$F^{U}_{\tilde{\mathcal{R}}_l(\tilde{\mathcal{H}}_l\tilde{\mathcal{G}}_l)} = \frac{F^{U}_{\tilde{\mathcal{R}}_l} + \frac{F^{U}_{\tilde{\mathcal{H}}_l} + F^{U}_{\tilde{\mathcal{G}}_l}}{2}}{2}. \tag{7}$$

From definition 3.2.1 for $(\tilde{\mathcal{R}} \cap'_M \tilde{\mathcal{H}}) \cap'_M \tilde{\mathcal{G}}$, we get

$$\breve{\mathcal{P}}_{l_{(\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l)\tilde{\mathcal{G}}_l}} = \left((\breve{\mathcal{P}}_{l_{\tilde{\mathcal{R}}_l}} \cap \breve{\mathcal{P}}_{l_{\tilde{\mathcal{H}}_l}}) \cap \breve{\mathcal{P}}_{l_{\tilde{\mathcal{G}}_l}}\right),\ \breve{\mathcal{P}}_{l_{T_{(\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l)\tilde{\mathcal{G}}_l}}} = \left((\breve{\mathcal{P}}_{l_{T\tilde{\mathcal{R}}_l}} \cap \breve{\mathcal{P}}_{l_{T\tilde{\mathcal{H}}_l}}) \cap \breve{\mathcal{P}}_{l_{T\tilde{\mathcal{G}}_l}}\right)$$

$$\breve{\mathcal{P}}_{l_{U_{(\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l)\tilde{\mathcal{G}}_l}}} = \left((\breve{\mathcal{P}}_{l_{U\tilde{\mathcal{R}}_l}} \cap \breve{\mathcal{P}}_{l_{U\tilde{\mathcal{H}}_l}}) \cap \breve{\mathcal{P}}_{l_{U\tilde{\mathcal{G}}_l}}\right), \breve{\mathcal{P}}_{l_{C_{(\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l)\tilde{\mathcal{G}}_l}}} = \left((\breve{\mathcal{P}}_{l_{C\tilde{\mathcal{R}}_l}} \cap \breve{\mathcal{P}}_{l_{C\tilde{\mathcal{H}}_l}}) \cap \breve{\mathcal{P}}_{l_{C\tilde{\mathcal{G}}_l}}\right)$$

$$\breve{\mathcal{P}}_{l_{F_{(\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l)\tilde{\mathcal{G}}_l}}} = \left((\breve{\mathcal{P}}_{l_{F\tilde{\mathcal{R}}_l}} \cap \breve{\mathcal{P}}_{l_{F\tilde{\mathcal{H}}_l}}) \cap \breve{\mathcal{P}}_{l_{F\tilde{\mathcal{G}}_l}}\right)$$

$$T^{L}_{(\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l)\tilde{\mathcal{G}}_l} = \frac{\frac{T^{L}_{\tilde{\mathcal{R}}_l} + T^{L}_{\tilde{\mathcal{H}}_l}}{2} + T^{L}_{\tilde{\mathcal{G}}_l}}{2}, \qquad T^{U}_{(\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l)\tilde{\mathcal{G}}_l} = \frac{\frac{T^{U}_{\tilde{\mathcal{R}}_l} + T^{U}_{\tilde{\mathcal{H}}_l}}{2} + T^{U}_{\tilde{\mathcal{G}}_l}}{2}$$

$$U^{L}_{(\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l)\tilde{\mathcal{G}}_l} = \frac{\frac{U^{L}_{\tilde{\mathcal{R}}_l} + U^{L}_{\tilde{\mathcal{H}}_l}}{2} + U^{L}_{\tilde{\mathcal{G}}_l}}{2}, \qquad U^{U}_{(\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l)\tilde{\mathcal{G}}_l} = \frac{\frac{U^{U}_{\tilde{\mathcal{R}}_l} + U^{U}_{\tilde{\mathcal{H}}_l}}{2} + U^{U}_{\tilde{\mathcal{G}}_l}}{2}$$

$$C^{L}_{(\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l)\tilde{\mathcal{G}}_l} = \frac{\frac{C^{L}_{\tilde{\mathcal{R}}_l} + C^{L}_{\tilde{\mathcal{H}}_l}}{2} + C^{L}_{\tilde{\mathcal{G}}_l}}{2}, \qquad C^{U}_{(\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l)\tilde{\mathcal{G}}_l} = \frac{\frac{C^{U}_{\tilde{\mathcal{R}}_l} + C^{U}_{\tilde{\mathcal{H}}_l}}{2} + C^{U}_{\tilde{\mathcal{G}}_l}}{2}$$

$$F^{L}_{(\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l)\tilde{\mathcal{G}}_l} = \frac{\frac{F^{L}_{\tilde{\mathcal{R}}_l} + F^{L}_{\tilde{\mathcal{H}}_l}}{2} + F^{L}_{\tilde{\mathcal{G}}_l}}{2},$$

$$F^{U}_{(\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l)\tilde{\mathcal{G}}_l} = \frac{\frac{F^{U}_{\tilde{\mathcal{R}}_l} + F^{U}_{\tilde{\mathcal{H}}_l}}{2} + F^{U}_{\tilde{\mathcal{G}}_l}}{2}. \tag{8}$$

Definition 3.1.1, from (7) and (8) we get

$$\tilde{\mathcal{R}} \cap'_M (\tilde{\mathcal{H}} \cap'_M \tilde{\mathcal{G}}) =' (\tilde{\mathcal{R}} \cap'_M \tilde{\mathcal{H}}) \cap'_M \tilde{\mathcal{G}}$$

Proof of {xi, xii} are obtained similar to x.

xiii. From Definition 3.2.1 for $\tilde{\mathcal{R}} \cap'_M (\tilde{\mathcal{H}} \cup'_M \tilde{\mathcal{G}})$, we get

$$\breve{\mathcal{P}}_{l_{\tilde{\mathcal{R}}_l(\tilde{\mathcal{H}}_l\tilde{\mathcal{G}}_l)}} = \left(\breve{\mathcal{P}}_{l_{\tilde{\mathcal{R}}_l}} \cap \left(\breve{\mathcal{P}}_{l_{\tilde{\mathcal{H}}_l}} \cup \breve{\mathcal{P}}_{l_{\tilde{\mathcal{G}}_l}}\right)\right),\ \breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{R}}_l(\tilde{\mathcal{H}}_l\tilde{\mathcal{G}}_l)}}} = \left(\breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{R}}_l}}} \cap \left(\breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{H}}_l}}} \cup \breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{G}}_l}}}\right)\right)$$

$$\breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{R}}_l(\tilde{\mathcal{H}}_l\tilde{\mathcal{G}}_l)}}} = \left(\breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{R}}_l}}} \cap \left(\breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{H}}_l}}} \cup \breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{G}}_l}}}\right)\right),\ \breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{R}}_l(\tilde{\mathcal{H}}_l\tilde{\mathcal{G}}_l)}}} = \left(\breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{R}}_l}}} \cap \left(\breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{H}}_l}}} \cup \breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{G}}_l}}}\right)\right)$$

$$\breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{R}}_l(\tilde{\mathcal{H}}_l\tilde{\mathcal{G}}_l)}}} = \left(\breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{R}}_l}}} \cap \left(\breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{H}}_l}}} \cup \breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{G}}_l}}}\right)\right)$$

$$T^L_{\tilde{\mathcal{R}}_l(\tilde{\mathcal{H}}_l\tilde{\mathcal{G}}_l)} = \frac{T^L_{\tilde{\mathcal{R}}_l} + \frac{T^L_{\tilde{\mathcal{H}}_l} + T^L_{\tilde{\mathcal{G}}_l}}{2}}{2}, \qquad T^U_{\tilde{\mathcal{R}}_l(\tilde{\mathcal{H}}_l\tilde{\mathcal{G}}_l)} = \frac{T^U_{\tilde{\mathcal{R}}_l} + \frac{T^U_{\tilde{\mathcal{H}}_l} + T^U_{\tilde{\mathcal{G}}_l}}{2}}{2}$$

$$U^L_{\tilde{\mathcal{R}}_l(\tilde{\mathcal{H}}_l\tilde{\mathcal{G}}_l)} = \frac{U^L_{\tilde{\mathcal{R}}_l} + \frac{U^L_{\tilde{\mathcal{H}}_l} + U^L_{\tilde{\mathcal{G}}_l}}{2}}{2}, \qquad U^U_{\tilde{\mathcal{R}}_l(\tilde{\mathcal{H}}_l\tilde{\mathcal{G}}_l)} = \frac{U^U_{\tilde{\mathcal{R}}_l} + \frac{U^U_{\tilde{\mathcal{H}}_l} + U^U_{\tilde{\mathcal{G}}_l}}{2}}{2}$$

$$C^L_{\tilde{\mathcal{R}}_l(\tilde{\mathcal{H}}_l\tilde{\mathcal{G}}_l)} = \frac{C^L_{\tilde{\mathcal{R}}_l} + \frac{C^L_{\tilde{\mathcal{H}}_l} + C^L_{\tilde{\mathcal{G}}_l}}{2}}{2}, \qquad C^U_{\tilde{\mathcal{R}}_l(\tilde{\mathcal{H}}_l\tilde{\mathcal{G}}_l)} = \frac{C^U_{\tilde{\mathcal{R}}_l} + \frac{C^U_{\tilde{\mathcal{H}}_l} + C^U_{\tilde{\mathcal{G}}_l}}{2}}{2}$$

$$F^L_{\tilde{\mathcal{R}}_l(\tilde{\mathcal{H}}_l\tilde{\mathcal{G}}_l)} = \frac{F^L_{\tilde{\mathcal{R}}_l} + \frac{F^L_{\tilde{\mathcal{H}}_l} + F^L_{\tilde{\mathcal{G}}_l}}{2}}{2},$$

$$F^U_{\tilde{\mathcal{R}}_l(\tilde{\mathcal{H}}_l\tilde{\mathcal{G}}_l)} = \frac{F^U_{\tilde{\mathcal{R}}_l} + \frac{F^U_{\tilde{\mathcal{H}}_l} + F^U_{\tilde{\mathcal{G}}_l}}{2}}{2}. \tag{9}$$

Definition 3.2.1 for $(\tilde{\mathcal{R}} \cap'_M \tilde{\mathcal{H}}) \cup'_M (\tilde{\mathcal{R}} \cap'_M \tilde{\mathcal{G}})$, we get

$$\breve{\mathcal{P}}_{l_{(\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l)(\tilde{\mathcal{R}}_l\tilde{\mathcal{G}}_l)}} = \left(\left(\breve{\mathcal{P}}_{l_{\tilde{\mathcal{R}}_l}} \cap \breve{\mathcal{P}}_{l_{\tilde{\mathcal{H}}_l}}\right) \cup \left(\breve{\mathcal{P}}_{l_{\tilde{\mathcal{R}}_l}} \cap \breve{\mathcal{P}}_{l_{\tilde{\mathcal{G}}_l}}\right)\right),$$

$$\breve{\mathcal{P}}_{l_{T_{(\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l)(\tilde{\mathcal{R}}_l\tilde{\mathcal{G}}_l)}}} = \left(\left(\breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{R}}_l}}} \cap \breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{H}}_l}}}\right) \cup \left(\breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{R}}_l}}} \cap \breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{G}}_l}}}\right)\right),$$

$$\breve{\mathcal{P}}_{l_{U_{(\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l)(\tilde{\mathcal{R}}_l\tilde{\mathcal{G}}_l)}}} = \left(\left(\breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{R}}_l}}} \cap \breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{H}}_l}}}\right) \cup \left(\breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{R}}_l}}} \cap \breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{G}}_l}}}\right)\right),$$

$$\breve{\mathcal{P}}_{l_{C_{(\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l)(\tilde{\mathcal{R}}_l\tilde{\mathcal{G}}_l)}}} = \left(\left(\breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{R}}_l}}} \cap \breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{H}}_l}}}\right) \cup \left(\breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{R}}_l}}} \cap \breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{G}}_l}}}\right)\right),$$

$$\breve{\mathcal{P}}_{l_{F_{(\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l)(\tilde{\mathcal{R}}_l\tilde{\mathcal{G}}_l)}}} = \left(\left(\breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{R}}_l}}} \cap \breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{H}}_l}}}\right) \cup \left(\breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{R}}_l}}} \cap \breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{G}}_l}}}\right)\right),$$

$$T^L_{(\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l)(\tilde{\mathcal{R}}_l\tilde{\mathcal{G}}_l)} = \frac{\frac{T^L_{\tilde{\mathcal{R}}_l} + T^L_{\tilde{\mathcal{H}}_l}}{2} + \frac{T^L_{\tilde{\mathcal{R}}_l} + T^L_{\tilde{\mathcal{G}}_l}}{2}}{2}, \qquad T^U_{(\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l)(\tilde{\mathcal{R}}_l\tilde{\mathcal{G}}_l)} = \frac{\frac{T^U_{\tilde{\mathcal{R}}_l} + T^U_{\tilde{\mathcal{H}}_l}}{2} + \frac{T^U_{\tilde{\mathcal{R}}_l} + T^U_{\tilde{\mathcal{G}}_l}}{2}}{2}$$

$$U^L_{(\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l)(\tilde{\mathcal{R}}_l\tilde{\mathcal{G}}_l)} = \frac{\frac{U^L_{\tilde{\mathcal{R}}_l} + U^L_{\tilde{\mathcal{H}}_l}}{2} + \frac{U^L_{\tilde{\mathcal{R}}_l} + U^L_{\tilde{\mathcal{G}}_l}}{2}}{2}, \qquad U^U_{(\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l)(\tilde{\mathcal{R}}_l\tilde{\mathcal{G}}_l)} = \frac{\frac{U^U_{\tilde{\mathcal{R}}_l} + U^U_{\tilde{\mathcal{H}}_l}}{2} + \frac{U^U_{\tilde{\mathcal{R}}_l} + U^U_{\tilde{\mathcal{G}}_l}}{2}}{2}$$

$$C^{L}{}_{(\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l)(\tilde{\mathcal{R}}_l\tilde{\mathcal{G}}_l)} = \frac{\frac{C^{L}{}_{\tilde{\mathcal{R}}_l} + C^{L}{}_{\tilde{\mathcal{H}}_l}}{2} + \frac{C^{L}{}_{\tilde{\mathcal{R}}_l} + C^{L}{}_{\tilde{\mathcal{G}}_l}}{2}}{2}, \qquad C^{U}{}_{(\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l)(\tilde{\mathcal{R}}_l\tilde{\mathcal{G}}_l)} = \frac{\frac{C^{U}{}_{\tilde{\mathcal{R}}_l} + C^{U}{}_{\tilde{\mathcal{H}}_l}}{2} + \frac{C^{U}{}_{\tilde{\mathcal{R}}_l} + C^{U}{}_{\tilde{\mathcal{G}}_l}}{2}}{2}$$

$$F^{L}{}_{(\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l)(\tilde{\mathcal{R}}_l\tilde{\mathcal{G}}_l)} = \frac{\frac{F^{L}{}_{\tilde{\mathcal{R}}_l} + F^{L}{}_{\tilde{\mathcal{H}}_l}}{2} + \frac{F^{L}{}_{\tilde{\mathcal{R}}_l} + F^{L}{}_{\tilde{\mathcal{G}}_l}}{2}}{2},$$

$$F^{U}{}_{(\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l)(\tilde{\mathcal{R}}_l\tilde{\mathcal{G}}_l)} = \frac{\frac{F^{U}{}_{\tilde{\mathcal{R}}_l} + F^{U}{}_{\tilde{\mathcal{H}}_l}}{2} + \frac{F^{U}{}_{\tilde{\mathcal{R}}_l} + F^{U}{}_{\tilde{\mathcal{G}}_l}}{2}}{2} \tag{10}$$

From Definition 3.1.1, (9) and (10) we get

$$\tilde{\mathcal{R}} \cap'_M (\tilde{\mathcal{H}} \cup'_M \tilde{\mathcal{G}}) =' (\tilde{\mathcal{R}} \cap'_M \tilde{\mathcal{H}}) \cup'_M (\tilde{\mathcal{R}} \cap'_M \tilde{\mathcal{G}}).$$

Proof of {xiv, xv} are obtained similar to xiii.

xvi. From Definition 3.2.1 for $\tilde{\mathcal{R}} \cup'_M (\tilde{\mathcal{H}} \cap'_M \tilde{\mathcal{G}})$, we get

$$\breve{\mathcal{P}}_{l_{\tilde{\mathcal{R}}_l(\tilde{\mathcal{H}}_l\tilde{\mathcal{G}}_l)}} = \left(\breve{\mathcal{P}}_{l_{\tilde{\mathcal{R}}_l}} \cup \left(\breve{\mathcal{P}}_{l_{\tilde{\mathcal{H}}_l}} \cap \breve{\mathcal{P}}_{l_{\tilde{\mathcal{G}}_l}}\right)\right), \qquad \breve{\mathcal{P}}_{l_{T\tilde{\mathcal{R}}_l(\tilde{\mathcal{H}}_l\tilde{\mathcal{G}}_l)}} = \left(\breve{\mathcal{P}}_{l_{T\tilde{\mathcal{R}}_l}} \cup \left(\breve{\mathcal{P}}_{l_{T\tilde{\mathcal{H}}_l}} \cap \breve{\mathcal{P}}_{l_{T\tilde{\mathcal{G}}_l}}\right)\right)$$

$$\breve{\mathcal{P}}_{l_{U\tilde{\mathcal{R}}_l(\tilde{\mathcal{H}}_l\tilde{\mathcal{G}}_l)}} = \left(\breve{\mathcal{P}}_{l_{U\tilde{\mathcal{R}}_l}} \cup \left(\breve{\mathcal{P}}_{l_{U\tilde{\mathcal{H}}_l}} \cap \breve{\mathcal{P}}_{l_{U\tilde{\mathcal{G}}_l}}\right)\right), \qquad \breve{\mathcal{P}}_{l_{C\tilde{\mathcal{R}}_l(\tilde{\mathcal{H}}_l\tilde{\mathcal{G}}_l)}} = \left(\breve{\mathcal{P}}_{l_{C\tilde{\mathcal{R}}_l}} \cup \left(\breve{\mathcal{P}}_{l_{C\tilde{\mathcal{H}}_l}} \cap \breve{\mathcal{P}}_{l_{C\tilde{\mathcal{G}}_l}}\right)\right)$$

$$\breve{\mathcal{P}}_{l_{F\tilde{\mathcal{R}}_l(\tilde{\mathcal{H}}_l\tilde{\mathcal{G}}_l)}} = \left(\breve{\mathcal{P}}_{l_{F\tilde{\mathcal{R}}_l}} \cup \left(\breve{\mathcal{P}}_{l_{F\tilde{\mathcal{H}}_l}} \cap \breve{\mathcal{P}}_{l_{F\tilde{\mathcal{G}}_l}}\right)\right)$$

$$T^{L}{}_{\tilde{\mathcal{R}}_l(\tilde{\mathcal{H}}_l\tilde{\mathcal{G}}_l)} = \frac{T^{L}{}_{\tilde{\mathcal{R}}_l} + \frac{T^{L}{}_{\tilde{\mathcal{H}}_l} + T^{L}{}_{\tilde{\mathcal{G}}_l}}{2}}{2}, \qquad T^{U}{}_{\tilde{\mathcal{R}}_l(\tilde{\mathcal{H}}_l\tilde{\mathcal{G}}_l)} = \frac{T^{U}{}_{\tilde{\mathcal{R}}_l} + \frac{T^{U}{}_{\tilde{\mathcal{H}}_l} + T^{U}{}_{\tilde{\mathcal{G}}_l}}{2}}{2}$$

$$U^{L}{}_{\tilde{\mathcal{R}}_l(\tilde{\mathcal{H}}_l\tilde{\mathcal{G}}_l)} = \frac{U^{L}{}_{\tilde{\mathcal{R}}_l} + \frac{U^{L}{}_{\tilde{\mathcal{H}}_l} + U^{L}{}_{\tilde{\mathcal{G}}_l}}{2}}{2}, \qquad U^{U}{}_{\tilde{\mathcal{R}}_l(\tilde{\mathcal{H}}_l\tilde{\mathcal{G}}_l)} = \frac{U^{U}{}_{\tilde{\mathcal{R}}_l} + \frac{U^{U}{}_{\tilde{\mathcal{H}}_l} + U^{U}{}_{\tilde{\mathcal{G}}_l}}{2}}{2}$$

$$C^{L}{}_{\tilde{\mathcal{R}}_l(\tilde{\mathcal{H}}_l\tilde{\mathcal{G}}_l)} = \frac{C^{L}{}_{\tilde{\mathcal{R}}_l} + \frac{C^{L}{}_{\tilde{\mathcal{H}}_l} + C^{L}{}_{\tilde{\mathcal{G}}_l}}{2}}{2}, \qquad C^{U}{}_{\tilde{\mathcal{R}}_l(\tilde{\mathcal{H}}_l\tilde{\mathcal{G}}_l)} = \frac{C^{U}{}_{\tilde{\mathcal{R}}_l} + \frac{C^{U}{}_{\tilde{\mathcal{H}}_l} + C^{U}{}_{\tilde{\mathcal{G}}_l}}{2}}{2}$$

$$F^{L}{}_{\tilde{\mathcal{R}}_l(\tilde{\mathcal{H}}_l\tilde{\mathcal{G}}_l)} = \frac{F^{L}{}_{\tilde{\mathcal{R}}_l} + \frac{F^{L}{}_{\tilde{\mathcal{H}}_l} + F^{L}{}_{\tilde{\mathcal{G}}_l}}{2}}{2},$$

$$F^{U}{}_{\tilde{\mathcal{R}}_l(\tilde{\mathcal{H}}_l\tilde{\mathcal{G}}_l)} = \frac{F^{U}{}_{\tilde{\mathcal{R}}_l} + \frac{F^{U}{}_{\tilde{\mathcal{H}}_l} + F^{U}{}_{\tilde{\mathcal{G}}_l}}{2}}{2}. \tag{11}$$

Definition 3.2.1 for $(\tilde{\mathcal{R}} \cup'_M \tilde{\mathcal{H}}) \cap'_M (\tilde{\mathcal{R}} \cup'_M \tilde{\mathcal{G}})$ we get

$$\breve{\mathcal{P}}_{l_{(\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l)(\tilde{\mathcal{R}}_l\tilde{\mathcal{G}}_l)}} = \left(\left(\breve{\mathcal{P}}_{l_{\tilde{\mathcal{R}}_l}} \cup \breve{\mathcal{P}}_{l_{\tilde{\mathcal{H}}_l}}\right) \cap \left(\breve{\mathcal{P}}_{l_{\tilde{\mathcal{R}}_l}} \cup \breve{\mathcal{P}}_{l_{\tilde{\mathcal{G}}_l}}\right)\right),$$

$$\breve{\mathcal{P}}_{l_{T(\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l)(\tilde{\mathcal{R}}_l\tilde{\mathcal{G}}_l)}} = \left(\left(\breve{\mathcal{P}}_{l_{T\tilde{\mathcal{R}}_l}} \cup \breve{\mathcal{P}}_{l_{T\tilde{\mathcal{H}}_l}}\right) \cap \left(\breve{\mathcal{P}}_{l_{T\tilde{\mathcal{R}}_l}} \cup \breve{\mathcal{P}}_{l_{T\tilde{\mathcal{G}}_l}}\right)\right)$$

$$\breve{\mathcal{P}}_{l_{U(\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l)(\tilde{\mathcal{R}}_l\tilde{\mathcal{G}}_l)}} = \left(\left(\breve{\mathcal{P}}_{l_{U\tilde{\mathcal{R}}_l}} \cup \breve{\mathcal{P}}_{l_{U\tilde{\mathcal{H}}_l}}\right) \cap \left(\breve{\mathcal{P}}_{l_{U\tilde{\mathcal{R}}_l}} \cup \breve{\mathcal{P}}_{l_{U\tilde{\mathcal{G}}_l}}\right)\right),$$

$$\breve{\mathcal{P}}_{l_{C(\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l)(\tilde{\mathcal{R}}_l\tilde{\mathcal{G}}_l)}} = \left(\left(\breve{\mathcal{P}}_{l_{C\tilde{\mathcal{R}}_l}} \cup \breve{\mathcal{P}}_{l_{C\tilde{\mathcal{H}}_l}}\right) \cap \left(\breve{\mathcal{P}}_{l_{C\tilde{\mathcal{R}}_l}} \cup \breve{\mathcal{P}}_{l_{C\tilde{\mathcal{G}}_l}}\right)\right)$$

$$\breve{\mathcal{P}}_{l_{F(\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l)(\tilde{\mathcal{R}}_l\tilde{\mathcal{G}}_l)}} = \left(\left(\breve{\mathcal{P}}_{l_{F\tilde{\mathcal{R}}_l}} \cup \breve{\mathcal{P}}_{l_{F\tilde{\mathcal{H}}_l}}\right) \cap \left(\breve{\mathcal{P}}_{l_{F\tilde{\mathcal{R}}_l}} \cup \breve{\mathcal{P}}_{l_{F\tilde{\mathcal{G}}_l}}\right)\right)$$

$$T^L_{(\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l)(\tilde{\mathcal{R}}_l\tilde{\mathcal{G}}_l)} = \frac{\frac{T^L_{\tilde{\mathcal{R}}_l} + T^L_{\tilde{\mathcal{H}}_l}}{2} + \frac{T^L_{\tilde{\mathcal{R}}_l} + T^L_{\tilde{\mathcal{G}}_l}}{2}}{2}, \qquad T^U_{(\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l)(\tilde{\mathcal{R}}_l\tilde{\mathcal{G}}_l)} = \frac{\frac{T^U_{\tilde{\mathcal{R}}_l} + T^U_{\tilde{\mathcal{H}}_l}}{2} + \frac{T^U_{\tilde{\mathcal{R}}_l} + T^U_{\tilde{\mathcal{G}}_l}}{2}}{2}$$

$$U^L_{(\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l)(\tilde{\mathcal{R}}_l\tilde{\mathcal{G}}_l)} = \frac{\frac{U^L_{\tilde{\mathcal{R}}_l} + U^L_{\tilde{\mathcal{H}}_l}}{2} + \frac{U^L_{\tilde{\mathcal{R}}_l} + U^L_{\tilde{\mathcal{G}}_l}}{2}}{2}, \qquad U^U_{(\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l)(\tilde{\mathcal{R}}_l\tilde{\mathcal{G}}_l)} = \frac{\frac{U^U_{\tilde{\mathcal{R}}_l} + U^U_{\tilde{\mathcal{H}}_l}}{2} + \frac{U^U_{\tilde{\mathcal{R}}_l} + U^U_{\tilde{\mathcal{G}}_l}}{2}}{2}$$

$$C^L_{(\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l)(\tilde{\mathcal{R}}_l\tilde{\mathcal{G}}_l)} = \frac{\frac{C^L_{\tilde{\mathcal{R}}_l} + C^L_{\tilde{\mathcal{H}}_l}}{2} + \frac{C^L_{\tilde{\mathcal{R}}_l} + C^L_{\tilde{\mathcal{G}}_l}}{2}}{2}, \qquad C^U_{(\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l)(\tilde{\mathcal{R}}_l\tilde{\mathcal{G}}_l)} = \frac{\frac{C^U_{\tilde{\mathcal{R}}_l} + C^U_{\tilde{\mathcal{H}}_l}}{2} + \frac{C^U_{\tilde{\mathcal{R}}_l} + C^U_{\tilde{\mathcal{G}}_l}}{2}}{2}$$

$$F^L_{(\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l)(\tilde{\mathcal{R}}_l\tilde{\mathcal{G}}_l)} = \frac{\frac{F^L_{\tilde{\mathcal{R}}_l} + F^L_{\tilde{\mathcal{H}}_l}}{2} + \frac{F^L_{\tilde{\mathcal{R}}_l} + F^L_{\tilde{\mathcal{G}}_l}}{2}}{2},$$

$$F^U_{(\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l)(\tilde{\mathcal{R}}_l\tilde{\mathcal{G}}_l)} = \frac{\frac{F^U_{\tilde{\mathcal{R}}_l} + F^U_{\tilde{\mathcal{H}}_l}}{2} + \frac{F^U_{\tilde{\mathcal{R}}_l} + F^U_{\tilde{\mathcal{G}}_l}}{2}}{2}. \qquad (12)$$

Definition 3.1.1, from (11) and (12), we get

$$\tilde{\mathcal{R}} \cup'_M \left(\tilde{\mathcal{H}} \cap'_M \tilde{\mathcal{G}}\right) =' \left(\tilde{\mathcal{R}} \cup'_M \tilde{\mathcal{H}}\right) \cap'_M \left(\tilde{\mathcal{R}} \cup'_M \tilde{\mathcal{G}}\right).$$

Proof of {xvii, xviii} are obtained similar to xvi.

xix. We assume that $\tilde{\mathcal{R}} =' \tilde{\mathcal{H}}$. From, Definition 3.1.1, Definition 3.2.1, Definition 3.2.3, Definition 3.2.5 we obtain

$$T^L_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \min\{T^L_{\tilde{\mathcal{R}}_l}, T^L_{\tilde{\mathcal{H}}_l}\} = \frac{T^L_{\tilde{\mathcal{R}}_l} + T^L_{\tilde{\mathcal{H}}_l}}{2} = \max\{T^L_{\tilde{\mathcal{R}}_l}, T^L_{\tilde{\mathcal{H}}_l}\}$$

$$T^U_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \min\{T^U_{\tilde{\mathcal{R}}_l}, T^U_{\tilde{\mathcal{H}}_l}\} = \frac{T^U_{\tilde{\mathcal{R}}_l} + T^U_{\tilde{\mathcal{H}}_l}}{2} = \max\{T^U_{\tilde{\mathcal{R}}_l}, T^U_{\tilde{\mathcal{H}}_l}\}$$

$$U^L_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \max\{U^L_{\tilde{\mathcal{R}}_l}, U^L_{\tilde{\mathcal{H}}_l}\} = \frac{U^L_{\tilde{\mathcal{R}}_l} + U^L_{\tilde{\mathcal{H}}_l}}{2} = \min\{U^L_{\tilde{\mathcal{R}}_l}, U^L_{\tilde{\mathcal{H}}_l}\}$$

$$U^U_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \max\{U^U_{\tilde{\mathcal{R}}_l}, U^U_{\tilde{\mathcal{H}}_l}\} = \frac{U^U_{\tilde{\mathcal{R}}_l} + U^U_{\tilde{\mathcal{H}}_l}}{2} = \min\{U^U_{\tilde{\mathcal{R}}_l}, U^U_{\tilde{\mathcal{H}}_l}\}$$

$$C^L_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \max\ \{C^L_{\tilde{\mathcal{R}}_l}, C^L_{\tilde{\mathcal{H}}_l}\} = \frac{C^L_{\tilde{\mathcal{R}}_l} + C^L_{\tilde{\mathcal{H}}_l}}{2} = \min\{C^L_{\tilde{\mathcal{R}}_l}, C^L_{\tilde{\mathcal{H}}_l}\}$$

$$C^U{}_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \max\{C^U{}_{\tilde{\mathcal{R}}_l}, C^U{}_{\tilde{\mathcal{H}}_l}\} = \frac{C^U{}_{\tilde{\mathcal{R}}_l} + C^U{}_{\tilde{\mathcal{H}}_l}}{2} = \min\{C^U{}_{\tilde{\mathcal{R}}_l}, C^U{}_{\tilde{\mathcal{H}}_l}\}$$

$$F^L{}_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \max\{F^L{}_{\tilde{\mathcal{R}}_l}, F^L{}_{\tilde{\mathcal{H}}_l}\} = \frac{F^L{}_{\tilde{\mathcal{R}}_l} + F^L{}_{\tilde{\mathcal{H}}_l}}{2} = \min\{F^L{}_{\tilde{\mathcal{R}}_l}, F^L{}_{\mathcal{H}_l}\}$$

$$F^U{}_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \max\{F^U{}_{\tilde{\mathcal{R}}_l}, F^U{}_{\tilde{\mathcal{H}}_l}\} = \frac{F^U{}_{\tilde{\mathcal{R}}_l} + F^U{}_{\tilde{\mathcal{H}}_l}}{2} = \min\{F^U{}_{\mathcal{R}_l}, F^U{}_{\tilde{\mathcal{H}}_l}\}. \quad (13)$$

Also,

$$\breve{\mathcal{P}}_{l_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}} = \left(\breve{\mathcal{P}}_{l_{\tilde{\mathcal{R}}_l}} \cup \breve{\mathcal{P}}_{l_{\tilde{\mathcal{H}}_l}}\right),\ \breve{\mathcal{P}}_{l_{T\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}} = \left(\breve{\mathcal{P}}_{l_{T\tilde{\mathcal{R}}_l}} \cup \breve{\mathcal{P}}_{l_{T\tilde{\mathcal{H}}_l}}\right),\ \breve{\mathcal{P}}_{l_{U\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}} = \left(\breve{\mathcal{P}}_{l_{U\tilde{\mathcal{R}}_l}} \cup \breve{\mathcal{P}}_{l_{U\tilde{\mathcal{H}}_l}}\right)$$

$$\breve{\mathcal{P}}_{l_{C\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}} = \left(\breve{\mathcal{P}}_{l_{C\tilde{\mathcal{R}}_l}} \cup \breve{\mathcal{P}}_{l_{C\tilde{\mathcal{H}}_l}}\right),\ \breve{\mathcal{P}}_{l_{F\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}} = \left(\breve{\mathcal{P}}_{l_{F\tilde{\mathcal{R}}_l}} \cup \breve{\mathcal{P}}_{l_{F\tilde{\mathcal{H}}_l}}\right) \quad (14)$$

Thusi from (13) and(14) we get

$$(\mathcal{R} \cup'_M \mathcal{H}) =' (\mathcal{R} \cup'_O \mathcal{H}) =' (\mathcal{R} \cup'_P \mathcal{H}).$$

Proof of xx are obtained similar to xix.

Theorem 3.2.10: Let

$$\tilde{\mathcal{R}} = \{< \breve{\mathcal{P}}_{1_{\tilde{\mathcal{R}}_1}},\ \breve{\mathcal{P}}_{1_{T\tilde{\mathcal{R}}_1}}[\ T^L{}_{\tilde{\mathcal{R}}_1}, T^U{}_{\tilde{\mathcal{R}}_1}],\ \breve{\mathcal{P}}_{1_{U\tilde{\mathcal{R}}_1}}[U^L{}_{\tilde{\mathcal{R}}_1},\ U^U{}_{\tilde{\mathcal{R}}_1}],\ \breve{\mathcal{P}}_{1_{C\tilde{\mathcal{R}}_1}}[C^L{}_{\tilde{\mathcal{R}}_1},\ C^U{}_{\tilde{\mathcal{R}}_1}], \breve{\mathcal{P}}_{1_{F\tilde{\mathcal{R}}_1}}[F^L{}_{\tilde{\mathcal{R}}_1}, F^U{}_{\tilde{\mathcal{R}}_1}];$$

$$\breve{\mathcal{P}}_{2_{\tilde{\mathcal{R}}_2}},\ \breve{\mathcal{P}}_{2_{T\tilde{\mathcal{R}}_2}}[\ T^L{}_{\tilde{\mathcal{R}}_2}, T^U{}_{\tilde{\mathcal{R}}_2}],\ \breve{\mathcal{P}}_{2_{U\tilde{\mathcal{R}}_2}}[U^L{}_{\tilde{\mathcal{R}}_2},\ U^U{}_{\tilde{\mathcal{R}}_2}],\ \breve{\mathcal{P}}_{2_{C\tilde{\mathcal{R}}_2}}[C^L{}_{\tilde{\mathcal{R}}_2},\ C^U{}_{\tilde{\mathcal{R}}_2}], \breve{\mathcal{P}}_{2_{F\tilde{\mathcal{R}}_2}}[F^L{}_{\tilde{\mathcal{R}}_2}, F^U{}_{\tilde{\mathcal{R}}_2}];$$

...

$$\breve{\mathcal{P}}_{s_{\tilde{\mathcal{R}}_s}},\ \breve{\mathcal{P}}_{s_{T\tilde{\mathcal{R}}_s}}[\ T^L{}_{\tilde{\mathcal{R}}_s}, T^U{}_{\tilde{\mathcal{R}}_s}],\ \breve{\mathcal{P}}_{s_{U\tilde{\mathcal{R}}_s}}[U^L{}_{\tilde{\mathcal{R}}_s},\ U^U{}_{\tilde{\mathcal{R}}_s}],\ \breve{\mathcal{P}}_{s_{C\tilde{\mathcal{R}}_s}}[C^L{}_{\tilde{\mathcal{R}}_s},\ C^U{}_{\tilde{\mathcal{R}}_s}], \breve{\mathcal{P}}_{s_{F\tilde{\mathcal{R}}_s}}[F^L{}_{\tilde{\mathcal{R}}_s}, F^U{}_{\tilde{\mathcal{R}}_s}];>,$$

$$\breve{\mathcal{P}}_{l_{\tilde{\mathcal{R}}_l}}, \breve{\mathcal{P}}_{l_{T\tilde{\mathcal{R}}_l}}, \breve{\mathcal{P}}_{l_{U\tilde{\mathcal{R}}_l}}, \breve{\mathcal{P}}_{l_{C\tilde{\mathcal{R}}_l}}, \breve{\mathcal{P}}_{l_{F\tilde{\mathcal{R}}_l}} \in P(\mathcal{N}); l = 1,2,3,\dots,s\}$$

and

$$\tilde{\mathcal{H}} = \{< \breve{\mathcal{P}}_{1_{\tilde{\mathcal{H}}_1}},\ \breve{\mathcal{P}}_{1_{T\tilde{\mathcal{H}}_1}}[\ T^L{}_{\tilde{\mathcal{H}}_1}, T^U{}_{\tilde{\mathcal{H}}_1}],\ \breve{\mathcal{P}}_{1_{U\tilde{\mathcal{H}}_1}}[U^L{}_{\tilde{\mathcal{H}}_1},\ U^U{}_{\tilde{\mathcal{H}}_1}],\ \breve{\mathcal{P}}_{1_{C\tilde{\mathcal{H}}_1}}[C^L{}_{\tilde{\mathcal{H}}_1},\ C^U{}_{\tilde{\mathcal{H}}_1}], \breve{\mathcal{P}}_{1_{F\tilde{\mathcal{H}}_1}}[F^L{}_{\tilde{\mathcal{H}}_1}, F^U{}_{\tilde{\mathcal{H}}_1}];$$

$$\breve{\mathcal{P}}_{2_{\tilde{\mathcal{H}}_2}},\ \breve{\mathcal{P}}_{2_{T\tilde{\mathcal{H}}_2}}[\ T^L{}_{\tilde{\mathcal{H}}_2}, T^U{}_{\tilde{\mathcal{H}}_2}],\ \breve{\mathcal{P}}_{2_{U\tilde{\mathcal{H}}_2}}[U^L{}_{\tilde{\mathcal{H}}_2},\ U^U{}_{\tilde{\mathcal{H}}_2}],\ \breve{\mathcal{P}}_{2_{C\tilde{\mathcal{H}}_2}}[C^L{}_{\tilde{\mathcal{H}}_2},\ C^U{}_{\tilde{\mathcal{H}}_2}], \breve{\mathcal{P}}_{2_{F\tilde{\mathcal{H}}_2}}[F^L{}_{\tilde{\mathcal{H}}_2}, F^U{}_{\tilde{\mathcal{H}}_2}];$$

...

$$\breve{\mathcal{P}}_{s_{\tilde{\mathcal{H}}_s}},\ \breve{\mathcal{P}}_{s_{T\tilde{\mathcal{H}}_s}}[\ T^L{}_{\tilde{\mathcal{H}}_s}, T^U{}_{\tilde{\mathcal{H}}_s}],\ \breve{\mathcal{P}}_{s_{U\tilde{\mathcal{H}}_s}}[U^L{}_{\tilde{\mathcal{H}}_s},\ U^U{}_{\tilde{\mathcal{H}}_s}],\ \breve{\mathcal{P}}_{s_{C\tilde{\mathcal{H}}_s}}[C^L{}_{\tilde{\mathcal{H}}_s},\ C^U{}_{\tilde{\mathcal{H}}_s}], \breve{\mathcal{P}}_{s_{F\tilde{\mathcal{H}}_s}}[F^L{}_{\tilde{\mathcal{H}}_s}, F^U{}_{\tilde{\mathcal{H}}_s}];>,$$

$$\breve{\mathcal{P}}_{l_{\tilde{\mathcal{H}}_l}}, \breve{\mathcal{P}}_{l_{T\tilde{\mathcal{H}}_l}}, \breve{\mathcal{P}}_{l_{U\tilde{\mathcal{H}}_l}}, \breve{\mathcal{P}}_{l_{C\tilde{\mathcal{H}}_l}}, \breve{\mathcal{P}}_{l_{F\tilde{\mathcal{H}}_l}} \in P(\mathcal{N}); l = 1,2,3,\dots,s\}$$

be two IGSVNQSs. Then, following conditions are satisfied.

i. $\left(\tilde{\mathcal{R}} \cap'_P \tilde{\mathcal{H}}\right) \subset' \left(\tilde{\mathcal{R}} \cap'_M \tilde{\mathcal{H}}\right) \subset' \left(\tilde{\mathcal{R}} \cap'_O \tilde{\mathcal{H}}\right)$

ii. $\left(\tilde{\mathcal{R}} \cup'_P \tilde{\mathcal{H}}\right) \subset' \left(\tilde{\mathcal{R}} \cup'_M \tilde{\mathcal{H}}\right) \subset' \left(\tilde{\mathcal{R}} \cup'_O \tilde{\mathcal{H}}\right)$

iii. $(\tilde{\mathcal{R}} \cap'_M \tilde{\mathcal{H}}) \subset' (\tilde{\mathcal{R}} \cup'_M \tilde{\mathcal{H}})$

$(\tilde{\mathcal{R}} \cap'_O \tilde{\mathcal{H}}) \subset' (\tilde{\mathcal{R}} \cup'_O \tilde{\mathcal{H}})$

$(\tilde{\mathcal{R}} \cap'_P \tilde{\mathcal{H}}) \subset' (\tilde{\mathcal{R}} \cup'_P \tilde{\mathcal{H}})$

Proof:

i. From Definition 3.1.1, Definition 3.2.1, Definition 3.2.3 and Definition 3.2.5, we get

$$\min\{T^L_{\tilde{\mathcal{R}}_l}, T^L_{\tilde{\mathcal{H}}_l}\} \leq \frac{T^L_{\tilde{\mathcal{R}}_l} + T^L_{\tilde{\mathcal{H}}_l}}{2} \leq \max\{T^L_{\tilde{\mathcal{R}}_l}, T^L_{\tilde{\mathcal{H}}_l}\}$$

$$\min\{T^U_{\tilde{\mathcal{R}}_l}, T^U_{\tilde{\mathcal{H}}_l}\} \leq \frac{T^U_{\tilde{\mathcal{R}}_l} + T^U_{\tilde{\mathcal{H}}_l}}{2} \leq \max\{T^U_{\tilde{\mathcal{R}}_l}, T^U_{\tilde{\mathcal{H}}_l}\}$$

$$\max\{U^L_{\tilde{\mathcal{R}}_l}, U^L_{\tilde{\mathcal{H}}_l}\} \geq \frac{U^L_{\tilde{\mathcal{R}}_l} + U^L_{\tilde{\mathcal{H}}_l}}{2} \geq \min\{U^L_{\tilde{\mathcal{R}}_l}, U^L_{\tilde{\mathcal{H}}_l}\}$$

$$\max\{U^U_{\tilde{\mathcal{R}}_l}, U^U_{\tilde{\mathcal{H}}_l}\} \geq \frac{U^U_{\tilde{\mathcal{R}}_l} + U^U_{\tilde{\mathcal{H}}_l}}{2} \geq \min\{U^U_{\tilde{\mathcal{R}}_l}, U^U_{\tilde{\mathcal{H}}_l}\}$$

$$\max\ \{C^L_{\tilde{\mathcal{R}}_l}, C^L_{\tilde{\mathcal{H}}_l}\} \geq \frac{C^L_{\tilde{\mathcal{R}}_l} + C^L_{\tilde{\mathcal{H}}_l}}{2} \geq \min\{C^L_{\tilde{\mathcal{R}}_l}, C^L_{\tilde{\mathcal{H}}_l}\}$$

$$\max\{C^U_{\tilde{\mathcal{R}}_l}, C^U_{\tilde{\mathcal{H}}_l}\} \geq \frac{C^U_{\tilde{\mathcal{R}}_l} + C^U_{\tilde{\mathcal{H}}_l}}{2} \geq \min\{C^U_{\tilde{\mathcal{R}}_l}, C^U_{\tilde{\mathcal{H}}_l}\}$$

$$\max\{F^L_{\tilde{\mathcal{R}}_l}, F^L_{\tilde{\mathcal{H}}_l}\} \geq \frac{F^L_{\tilde{\mathcal{R}}_l} + F^L_{\tilde{\mathcal{H}}_l}}{2} \geq \min\{F^L_{\tilde{\mathcal{R}}_l}, F^L_{\mathcal{H}_l}\}$$

$$\max\{F^U_{\tilde{\mathcal{R}}_l}, F^U_{\tilde{\mathcal{H}}_l}\} \geq \frac{F^U_{\tilde{\mathcal{R}}_l} + F^U_{\tilde{\mathcal{H}}_l}}{2} \geq \min\{F^U_{\mathcal{R}_l}, F^U_{\tilde{\mathcal{H}}_l}\}. \quad (15)$$

Also,

$$\breve{\mathcal{P}}_{l_{\tilde{\mathcal{R}}_l \tilde{\mathcal{H}}_l}} = \left(\breve{\mathcal{P}}_{l_{\tilde{\mathcal{R}}_l}} \cap \breve{\mathcal{P}}_{l_{\tilde{\mathcal{H}}_l}}\right),\ \breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{R}}_l \tilde{\mathcal{H}}_l}}} = \left(\breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{R}}_l}}} \cap \breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{H}}_l}}}\right),\ \breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{R}}_l \tilde{\mathcal{H}}_l}}} = \left(\breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{R}}_l}}} \cap \breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{H}}_l}}}\right)$$

$$\breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{R}}_l \tilde{\mathcal{H}}_l}}} = \left(\breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{R}}_l}}} \cap \breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{H}}_l}}}\right),\ \breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{R}}_l \tilde{\mathcal{H}}_l}}} = \left(\breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{R}}_l}}} \cap \breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{H}}_l}}}\right) \quad (16)$$

Thus, from (15), (16) and Definition 3.1.1, we get

$$\left(\tilde{\mathcal{R}} \cap'_P \tilde{\mathcal{H}}\right) \subset' \left(\tilde{\mathcal{R}} \cap'_M \tilde{\mathcal{H}}\right) \subset' \left(\tilde{\mathcal{R}} \cap'_O \tilde{\mathcal{H}}\right).$$

The proofs of {ii, iii} are similar to i.

Theorem 3.2.11: Let

$\tilde{\mathcal{R}} = \{< \breve{\mathcal{P}}_{1\tilde{\mathcal{R}}_1},\ \breve{\mathcal{P}}_{1T\tilde{\mathcal{R}}_1}[\ T^L_{\tilde{\mathcal{R}}_1}, T^U_{\tilde{\mathcal{R}}_1}],\ \breve{\mathcal{P}}_{1U\tilde{\mathcal{R}}_1}[U^L_{\tilde{\mathcal{R}}_1},\ U^U_{\tilde{\mathcal{R}}_1}],\ \breve{\mathcal{P}}_{1C\tilde{\mathcal{R}}_1}[C^L_{\tilde{\mathcal{R}}_1},\ C^U_{\tilde{\mathcal{R}}_1}], \breve{\mathcal{P}}_{1F\tilde{\mathcal{R}}_1}[F^L_{\tilde{\mathcal{R}}_1}, F^U_{\tilde{\mathcal{R}}_1}];$

$\breve{\mathcal{P}}_{2\tilde{\mathcal{R}}_2},\ \breve{\mathcal{P}}_{2T\tilde{\mathcal{R}}_2}[\ T^L_{\tilde{\mathcal{R}}_2}, T^U_{\tilde{\mathcal{R}}_2}],\ \breve{\mathcal{P}}_{2U\tilde{\mathcal{R}}_2}[U^L_{\tilde{\mathcal{R}}_2},\ U^U_{\tilde{\mathcal{R}}_2}],\ \breve{\mathcal{P}}_{2C\tilde{\mathcal{R}}_2}[C^L_{\tilde{\mathcal{R}}_2},\ C^U_{\tilde{\mathcal{R}}_2}], \breve{\mathcal{P}}_{2F\tilde{\mathcal{R}}_2}[F^L_{\tilde{\mathcal{R}}_2}, F^U_{\tilde{\mathcal{R}}_2}];$

…

$\breve{\mathcal{P}}_{s\tilde{\mathcal{R}}_s},\ \breve{\mathcal{P}}_{sT\tilde{\mathcal{R}}_s}[\ T^L_{\tilde{\mathcal{R}}_s}, T^U_{\tilde{\mathcal{R}}_s}],\ \breve{\mathcal{P}}_{sU\tilde{\mathcal{R}}_s}[U^L_{\tilde{\mathcal{R}}_s},\ U^U_{\tilde{\mathcal{R}}_s}],\ \breve{\mathcal{P}}_{sC\tilde{\mathcal{R}}_s}[C^L_{\tilde{\mathcal{R}}_s},\ C^U_{\tilde{\mathcal{R}}_s}], \breve{\mathcal{P}}_{sF\tilde{\mathcal{R}}_s}[F^L_{\tilde{\mathcal{R}}_s}, F^U_{\tilde{\mathcal{R}}_s}];>,$

$\breve{\mathcal{P}}_{l\tilde{\mathcal{R}}_l}, \breve{\mathcal{P}}_{lT\tilde{\mathcal{R}}_l}, \breve{\mathcal{P}}_{lU\tilde{\mathcal{R}}_l}, \breve{\mathcal{P}}_{lC\tilde{\mathcal{R}}_l}, \breve{\mathcal{P}}_{lF\tilde{\mathcal{R}}_l} \in P(\mathcal{N}); l = 1,2,3,\dots,s\}$

and

$\tilde{\mathcal{H}} = \{< \breve{\mathcal{P}}_{1\tilde{\mathcal{H}}_1},\ \breve{\mathcal{P}}_{1T\tilde{\mathcal{H}}_1}[\ T^L_{\tilde{\mathcal{H}}_1}, T^U_{\tilde{\mathcal{H}}_1}],\ \breve{\mathcal{P}}_{1U\tilde{\mathcal{H}}_1}[U^L_{\tilde{\mathcal{H}}_1},\ U^U_{\tilde{\mathcal{H}}_1}],\ \breve{\mathcal{P}}_{1C\tilde{\mathcal{H}}_1}[C^L_{\tilde{\mathcal{H}}_1},\ C^U_{\tilde{\mathcal{H}}_1}], \breve{\mathcal{P}}_{1F\tilde{\mathcal{H}}_1}[F^L_{\tilde{\mathcal{H}}_1}, F^U_{\tilde{\mathcal{H}}_1}];$

$\breve{\mathcal{P}}_{2\tilde{\mathcal{H}}_2},\ \breve{\mathcal{P}}_{2T\tilde{\mathcal{H}}_2}[\ T^L_{\tilde{\mathcal{H}}_2}, T^U_{\tilde{\mathcal{H}}_2}],\ \breve{\mathcal{P}}_{2U\tilde{\mathcal{H}}_2}[U^L_{\tilde{\mathcal{H}}_2},\ U^U_{\tilde{\mathcal{H}}_2}],\ \breve{\mathcal{P}}_{2C\tilde{\mathcal{H}}_2}[C^L_{\tilde{\mathcal{H}}_2},\ C^U_{\tilde{\mathcal{H}}_2}], \breve{\mathcal{P}}_{2F\tilde{\mathcal{H}}_2}[F^L_{\tilde{\mathcal{H}}_2}, F^U_{\tilde{\mathcal{H}}_2}];$

…

$\breve{\mathcal{P}}_{s\tilde{\mathcal{H}}_s},\ \breve{\mathcal{P}}_{sT\tilde{\mathcal{H}}_s}[\ T^L_{\tilde{\mathcal{H}}_s}, T^U_{\tilde{\mathcal{H}}_s}],\ \breve{\mathcal{P}}_{sU\tilde{\mathcal{H}}_s}[U^L_{\tilde{\mathcal{H}}_s},\ U^U_{\tilde{\mathcal{H}}_s}],\ \breve{\mathcal{P}}_{sC\tilde{\mathcal{H}}_s}[C^L_{\tilde{\mathcal{H}}_s},\ C^U_{\tilde{\mathcal{H}}_s}], \breve{\mathcal{P}}_{sF\tilde{\mathcal{H}}_s}[F^L_{\tilde{\mathcal{H}}_s}, F^U_{\tilde{\mathcal{H}}_s}];>,$

$\breve{\mathcal{P}}_{l\tilde{\mathcal{H}}_l}, \breve{\mathcal{P}}_{lT\tilde{\mathcal{H}}_l}, \breve{\mathcal{P}}_{lU\tilde{\mathcal{H}}_l}, \breve{\mathcal{P}}_{lC\tilde{\mathcal{H}}_l}, \breve{\mathcal{P}}_{lF\tilde{\mathcal{H}}_l} \in P(\mathcal{N}); l = 1,2,3,\dots,s\}$

be two IGSVNQSs and let $(\tilde{\mathcal{R}} \subset' \tilde{\mathcal{H}})$. Then, following conditions are satisfied.

i. $\tilde{\mathcal{R}} \subset' (\tilde{\mathcal{R}} \cap'_M \tilde{\mathcal{H}}) \subset' \tilde{\mathcal{H}}$

$\tilde{\mathcal{R}} \subset' \left(\tilde{\mathcal{R}} \cap'_O \tilde{\mathcal{H}}\right) =' \tilde{\mathcal{H}}$

$\tilde{\mathcal{R}} =' (\tilde{\mathcal{R}} \cap'_P \tilde{\mathcal{H}}) \subset' \tilde{\mathcal{H}}$

ii. $\tilde{\mathcal{R}} \subset' (\tilde{\mathcal{R}} \cup'_M \tilde{\mathcal{H}}) \subset' \tilde{\mathcal{H}}$

$\tilde{\mathcal{R}} \subset' \left(\tilde{\mathcal{R}} \cup'_O \tilde{\mathcal{H}}\right) =' \tilde{\mathcal{H}}$

$\tilde{\mathcal{R}} =' (\tilde{\mathcal{R}} \cup'_P \tilde{\mathcal{H}}) \subset' \tilde{\mathcal{H}}$

Proof:

i. From Definition 3.1.1, we get

$\breve{\mathcal{P}}_{l\tilde{\mathcal{R}}_l} \subset \breve{\mathcal{P}}_{l\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} \subset \breve{\mathcal{P}}_{l\tilde{\mathcal{H}}_l}$, $\breve{\mathcal{P}}_{lT\tilde{\mathcal{R}}_l} \subset \breve{\mathcal{P}}_{lT\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} \subset \breve{\mathcal{P}}_{lT\tilde{\mathcal{H}}_l}$, $\breve{\mathcal{P}}_{lU\tilde{\mathcal{R}}_l} \subset \breve{\mathcal{P}}_{lU\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} \subset \breve{\mathcal{P}}_{lU\tilde{\mathcal{H}}_l}$

$\breve{\mathcal{P}}_{lC\tilde{\mathcal{R}}_l} \subset \breve{\mathcal{P}}_{lC\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} \subset \breve{\mathcal{P}}_{lC\tilde{\mathcal{H}}_l}$, $\breve{\mathcal{P}}_{lF\tilde{\mathcal{R}}_l} \subset \breve{\mathcal{P}}_{lF\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} \subset \breve{\mathcal{P}}_{lF\tilde{\mathcal{H}}_l}$

$T^L_{\tilde{\mathcal{R}}_l} \leq T^L_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} \leq T^L_{\tilde{\mathcal{H}}_l}$, $T^U_{\tilde{\mathcal{R}}_l} \leq T^U_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} \leq T^U_{\tilde{\mathcal{H}}_l}$

$U^L_{\tilde{\mathcal{R}}_l} \geq U^L_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} \geq U^L_{\tilde{\mathcal{H}}_l}$, $U^U_{\tilde{\mathcal{R}}_l} \geq U^U_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} \geq U^U_{\tilde{\mathcal{H}}_l}$

$C^L_{\tilde{\mathcal{R}}_l} \geq C^L_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} \geq C^L_{\tilde{\mathcal{H}}_l}$, $C^U_{\tilde{\mathcal{R}}_l} \geq C^U_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} \geq C^U_{\tilde{\mathcal{H}}_l}$

$F^{L}_{\tilde{\mathcal{R}}_l} \geq F^{L}_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} \geq F^{L}_{\tilde{\mathcal{H}}_l}$,$F^{U}_{\tilde{\mathcal{R}}_l} \geq F^{U}_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} \geq F^{U}_{\tilde{\mathcal{H}}_l}$.

From Definition 3.2.1, we get

$$\breve{\mathcal{P}}_{l_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}} = \left(\breve{\mathcal{P}}_{l_{\tilde{\mathcal{R}}_l}} \cap \breve{\mathcal{P}}_{l_{\tilde{\mathcal{H}}_l}}\right),\ \breve{\mathcal{P}}_{l_{T\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}} = \left(\breve{\mathcal{P}}_{l_{T\tilde{\mathcal{R}}_l}} \cap \breve{\mathcal{P}}_{l_{T\tilde{\mathcal{H}}_l}}\right),\ \breve{\mathcal{P}}_{l_{U\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}} = \left(\breve{\mathcal{P}}_{l_{U\tilde{\mathcal{R}}_l}} \cap \breve{\mathcal{P}}_{l_{U\tilde{\mathcal{H}}_l}}\right)$$

$$\breve{\mathcal{P}}_{l_{C\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}} = \left(\breve{\mathcal{P}}_{l_{C\tilde{\mathcal{R}}_l}} \cap \breve{\mathcal{P}}_{l_{C\tilde{\mathcal{H}}_l}}\right),\ \breve{\mathcal{P}}_{l_{F\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}} = \left(\breve{\mathcal{P}}_{l_{F\tilde{\mathcal{R}}_l}} \cap \breve{\mathcal{P}}_{l_{F\tilde{\mathcal{H}}_l}}\right).$$

Thus, from the proof of (i) in Theorem 3.2.10, conditions (15), Definition 3.1.1, Definition 3.2.1, Definition 3.2.3, Definition 3.2.5, (17) and (18), we get

$$\tilde{\mathcal{R}} \subset' (\tilde{\mathcal{R}} \cap'_M \tilde{\mathcal{H}}) \subset' \tilde{\mathcal{H}}$$

$$\tilde{\mathcal{R}} \subset' \left(\tilde{\mathcal{R}} \cap'_O \tilde{\mathcal{H}}\right) =' \tilde{\mathcal{H}}$$

$$\tilde{\mathcal{R}} =' \left(\tilde{\mathcal{R}} \cap'_P \tilde{\mathcal{H}}\right) \subset' \tilde{\mathcal{H}}.$$

The proof of {ii} is similar to i.

4. Application

4.1 Algorithm and Example for Operators Based on Interval Generalized Set-Valued Neutrosophic Quintuple Set

In this Section, an algorithm and its application are given simultaneously in an example to demonstrate the use of the operators obtained in the study in a decision-making application. Thanks to this application, researchers will be able to use operators to solve decision-making problems.

Example 4.2: A decision-making exercise will be carried out in which three experts evaluate three alternatives on the basis of seven different criteria.

Step 1: Experts, alternatives and criteria required for the application are determined.

Let $\boldsymbol{\mathcal{D}} = \{\breve{\boldsymbol{\mathcal{D}}}_1, \breve{\boldsymbol{\mathcal{D}}}_2, \breve{\boldsymbol{\mathcal{D}}}_3\}$ be the set of experts.

The criteria to be used in the application are determined. Set of criteria

Let $\boldsymbol{\mathcal{Y}} = \{\ddot{\boldsymbol{y}}_1, \ddot{\boldsymbol{y}}_2, \ddot{\boldsymbol{y}}_3, \ddot{\boldsymbol{y}}_4, \ddot{\boldsymbol{y}}_5, \ddot{\boldsymbol{y}}_6, \ddot{\boldsymbol{y}}_7\}$ be the set of criterias.

Alternatives are identified. Let $\boldsymbol{\mathcal{S}} = \{\dot{\boldsymbol{\mathcal{S}}}^1, \dot{\boldsymbol{\mathcal{S}}}^2, \dot{\boldsymbol{\mathcal{S}}}^3\}$ be the set of alternatives.

Step 2: Opinions of experts about the alternatives are obtained.

Experts' opinions about an alternative are expressed using IGSVNQS and denoted by $\dot{\boldsymbol{S}}^{\boldsymbol{i}}{}_{\breve{\boldsymbol{D}}_{\boldsymbol{j}}}$.Where $\boldsymbol{i},\boldsymbol{j}=\{\mathbf{1},\mathbf{2},\mathbf{3}\}$, $\boldsymbol{i}$ stands for alternatives and $\boldsymbol{j}$ stands for experts. Opinions of experts on alternatives are expressed using IGSVNQS and denoted by $\widetilde{\boldsymbol{\mathcal{G}}}_{\breve{\boldsymbol{D}}_{\boldsymbol{j}}}$.

The IGSVNQS expressing the opinion of expert $\breve{\boldsymbol{D}}_{\mathbf{1}}$ is denoted by $\widetilde{\boldsymbol{\mathcal{G}}}_{\breve{\boldsymbol{D}}_{\mathbf{1}}}$.This set is

$$\widetilde{\boldsymbol{\mathcal{G}}}_{\breve{\boldsymbol{D}}_{\mathbf{1}}}=\left\{\begin{array}{c}\dot{\boldsymbol{S}}^{\mathbf{1}}{}_{\breve{\boldsymbol{D}}_{\mathbf{1}}}=\{<\{\ddot{\boldsymbol{y}}_{\mathbf{1}},\ddot{\boldsymbol{y}}_{\mathbf{2}},\ddot{\boldsymbol{y}}_{\mathbf{3}},\ddot{\boldsymbol{y}}_{\mathbf{4}},\ddot{\boldsymbol{y}}_{\mathbf{5}},\ddot{\boldsymbol{y}}_{\mathbf{6}},\ddot{\boldsymbol{y}}_{\mathbf{7}}\},\{\ddot{\boldsymbol{y}}_{\mathbf{1}},\ddot{\boldsymbol{y}}_{\mathbf{3}},\ddot{\boldsymbol{y}}_{\mathbf{4}},\ddot{\boldsymbol{y}}_{\mathbf{7}}\}[\mathbf{0.7},\mathbf{0.8}],\{\ddot{\boldsymbol{y}}_{\mathbf{3}},\ddot{\boldsymbol{y}}_{\mathbf{4}},\ddot{\boldsymbol{y}}_{\mathbf{7}}\}[\mathbf{0.4},\mathbf{0.5}],\\ \{\ddot{\boldsymbol{y}}_{\mathbf{1}},\ddot{\boldsymbol{y}}_{\mathbf{2}}\}[\mathbf{0.55},\mathbf{0.6}],\{\ddot{\boldsymbol{y}}_{\mathbf{5}},\ddot{\boldsymbol{y}}_{\mathbf{6}}\}[\mathbf{0.3},\mathbf{0.45}]>\}\\ \dot{\boldsymbol{S}}^{\mathbf{2}}{}_{\breve{\boldsymbol{D}}_{\mathbf{1}}}=\{<\{\ddot{\boldsymbol{y}}_{\mathbf{1}},\ddot{\boldsymbol{y}}_{\mathbf{2}},\ddot{\boldsymbol{y}}_{\mathbf{3}},\ddot{\boldsymbol{y}}_{\mathbf{4}},\ddot{\boldsymbol{y}}_{\mathbf{5}},\ddot{\boldsymbol{y}}_{\mathbf{6}},\ddot{\boldsymbol{y}}_{\mathbf{7}}\},\{\ddot{\boldsymbol{y}}_{\mathbf{1}},\ddot{\boldsymbol{y}}_{\mathbf{2}},\ddot{\boldsymbol{y}}_{\mathbf{5}}\}[\mathbf{0.7},\mathbf{0.85}],\{\ddot{\boldsymbol{y}}_{\mathbf{2}},\ddot{\boldsymbol{y}}_{\mathbf{4}},\ddot{\boldsymbol{y}}_{\mathbf{5}}\}[\mathbf{0.25},\mathbf{0.5}],\\ \{\ddot{\boldsymbol{y}}_{\mathbf{1}},\ddot{\boldsymbol{y}}_{\mathbf{2}}\}[\mathbf{0.45},\mathbf{0.7}],\{\ddot{\boldsymbol{y}}_{\mathbf{3}},\ddot{\boldsymbol{y}}_{\mathbf{6}},\ddot{\boldsymbol{y}}_{\mathbf{7}}\}[\mathbf{0.1},\mathbf{0.2}]>\}\\ \dot{\boldsymbol{S}}^{\mathbf{3}}{}_{\breve{\boldsymbol{D}}_{\mathbf{1}}}=\{<\{\ddot{\boldsymbol{y}}_{\mathbf{1}},\ddot{\boldsymbol{y}}_{\mathbf{2}},\ddot{\boldsymbol{y}}_{\mathbf{3}},\ddot{\boldsymbol{y}}_{\mathbf{4}},\ddot{\boldsymbol{y}}_{\mathbf{5}},\ddot{\boldsymbol{y}}_{\mathbf{6}},\ddot{\boldsymbol{y}}_{\mathbf{7}}\},\{\ddot{\boldsymbol{y}}_{\mathbf{1}},\ddot{\boldsymbol{y}}_{\mathbf{6}},\ddot{\boldsymbol{y}}_{\mathbf{7}}\}[\mathbf{0.5},\mathbf{0.65}],\{\ddot{\boldsymbol{y}}_{\mathbf{1}},\ddot{\boldsymbol{y}}_{\mathbf{2}},\ddot{\boldsymbol{y}}_{\mathbf{6}}\}[\mathbf{0.35},\mathbf{0.5}],\\ \{\ddot{\boldsymbol{y}}_{\mathbf{3}},\ddot{\boldsymbol{y}}_{\mathbf{5}},\ddot{\boldsymbol{y}}_{\mathbf{6}}\}[\mathbf{0.4},\mathbf{0.6}],\{\ddot{\boldsymbol{y}}_{\mathbf{4}},\ddot{\boldsymbol{y}}_{\mathbf{7}}\}[\mathbf{0.2},\mathbf{0.3}]>\}\end{array}\right\}$$

The expressing the opinion of the expert $\breve{\boldsymbol{D}}_{\mathbf{2}}$ is denoted by the IGSVNQS $\widetilde{\boldsymbol{\mathcal{G}}}_{\breve{\boldsymbol{D}}_{\mathbf{2}}}$. This set is

$$\widetilde{\boldsymbol{\mathcal{G}}}_{\breve{\boldsymbol{D}}_{\mathbf{2}}}=\left\{\begin{array}{c}\dot{\boldsymbol{S}}^{\mathbf{1}}{}_{\breve{\boldsymbol{D}}_{\mathbf{2}}}=\{<\{\ddot{\boldsymbol{y}}_{\mathbf{1}},\ddot{\boldsymbol{y}}_{\mathbf{2}},\ddot{\boldsymbol{y}}_{\mathbf{3}},\ddot{\boldsymbol{y}}_{\mathbf{4}},\ddot{\boldsymbol{y}}_{\mathbf{6}},\ddot{\boldsymbol{y}}_{\mathbf{7}}\},\{\ddot{\boldsymbol{y}}_{\mathbf{2}},\ddot{\boldsymbol{y}}_{\mathbf{3}},\ddot{\boldsymbol{y}}_{\mathbf{4}}\}[\mathbf{0.5},\mathbf{0.75}],\{\ddot{\boldsymbol{y}}_{\mathbf{1}},\ddot{\boldsymbol{y}}_{\mathbf{4}},\ddot{\boldsymbol{y}}_{\mathbf{7}}\}[\mathbf{0.35},\mathbf{0.6}],\\ \{\ddot{\boldsymbol{y}}_{\mathbf{1}},\ddot{\boldsymbol{y}}_{\mathbf{2}},\ddot{\boldsymbol{y}}_{\mathbf{7}}\}[\mathbf{0.4},\mathbf{0.7}],\{\ddot{\boldsymbol{y}}_{\mathbf{3}},\ddot{\boldsymbol{y}}_{\mathbf{6}}\}[\mathbf{0.1},\mathbf{0.25}]>\}\\ \dot{\boldsymbol{S}}^{\mathbf{2}}{}_{\breve{\boldsymbol{D}}_{\mathbf{2}}}=\{<\{\ddot{\boldsymbol{y}}_{\mathbf{1}},\ddot{\boldsymbol{y}}_{\mathbf{2}},\ddot{\boldsymbol{y}}_{\mathbf{3}},\ddot{\boldsymbol{y}}_{\mathbf{4}},\ddot{\boldsymbol{y}}_{\mathbf{5}},\ddot{\boldsymbol{y}}_{\mathbf{7}}\},\{\ddot{\boldsymbol{y}}_{\mathbf{2}},\ddot{\boldsymbol{y}}_{\mathbf{4}},\ddot{\boldsymbol{y}}_{\mathbf{5}},\ddot{\boldsymbol{y}}_{\mathbf{7}}\}[\mathbf{0.5},\mathbf{0.9}],\{\ddot{\boldsymbol{y}}_{\mathbf{2}},\ddot{\boldsymbol{y}}_{\mathbf{3}},\ddot{\boldsymbol{y}}_{\mathbf{5}}\}[\mathbf{0.2},\mathbf{0.4}],\\ \{\ddot{\boldsymbol{y}}_{\mathbf{2}},\ddot{\boldsymbol{y}}_{\mathbf{4}}\}[\mathbf{0.3},\mathbf{0.5}],\{\ddot{\boldsymbol{y}}_{\mathbf{1}},\ddot{\boldsymbol{y}}_{\mathbf{3}},\ddot{\boldsymbol{y}}_{\mathbf{7}}\}[\mathbf{0.15},\mathbf{0.3}]>\}\\ \dot{\boldsymbol{S}}^{\mathbf{3}}{}_{\breve{\boldsymbol{D}}_{\mathbf{2}}}=\{<\{\ddot{\boldsymbol{y}}_{\mathbf{1}},\ddot{\boldsymbol{y}}_{\mathbf{3}},\ddot{\boldsymbol{y}}_{\mathbf{5}},\ddot{\boldsymbol{y}}_{\mathbf{6}},\ddot{\boldsymbol{y}}_{\mathbf{7}}\},\{\ddot{\boldsymbol{y}}_{\mathbf{5}},\ddot{\boldsymbol{y}}_{\mathbf{6}}\}[\mathbf{0.7},\mathbf{0.95}],\{\ddot{\boldsymbol{y}}_{\mathbf{1}},\ddot{\boldsymbol{y}}_{\mathbf{6}}\}[\mathbf{0.3},\mathbf{0.45}],\\ \{\ddot{\boldsymbol{y}}_{\mathbf{3}},\ddot{\boldsymbol{y}}_{\mathbf{5}},\ddot{\boldsymbol{y}}_{\mathbf{7}}\}[\mathbf{0.3},\mathbf{0.5}],\{\ddot{\boldsymbol{y}}_{\mathbf{1}},\ddot{\boldsymbol{y}}_{\mathbf{7}}\}[\mathbf{0.15},\mathbf{0.2}]>\}\end{array}\right\}$$

The expressing the opinion of the expert $\breve{\boldsymbol{D}}_{\mathbf{3}}$ is denoted by the The expressing the opinion of the expert $\breve{\boldsymbol{D}}_{\mathbf{2}}$ is denoted by the IGSVNQS $\widetilde{\boldsymbol{\mathcal{G}}}_{\breve{\boldsymbol{D}}_{\mathbf{2}}}$. This set is

$$\widetilde{\boldsymbol{\mathcal{G}}}_{\breve{\boldsymbol{D}}_{\mathbf{3}}}=\left\{\begin{array}{c}\dot{\boldsymbol{S}}^{\mathbf{1}}{}_{\breve{\boldsymbol{D}}_{\mathbf{3}}}=\{<\{\ddot{\boldsymbol{y}}_{\mathbf{1}},\ddot{\boldsymbol{y}}_{\mathbf{3}},\ddot{\boldsymbol{y}}_{\mathbf{4}},\ddot{\boldsymbol{y}}_{\mathbf{6}},\ddot{\boldsymbol{y}}_{\mathbf{7}}\},\{\ddot{\boldsymbol{y}}_{\mathbf{1}},\ddot{\boldsymbol{y}}_{\mathbf{3}},\ddot{\boldsymbol{y}}_{\mathbf{4}}\}[\mathbf{0.6},\mathbf{0.9}],\{\ddot{\boldsymbol{y}}_{\mathbf{1}},\ddot{\boldsymbol{y}}_{\mathbf{3}},\ddot{\boldsymbol{y}}_{\mathbf{7}}\}[\mathbf{0.3},\mathbf{0.55}],\\ \{\ddot{\boldsymbol{y}}_{\mathbf{1}},\ddot{\boldsymbol{y}}_{\mathbf{7}}\}[\mathbf{0.3},\mathbf{0.5}],\{\ddot{\boldsymbol{y}}_{\mathbf{4}},\ddot{\boldsymbol{y}}_{\mathbf{6}}\}[\mathbf{0.2},\mathbf{0.5}]>\}\\ \dot{\boldsymbol{S}}^{\mathbf{2}}{}_{\breve{\boldsymbol{D}}_{\mathbf{3}}}=\{<\{\ddot{\boldsymbol{y}}_{\mathbf{1}},\ddot{\boldsymbol{y}}_{\mathbf{2}},\ddot{\boldsymbol{y}}_{\mathbf{3}},\ddot{\boldsymbol{y}}_{\mathbf{4}},\ddot{\boldsymbol{y}}_{\mathbf{5}},\ddot{\boldsymbol{y}}_{\mathbf{6}}\},\{\ddot{\boldsymbol{y}}_{\mathbf{2}},\ddot{\boldsymbol{y}}_{\mathbf{4}},\ddot{\boldsymbol{y}}_{\mathbf{6}}\}[\mathbf{0.8},\mathbf{1.0}],\{\ddot{\boldsymbol{y}}_{\mathbf{2}},\ddot{\boldsymbol{y}}_{\mathbf{3}},\ddot{\boldsymbol{y}}_{\mathbf{4}},\ddot{\boldsymbol{y}}_{\mathbf{5}}\}[\mathbf{0.1},\mathbf{0.6}],\\ \{\ddot{\boldsymbol{y}}_{\mathbf{2}},\ddot{\boldsymbol{y}}_{\mathbf{4}},\ddot{\boldsymbol{y}}_{\mathbf{6}}\}[\mathbf{0.2},\mathbf{0.6}],\{\ddot{\boldsymbol{y}}_{\mathbf{1}},\ddot{\boldsymbol{y}}_{\mathbf{3}}\}[\mathbf{0.2},\mathbf{0.35}]>\}\\ \dot{\boldsymbol{S}}^{\mathbf{3}}{}_{\breve{\boldsymbol{D}}_{\mathbf{3}}}=\{<\{\ddot{\boldsymbol{y}}_{\mathbf{1}},\ddot{\boldsymbol{y}}_{\mathbf{2}},\ddot{\boldsymbol{y}}_{\mathbf{3}},\ddot{\boldsymbol{y}}_{\mathbf{5}},\ddot{\boldsymbol{y}}_{\mathbf{6}},\ddot{\boldsymbol{y}}_{\mathbf{7}}\},\{\ddot{\boldsymbol{y}}_{\mathbf{1}},\ddot{\boldsymbol{y}}_{\mathbf{4}},\ddot{\boldsymbol{y}}_{\mathbf{7}}\}[\mathbf{0.6},\mathbf{0.8}],\{\ddot{\boldsymbol{y}},\ddot{\boldsymbol{y}}_{\mathbf{6}}\}[\mathbf{0.4},\mathbf{0.5}],\\ \{\ddot{\boldsymbol{y}}_{\mathbf{5}},\ddot{\boldsymbol{y}}_{\mathbf{7}}\}[\mathbf{0.35},\mathbf{0.55}],\{\ddot{\boldsymbol{y}}_{\mathbf{4}},\ddot{\boldsymbol{y}}_{\mathbf{7}}\}[\mathbf{0.3},\mathbf{0.45}]>\}\end{array}\right\}$$

Step 3: The opinions of the experts are expressed as a common opinion using one operators which is obtained Section 3.

Using the mean intersection operator in Definition 3.2.1, the consensus opinion of the experts is expressed as an IGSVNQS. It is denoted by $\widetilde{\boldsymbol{\mathcal{G}}}_{\cap'_{\boldsymbol{M}}}$ and calculated such that

$$\widetilde{\boldsymbol{\mathcal{G}}}_{\cap'_{\boldsymbol{M}}}=\left(\widetilde{\boldsymbol{\mathcal{G}}}_{\breve{\boldsymbol{D}}_{\mathbf{1}}}\cap\left(\widetilde{\boldsymbol{\mathcal{G}}}_{\breve{\boldsymbol{D}}_{\mathbf{2}}}\cap\widetilde{\boldsymbol{\mathcal{G}}}_{\breve{\boldsymbol{D}}_{\mathbf{3}}}\right)\right)=(\dot{\boldsymbol{S}}^{\mathbf{1}}{}_{\cap'_{\boldsymbol{M}}},\ \dot{\boldsymbol{S}}^{\mathbf{2}}{}_{\cap'_{\boldsymbol{M}}},\ \dot{\boldsymbol{S}}^{\mathbf{3}}{}_{\cap'_{\boldsymbol{M}}})$$

Also, we calculate

$$\dot{\boldsymbol{S}}^{\mathbf{1}}{}_{\cap'_{\boldsymbol{M}}}=\left(\dot{\boldsymbol{S}}^{\mathbf{1}}{}_{\breve{\boldsymbol{D}}_{\mathbf{1}}}\cap(\dot{\boldsymbol{S}}^{\mathbf{1}}{}_{\breve{\boldsymbol{D}}_{\mathbf{2}}}\cap\dot{\boldsymbol{S}}^{\mathbf{1}}{}_{\breve{\boldsymbol{D}}_{\mathbf{3}}})\right)$$

$$=\{<\{\ddot{\boldsymbol{y}}_{\mathbf{1}},\ddot{\boldsymbol{y}}_{\mathbf{3}},\ddot{\boldsymbol{y}}_{\mathbf{4}},\ddot{\boldsymbol{y}}_{\mathbf{6}},\ddot{\boldsymbol{y}}_{\mathbf{7}}\},\{\ddot{\boldsymbol{y}}_{\mathbf{3}},\ddot{\boldsymbol{y}}_{\mathbf{4}},\ddot{\boldsymbol{y}}_{\mathbf{7}}\}[\mathbf{0.62},\mathbf{0.81}],\{\ddot{\boldsymbol{y}}_{\mathbf{4}},\ddot{\boldsymbol{y}}_{\mathbf{7}}\}[\mathbf{0.36},\mathbf{0.53}],\\ \{\ddot{\boldsymbol{y}}_{\mathbf{1}}\}[\mathbf{0.45},\mathbf{0.6}],\{\ddot{\boldsymbol{y}}_{\mathbf{6}}\}[\mathbf{0.22},\mathbf{0.41}]>\}$$

such that

$$\{\ddot{\boldsymbol{y}}_{\mathbf{1}},\ddot{\boldsymbol{y}}_{\mathbf{3}},\ddot{\boldsymbol{y}}_{\mathbf{4}},\ddot{\boldsymbol{y}}_{\mathbf{6}},\ddot{\boldsymbol{y}}_{\mathbf{7}}\}=\left(\{\ddot{\boldsymbol{y}}_{\mathbf{1}},\ddot{\boldsymbol{y}}_{\mathbf{2}},\ddot{\boldsymbol{y}}_{\mathbf{3}},\ddot{\boldsymbol{y}}_{\mathbf{4}},\ddot{\boldsymbol{y}}_{\mathbf{5}},\ddot{\boldsymbol{y}}_{\mathbf{6}},\ddot{\boldsymbol{y}}_{\mathbf{7}}\}\cap(\{\ddot{\boldsymbol{y}}_{\mathbf{1}},\ddot{\boldsymbol{y}}_{\mathbf{2}},\ddot{\boldsymbol{y}}_{\mathbf{3}},\ddot{\boldsymbol{y}}_{\mathbf{4}},\ddot{\boldsymbol{y}}_{\mathbf{6}},\ddot{\boldsymbol{y}}_{\mathbf{7}}\}\cap\{\ddot{\boldsymbol{y}}_{\mathbf{1}},\ddot{\boldsymbol{y}}_{\mathbf{3}},\ddot{\boldsymbol{y}}_{\mathbf{4}},\ddot{\boldsymbol{y}}_{\mathbf{6}},\ddot{\boldsymbol{y}}_{\mathbf{7}}\})\right)$$

$$\{\ddot{y}_3, \ddot{y}_4, \ddot{y}_7\} = \left(\{\ddot{y}_1, \ddot{y}_3, \ddot{y}_4, \ddot{y}_7\} \cap (\{\ddot{y}_2, \ddot{y}_3, \ddot{y}_4\} \cap \{\ddot{y}_1, \ddot{y}_3, \ddot{y}_4\})\right)$$

$$\{\ddot{y}_4, \ddot{y}_7\} = \left(\{\ddot{y}_3, \ddot{y}_4, \ddot{y}_7\} \cap (\{\ddot{y}_1, \ddot{y}_4, \ddot{y}_7\} \cap \{\ddot{y}_1, \ddot{y}_3, \ddot{y}_7\})\right)$$

$$\{\ddot{y}_1\} = \left(\{\ddot{y}_1, \ddot{y}_2\} \cap (\{\ddot{y}_1, \ddot{y}_2, \ddot{y}_7\} \cap \{\ddot{y}_1, \ddot{y}_7\})\right)$$

$$\{\ddot{y}_6\} = \left(\{\ddot{y}_5, \ddot{y}_6\} \cap (\{\ddot{y}_3, \ddot{y}_6\} \cap \{\ddot{y}_4, \ddot{y}_6\})\right)$$

$$0.62 = \frac{0.7 + \frac{0.5 + 0.6}{2}}{2}, 0.81 = \frac{0.8 + \frac{0.75 + 0,9}{2}}{2},$$

$$0.36 = \frac{0.4 + \frac{0.35 + 0.3}{2}}{2}, 0.53 = \frac{0.5 + \frac{0.6 + 0.55}{2}}{2}$$

$$0.45 = \frac{0.55 + \frac{0.4 + 0.3}{2}}{2}, 0.6 = \frac{0.6 + \frac{0.7 + 0.5}{2}}{2},$$

$$0.22 = \frac{0.3 + \frac{0.1 + 0.2}{2}}{2}, 0.41 = \frac{0.45 + \frac{0.25 + 0.5}{2}}{2}$$

Similarly, $\dot{S}^2{}_{\cap'_M}$ and $\dot{S}^3{}_{\cap'_M}$ are obtained such that

$$\widetilde{\mathcal{G}}_{\cap'_M} = \left\{ \begin{matrix} \dot{S}^1{}_{\cap'_M} = \{< \{\ddot{y}_1, \ddot{y}_3, \ddot{y}_4, \ddot{y}_6, \ddot{y}_7\}, \{\ddot{y}_3, \ddot{y}_4, \ddot{y}_7\}[0.62, 0.81], \{\ddot{y}_4, \ddot{y}_7\}[0.36, 0.53], \\ \{\ddot{y}_1\}[0.45, 0.6], \{\ddot{y}_6\}[0.22, 0.41] >\} \\ \dot{S}^2{}_{\cap'_M} = \{< \{\ddot{y}_1, \ddot{y}_2, \ddot{y}_3, \ddot{y}_4, \ddot{y}_5\}, \{\ddot{y}_2\}[0.67, 0.9], \{\ddot{y}_2, \ddot{y}_5\}[0.2, 0.5], \{\ddot{y}_2\} \\ [0.35, 0.62], \{\ddot{y}_3\}[0.13, 0.26] >\} \\ \dot{S}^3{}_{\cap'_M} = \{< \{\ddot{y}_1, \ddot{y}_3, \ddot{y}_4, \ddot{y}_5\}, \emptyset[0.57, 0.76], \{\ddot{y}_6\}[0.35, 0.48], \{\ddot{y}_5\}[0.36, 0.56], \\ \{\ddot{y}_7\}[0.21, 0.31] >\} \end{matrix} \right\}$$

Using the optimistic intersection operator in Definition 3.2.3, the consensus opinions of the experts are expressed as an IGSVNQS and denoted by $\widetilde{\mathcal{G}}_{\cap'_O}$ and calculated by

$$\widetilde{\mathcal{G}}_{\cap'_O} = \left(\widetilde{\mathcal{G}}_{\tilde{\mathcal{D}}_1} \cap (\widetilde{\mathcal{G}}_{\tilde{\mathcal{D}}_2} \cap \widetilde{\mathcal{G}}_{\tilde{\mathcal{D}}_3})\right)$$

$$= \left\{ \begin{matrix} \dot{S}^1{}_{\cap'_O} = \{< \{\ddot{y}_1, \ddot{y}_3, \ddot{y}_4, \ddot{y}_6, \ddot{y}_7\}, \{\ddot{y}_3, \ddot{y}_4, \ddot{y}_7\}[0.7, 0.9], \{\ddot{y}_4, \ddot{y}_7\}[0.3, 0.5], \\ \{\ddot{y}_1\}[0.3, 0.5], \{\ddot{y}_6\}[0.1, 0.25] >\} \\ \dot{S}^2{}_{\cap'_O} = \{< \{\ddot{y}_1, \ddot{y}_2, \ddot{y}_3, \ddot{y}_4, \ddot{y}_5\}, \{\ddot{y}_2\}[0.8, 1.0], \{\ddot{y}_2, \ddot{y}_5\}[0.1, 0.4], \\ \{\ddot{y}_2\}[0.2, 0.5], \{\ddot{y}_3\}[0.1, 0.2] >\} \\ \dot{S}^3{}_{\cap'_O} = \{< \{\ddot{y}_1, \ddot{y}_3, \ddot{y}_4, \ddot{y}_5\}, \emptyset[0.7, 0.95], \{\ddot{y}_6\}[0.1, 0.45], \{\ddot{y}_5\}[0.3, 0.5], \\ \{\ddot{y}_7\}[0.15, 0.2] >\} \end{matrix} \right\}$$

Using the pessimistic union operatör in Definition 3.2.5, the consensus opinions of experts are expressed as an IGSVNQS is denoted by $\widetilde{\mathcal{G}}_{\cup'_K}$ and is calculated by

$$\widetilde{\mathcal{G}}_{\cup'_P} = \left(\widetilde{\mathcal{G}}_{\tilde{\mathcal{D}}_1} \cup (\widetilde{\mathcal{G}}_{\tilde{\mathcal{D}}_2} \cup \widetilde{\mathcal{G}}_{\tilde{\mathcal{D}}_3})\right)$$

$$= \left\{ \begin{array}{c} \dot{\mathcal{S}}^{1}{}_{\cup'_P} = \{< \{\ddot{y}_1, \ddot{y}_2, \ddot{y}_3, \ddot{y}_4, \ddot{y}_5, \ddot{y}_6, \ddot{y}_7\}, \{\ddot{y}_1, \ddot{y}_2, \ddot{y}_3, \ddot{y}_4, \ddot{y}_7\}[0.5, 0.75], \\ \{\ddot{y}_1, \ddot{y}_3, \ddot{y}_4, \ddot{y}_7\}[0.4, 0.6], \{\ddot{y}_1, \ddot{y}_2, \ddot{y}_7\}[0.55, 0.7], \{\ddot{y}_4, \ddot{y}_5, \ddot{y}_6\}[0.3, 0.5] >\} \\ \dot{\mathcal{S}}^{2}{}_{\cup'_P} = \{< \{\ddot{y}_1, \ddot{y}_2, \ddot{y}_3, \ddot{y}_4, \ddot{y}_5, \ddot{y}_6, \ddot{y}_7\}, \{\ddot{y}_1, \ddot{y}_2, \ddot{y}_4, \ddot{y}_5, \ddot{y}_6, \ddot{y}_7\}[0.5, 0.85], \\ \{\ddot{y}_2, \ddot{y}_3, \ddot{y}_4, \ddot{y}_5\}[0.25, 0.6], \{\ddot{y}_1, \ddot{y}_2, \ddot{y}_4, \ddot{y}_6\}[0.45, 0.7], \{\ddot{y}_1, \ddot{y}_3, \ddot{y}_6, \ddot{y}_7\}[0.2, 0.35] >\} \\ \dot{\mathcal{S}}^{3}{}_{\cup'_P} = \{< \{\ddot{y}_1, \ddot{y}_2, \ddot{y}_3, \ddot{y}_4, \ddot{y}_5, \ddot{y}_6, \ddot{y}_7\}, \{\ddot{y}_1, \ddot{y}_3, \ddot{y}_4, \ddot{y}_5, \ddot{y}_6, \ddot{y}_7\}[0.5, 0.65], \\ \{\ddot{y}_1, \ddot{y}_2, \ddot{y}_6\}[0.4, 0.5], \{\ddot{y}_3, \ddot{y}_5, \ddot{y}_6, \ddot{y}_7\}[0.4, 0.6], \{\ddot{y}_1, \ddot{y}_4, \ddot{y}_7\}[0.3, 0.45] >\} \end{array} \right\}$$

Step 4: The obtained consensus is expressed as net value by using score and accuracy functions.

The score values of the common opinion obtained by the $\widetilde{\mathcal{G}}_{\cap'_M}$ are calculated and the net values obtained by the mean intersection operator for the alternatives are given in Table 1.

Table 1: Net Values of The Views Obtained with The Mean Intercept Operator

Alternative	Net Value
$\dot{\mathcal{S}}^{1}{}_{\cap'_M}$	1.09
$\dot{\mathcal{S}}^{2}{}_{\cap'_M}$	1.44
$\dot{\mathcal{S}}^{3}{}_{\cap'_M}$	1.42

The score values of the common opinion obtained by the $\widetilde{\mathcal{G}}_{\cap'_O}$ are calculated and the net values obtained by the optimistic intersection operator for the alternatives are given in Table 2.

Table 2: Net Values of The Opinions Obtained by Optimistic Intersection Operation

Alternative	Net Value
$\dot{\mathcal{S}}^{1}{}_{\cap'_O}$	1.08
$\dot{\mathcal{S}}^{2}{}_{\cap'_O}$	1.7
$\dot{\mathcal{S}}^{3}{}_{\cap'_O}$	1.48

The score values of the common opinion obtained by the $\widetilde{\mathcal{G}}_{\cup'_P}$ are calculated as follows

$$\boldsymbol{S_D}\left(\dot{\mathcal{S}}^{1}{}_{\cup'_P}\right) = \frac{|5-4-3-3|}{7} + \frac{|0.5-0.4-0.55-0.3|}{3} - \frac{|0.75-0.6-0.7-0.5|}{3} + 1 = 1,61$$

$$\boldsymbol{S_D}\left(\dot{\mathcal{S}}^{2}{}_{\cup'_P}\right) = \frac{|6-4-4-4|}{7} + \frac{|0.5-0.25-0.45-0.2|}{3} - \frac{|0.85-0.6-0.7-0.35|}{3} + 1 = 1,72$$

$$\boldsymbol{S_D}\left(\dot{\mathcal{S}}^{3}{}_{\cup'_P}\right) = \frac{|5-3-4-3|}{7} + \frac{|0.5-0.4-0.4-0.3|}{3} - \frac{|0.65-0.5-0.6-0.45|}{3} + 1 = 1.61$$

Where $\boldsymbol{S_D}\left(\dot{\mathcal{S}}^{1}{}_{\cup'_K}\right)$ and $\boldsymbol{S_D}\left(\dot{\mathcal{S}}^{3}{}_{\cup'_K}\right)$ score values are equal. Thus, the accuracy values are calculated as follows

$$\boldsymbol{K_D}\left(\dot{\mathcal{S}}^{1}{}_{\cup'_P}\right) = \frac{|1+2+4|}{2*5} + 0.5 + 0.75 = \frac{7}{10} + 0.5 + 0.75 = 1.95$$

$$\boldsymbol{K_D}\left(\dot{\mathcal{S}}^{3}{}_{\cup'_P}\right) = \frac{|3+2+2|}{2*5} + 0.5 + 0.65 = \frac{7}{10} + 0.5 + 0.65 = 1.85$$

the net values are obtained with the pessimistic union operator for the alternatives in Table 3.

Table 3: Net Values of The Opinions Obtained with The Pessimistic Union Operator

Alternative	Net Value
$\dot{\mathcal{S}}^{1}{}_{\cup'_{P}}$	1.95
$\dot{\mathcal{S}}^{2}{}_{\cup'_{P}}$	1.72
$\dot{\mathcal{S}}^{3}{}_{\cup'_{P}}$	1.85

Step 5: Rankings are made among the alternatives with the net values obtained.

i) The ranking of the alternatives with the common opinion obtained by using the mean intersection operator from Table 1,

$$\dot{\mathcal{S}}^{2} > \dot{\mathcal{S}}^{3} > \dot{\mathcal{S}}^{1}.$$

According to this ranking, alternative $\dot{\mathcal{S}}^{2}$ is the best alternative while alternative $\dot{\mathcal{S}}^{1}$ is the least good alternative.

ii) The ranking of the alternatives with the common opinion obtained by using the optimistic intersection operatör from Table 2,

$$\dot{\mathcal{S}}^{2} > \dot{\mathcal{S}}^{3} > \dot{\mathcal{S}}^{1}.$$

According to this ranking, alternative $\dot{\mathcal{S}}^{2}$ is the best alternative while alternative $\dot{\mathcal{S}}^{1}$ is the least good alternative.

iii) The ranking of the alternatives with the common opinion obtained by using the pessimistic union operator from Table 3,

$$\dot{\mathcal{S}}^{1} > \dot{\mathcal{S}}^{3} > \dot{\mathcal{S}}^{2}.$$

According to this ranking, alternative $\dot{\mathcal{S}}^{1}$ is the best alternative while alternative $\dot{\mathcal{S}}^{2}$ is the least good alternative.

4.2 Comparision Method

From Step 5 in Section 4.1, we obtain Table 4.

Table 4: Rankings Obtained from The Operators

Operator	Obtained Sequence
Mean Intersection	$\dot{\mathcal{S}}^{2} > \dot{\mathcal{S}}^{3} > \dot{\mathcal{S}}^{1}$
Optimistic Intersection	$\dot{\mathcal{S}}^{2} > \dot{\mathcal{S}}^{3} > \dot{\mathcal{S}}^{1}$
Pessimistic Union	$\dot{\mathcal{S}}^{1} > \dot{\mathcal{S}}^{3} > \dot{\mathcal{S}}^{2}$

It is clearly seen in Table 4 that similar rankings are obtained with the mean intersection and optimistic intersection operators, but the ranking obtained with the pessimistic intersection operator is different from these rankings. These results show that different operators can give similar or different results on the same data sets. For example, the alternative $\dot{\mathcal{S}}^{2}$, which was determined as the best alternative when the mean intersection and optimistic intersection operators were used, was determined as the worst alternative when the pessimistic union operator was used. Similarly, the alternative $\dot{\mathcal{S}}^{1}$ was determined as the best alternative when the pessimistic union operator was used

and as the worst alternative when the mean intersection and optimistic intersection operators were used. Therefore, the best alternative according to one operator may be the worst alternative according to another operator. This shows that similar or different results can be obtained in the ranking of alternatives by using different operators. In this context, it should be kept in mind that each operator represents different operations and these operations may give different results according to the problem domain. Therefore, it is necessary to make an important decision about which operator to use in the decision-making process. In the process of determining the operator to be used, the characteristics of the problem domain and the objectives should be taken into consideration. Because the selected operator can significantly affect the results. In particular, the selection of the most appropriate operator is of great importance for the accuracy of the decision-making process and the reliability of the results. Therefore, careful consideration should be given to operator selection.

5. Conclusion

In this study, some structures related to IGSVNQS and score and certainty functions are defined. Moreover, optimistic union, pessimistic union, mean union, optimistic intersection, pessimistic intersection and mean intersection operators are defined for IGSVNQS. Thus, **IGSVNQS** are made available for decision making applications. An example application was made with these operators and functions and how to use operators and functions in decision making applications was shown through the example application. When different operators are used in decision making applications, similar or different results can be obtained. Therefore, the operator to be used in decision making applications should be selected by considering the characteristics and objectives of the problem domain. Because the selected operator can greatly affect the results. Researchers can obtain more objective results by using the operators and functions in this study in decision-making applications. Also, by using IGSVNQS and operators, distance and similarity measures based on IGSVNQS can be defined for decision making problems. They can also define arithmetic operations for some decision making applications (VIKOR, ELECTRE, PROMETHEE, ...).

Abbreviations

SVNS: Single Valued Neutrosophic Set

IQNS: Interval Quadripartitioned Neutrosophic Set

INS: Interval Neutrosophic Set

ISVNQN: Interval Set Valued Neutrosophic Quadruple Number

ISVNQS: Interval Set Valued Neutrosophic Quadruple Set

IGSVNQN: Generalized Set Valued Neutrosophic Quadruple Number

IGSVNQS: Generalized Set Valued Neutrosophic Quadruple Set

IGSVNQN: Interval Generalized Set Valued Neutrosophic Quadruple Number

IGSVNQS: Interval Generalized Set Valued Neutrosophic Quadruple Set

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Received: March 29, 2024. Accepted: July 26, 2024

Neutrosophic Sets and Systems, Vol. 70, 2024
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Neutrosophic Regular Rings and Properties of Their Ideals

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Abstract: The goal of the research is to discover the required and sufficient conditions for $R(I)$ to be a neutrosophic regular as well as the properties of the neutrosophic ideals in the neutrosophic regular ring. Despite differences in the form of the elements and ideals in neutrosophic and classical rings, we discovered that the necessary and sufficient requirements for the classical ring to be regular also apply to neutrosophic rings.

Keywords: Neutrosophic ring, Neutrosophic regular, center, Neutrosophic ideal.

1. Introduction

The notion of a regular element within rings was introduced by J. Von Neumann [1]. Since then, regular rings and their properties have been comprehensively studied by numerous scholars and researchers [2,3,4].

Neutosophy signifies an advanced interpretation of intuitionistic fuzzy logic. This concept significantly influences decision making [5], and medical studies [6]. Further applications of neutrosophy are detailed in [7,8,9,10,11].

Neutosophy, as a novel area of philosophy, may be used in algebraic structures, leading to a greater understanding and evolution of these structures. Smarandache initially proposed the neutrosophic idea in 1980. Neutrosophic structures, a new concept in algebraic structures, have many applications and discoveries such as neutrosophic topologies [12, 13], and rings [14,15].

Kandasamy and Smarandache introduced neutrosophic groups, rings, and fields [16], which has been widely investigated [17,18,19,20], and is still being studied.

Numerous intriguing discoveries about neutrosophic rings have recently been discussed [21,22,23,24,25]. This study is still ongoing among researchers. Alabdullah discussed the basic

features of regular neutrosophic elements and their relationship with the elements of the neutrosophic ring [21].

In our research, we present the required and sufficient conditions for neutrosophic ideals for $R(I)$ to be neutrosophic regular.

Our motivation is to the display of properties determining properties of neutrosophic ideals in neutrosophic regular rings.

2. Definitions and notations

Since academics conversant in classical regular rings possess comprehensive insight into their properties, this section encompasses definitions and essential findings concerning neutrosophic rings. Those interested in discovering more about neutrosophic rings can investigate the reference list.

Definition 2.1 [16] Supposing $\boldsymbol{R}$ is a ring. The set $\boldsymbol{R(I)} = \{\boldsymbol{a + bI}\ ; \boldsymbol{a, b} \in \boldsymbol{R}\ \textbf{and}\ \boldsymbol{I^2 = I}\}$, is known as a neutrosophic ring. When $\boldsymbol{R}$ is a field, $\boldsymbol{R(I)}$ is referred to as a neutrosophic field.

Properties 2.2 [16,17]

1) $\boldsymbol{R}$ is a unity commutative ring iff $\boldsymbol{R(I)}$ is a unity commutative neutrosophic ring with neutrosophic unity I.

2) $\boldsymbol{I^m = I}$ for every $\boldsymbol{m} \in \mathbb{Z}^+$

3) $\boldsymbol{aI = Ia}\ \forall \boldsymbol{a} \in \boldsymbol{R}$.

4) $\boldsymbol{0I = 0}$, $\underbrace{\boldsymbol{I + I + I} + \ldots\ . + \boldsymbol{I}}_{\boldsymbol{m\ time}} = \boldsymbol{mI}$

Definition 2.3 [16,17] In any neutrosophic ring $\boldsymbol{R(I)}$, an element $\boldsymbol{e}$ that satisfies $\boldsymbol{e^2 = e}$ is considered a neutrosophic idempotent.

Theorem 2.4 [25] An element $\boldsymbol{a + bI} \in \boldsymbol{R(I)}$ is idempotent iff $\boldsymbol{a}$ is idempotent and $\boldsymbol{b = c - a}$, where $\boldsymbol{c} \in \boldsymbol{R}$ is an idempotent.

Definition 2.5 [16] The neutrosophic ring's center is defined as $\mathrm{C}\big(R(I)\big) = \{x \in R(I) \mid yx = xy \text{ for all } y \in R(I)\}$.

Theorem 2.6 [23]

If $J + KI \subseteq R(\mathrm{I})$, then $J + KI$ is a neutrosophic ideal iff J *and* K are ideals in R, where $J \subseteq K$.

Definition 2.7 [21] An element $x \in R(\mathrm{I})$ is called neutrosophic regular if there is an element y where $x = xyx$. $R(\mathrm{I})$ is neutrosophic regular when all its elements are regular.

3. Results

In this section, all neutrosophic rings are unity. In the $R(I)$, we denote the collection of neutrosophic regular elements as $NReg_{R(I)}$, $NId_{R(I)}$ is the set of neutrosophic idempotent elements, $NC_{R(I)}$ is the center of $R(I)$, and $N\mathfrak{T}_{R(I)}$ the set of neutrosophic ideals. In classical ring R, let Reg_R denote the regular elements, Id_R the idempotents, C_R the center, and $\mathfrak{T}_R$ the ideals.

Theorem 3.1 In $R(\mathrm{I})$, $\forall r_1 + r_2 I \in NReg_{R(I)} \Leftrightarrow r_1 \; and \; r_1 + r_2 \in Reg_R$

Proof.

($\Rightarrow$): If $r_1 + r_2 I \in NReg_{R(I)}$, then $\exists x + yI \in R(\mathrm{I})$, where, $r_1 + r_2 I = (r_1 + r_2 I)(x + yI)(r_1 + r_2 I) = r_1 x r_1 + [r_1 x r_2 + r_1 y r_1 + r_1 y r_2 + r_2 x r_1 + r_2 x r_2 + r_2 y r_1 + r_2 y r_2]I$

So $r_1 = r_1 x r_1 \Rightarrow r_1 \in Reg_R$

And $r_2 = r_1 x r_2 + r_1 y r_1 + r_1 y r_2 + r_2 x r_1 + r_2 x r_2 + r_2 y r_1 + r_2 y r_2$

$\Rightarrow r_1 + r_2 = r_1 x r_1 + r_1 x r_2 + r_1 y r_1 + r_1 y r_2 + r_2 x r_1 + r_2 x r_2 + r_2 y r_1 + r_2 y r_2 = (r_1 + r_2)(x + y)(r_1 + r_2)$.

Thus $r_1 + r_2 \in Reg_R$

($\Leftarrow$): If $r_1 \; and \; r_1 + r_2 \in Reg_R$, then $\exists x, r \in R$, where $r_1 = r_1 x r_1 \; and \; r_1 + r_2 = (r_1 + r_2) r (r_1 + r_2)$

Firstly, Let's put $y = r - x \in R \Rightarrow r = x + y$. Subsequently, $r_1 + r_2 = (r_1 + r_2)(x + y)(r_1 + r_2)$

$$\Rightarrow r_1 + r_2 = r_1 x r_1 + r_1 x r_2 + r_1 y r_1 + r_1 y r_2 + r_2 x r_1 + r_2 x r_2 + r_2 y r_1 + r_2 y r_2 \Rightarrow r_2$$
$$= r_1 x r_2 + r_1 y r_1 + r_1 y r_2 + r_2 x r_1 + r_2 x r_2 + r_2 y r_1 + r_2 y r_2$$

At other hand, we have $r_1 + r_2 I = r_1 x r_1 + [r_1 x r_2 + r_1 y r_1 + r_1 y r_2 + r_2 x r_1 + r_2 x r_2 + r_2 y r_1 + r_2 y r_2]I = (r_1 + r_2 I)(x + yI)(r_1 + r_2 I)$, wherefore $r_1 + r_2 I \in NReg_{R(I)}$

The following corollary was mentioned in reference [21], and we have presented another proof for it:

Corollary 3.2 Every neutrosophic field $R(\mathrm{I})$ is a regular.

Proof. $\forall r_1 + r_2 I \in R(I)$, where $r_1 and \; r_2 \in R$.

We know that every classical field is regular, so $r_1 \; and \; r_1 + r_2 \in Reg_R$. Subsequently, $r_1 + r_2 I \in NReg_{R(I)}$, according to theorem.3.1.

Example 3.3 $\mathbb{R}(I), \; \mathbb{Q}(I), \mathbb{C}(I), \; and \; \mathbb{Z}_p(I)$ are neutrosophic regular, where p is the prime.

Corollary 3.4 $R(I)$ is regular iff R is regular. (The proof results directly from the theorem.3.1)

Theorem 3.5 In $R(\mathrm{I})$, $\forall r_1 + r_2 I \in NC_{R(I)} \Leftrightarrow r_1 \; and \; r_1 + r_2 \in C_R$

Proof.

($\Rightarrow$): If $r_1 + r_2 I \in NC_{R(I)}$, then $\forall x + yI \in R(\mathrm{I}); \; (r_1 + r_2 I)(x + yI) = (x + yI)(r_1 + r_2 I)$

$\Rightarrow r_1 x + [r_1 y + r_2 x + r_2 y]I = x r_1 + [y r_1 + x r_2 + y r_2]I$. So $r_1 x = x r_1 \forall x \in R \Rightarrow r_1 \in C_R$

And $r_1 y + r_2 x + r_2 y = y r_1 + x r_2 + y r_2$

$$since \; r_1 x = x r_1 \Rightarrow r_1 x + r_1 y + r_2 x + r_2 y = x r_1 + y r_1 + x r_2 + y r_2$$
$$\Rightarrow (r_1 + r_2)(x + y) = (x + y)(r_1 + r_2), \forall x + y = r \in R$$
$$\Rightarrow r_1 + r_2 \in C_R$$

($\Leftarrow$): If $r_1 \; and \; r_1 + r_2 \in C_R$, then $\forall x, r \in R; \; r_1 x = x r_1 \; and \; (r_1 + r_2) r = r(r_1 + r_2)$

First, let us put $y = r - x \in R$. Subsequently, $r = x + y$

Subsequently, $(r_1 + r_2)(x + y) = (x + y)(r_1 + r_2) \Rightarrow r_1 x + r_1 y + r_2 x + r_2 y = x r_1 + y r_1 + x r_2 + y r_2$

$$since \; r_1 x = x r_1 \Rightarrow r_1 y + r_2 x + r_2 y = y r_1 + x r_2 + y r_2$$

At other hand, we have $(r_1 + r_2I)(x + yI) = r_1x + [r_1y + r_2x + r_2y]I = xr_1 + [yr_1 + xr_2 + yr_2]I$ $= (x + yI)(r_1 + r_2I)\ \forall x + yI \in R(\mathrm{I})$, therefore $r_1 + r_2I \in NC_{R(I)}$

Theorem 3.6 The center of the neutrosophic regular ring is also a regular.

Proof.

Suppose $R(I)$ is a neutrosophic regular. We have $NC_{R(I)}$ is a neutrosophic subring. Using theorem 3.5, if $r_1 + r_2I \in NC_{R(I)}$, then $r_1 and\ r_1 + r_2 \in C_R$.

Since $R(I)$ is regular, so R also is regular. Therefore C_R is regular.

Thus $r_1 and\ r_1 + r_2 \in Reg_R$. Subsequently $r_1 + r_2I \in NReg_{R(I)}$, according to theorem.3.1. Thus $NC_{R(I)}$ is regular.

Theorem 3.7 Assuming that $\boldsymbol{R(I)}$ is a neutrosophic ring, then $NC_{R(I)}$ is regular $\Leftrightarrow$ C_R is regular.

Proof.

($\Rightarrow$): $\forall r \in C_R \Rightarrow r\ and\ r + 0 \in C_R \Rightarrow \mathrm{r} + 0\mathrm{I} \in NC_{R(I)}$, according to theorem.3.5. Since $NC_{R(I)}$ is regular, so r is regular according to theorem.3.1. Thus C_R is regular.

($\Leftarrow$): $\forall r_1 + r_2I \in NC_{R(I)} \Rightarrow r_1\ and\ r_1 + r_2 \in C_R$, according to theorem.3.5. Since C_R is regular, so $r_1\ and\ r_1 + r_2$ is regular. Therefore $r_1 + r_2I$ is regular according to theorem.3.1. Thus $NC_{R(I)}$ is regular

Remark.3.8 Figure 1 shows the resulting relationship between the neutrosophic and classical regular ring, as follows:

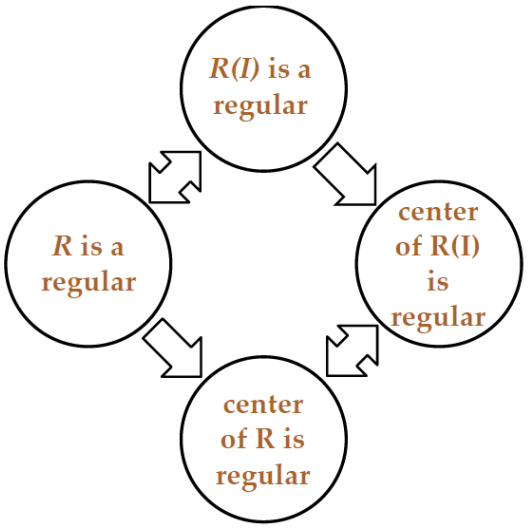

Figure 1. Relationship between the neutrosophic and classical regular rings.

Theorem 3.9 $R(I)$ is a regular $\Leftrightarrow$ every neutrosophic principal ideal is generated by a neutrosophic idempotent element.

Proof.

($\Rightarrow$): If $R(I) = NReg_{R(I)}$ and $J + KI$ is a neutrosophic ideal generated by $r_1 + r_2I$. So $J + KI = (r_1 + r_2I)R(I) = (r_1 + r_2I)(R + RI) = r_1R + [(r_1 + r_2)R]I$. Therefore, $J = r_1R\ and\ K = (r_1 + r_2)R$ are principal ideals of R. Because $R(I) = NReg_{R(I)}$, R is regular according to corollary.3.4. Therefore,

$J = r_1R = e_1R \; and \; K = (r_1 + r_2)R = e_2R, where, e_1, e_2 \in Id_R$. Thus $J + KI = e_1R + e_2RI = e_1R + e_2RI - e_1RI + e_1RI = e_1R + [e_1R + (e_2 - e_1)R]I = e_1R + [e_1R + (e_2 - e_1)R + (e_2 - e_1)R]I = [e_1 + (e_2 - e_1)I][R + RI]$

It can easily be verified that $e_1 + (e_2 - e_1)I \in NId_{R(I)}$. Therefore, the desired result is achieved.

(⇐): If $r_1 + r_2I \in R(I)$, then $(r_1 + r_2I)$R(I) $= (e_1 + e_2I)R(I)$ is the principal ideal of $R(I)$ generated by $e_1 + e_2I \in NId_{R(I)}$. First, $(r_1 + r_2I) \in (e_1 + e_2I)R(I) \Rightarrow \exists s_1 + s_2I \in R(I), where \; r_1 + r_2I = (e_1 + e_2I)(s_1 + s_2I) \Rightarrow (e_1 + e_2I)(r_1 + r_2I) = (e_1 + e_2I)(s_1 + s_2I) = r_1 + r_2I$

At other hand, $e_1 + e_2I \in (e_1 + e_2I)R(I) = (r_1 + r_2I)$R(I) $\Rightarrow \exists s_3 + s_4I \in R(I), where, e_1 + e_2I = (r_1 + r_2I)(s_3 + s_4I) \Rightarrow (e_1 + e_2I)(r_1 + r_2I) = (r_1 + r_2I)(s_3 + s_4I)(r_1 + r_2I)$

$\Rightarrow r_1 + r_2I = (r_1 + r_2I)(s_3 + s_4I)(r_1 + r_2I) \Rightarrow r_1 + r_2I \in NReg_{R(I)} \Rightarrow R(I) = NReg_{R(I)}$

Example 3.10 In any neutrosophic field $R(I)$, we have only three ideals. $\{0\}, RI, and \; R(I)$.

We note that each of the previous ideals is generated by the idempotent elements $0, I \; and \; 1$, respectively.

Theorem 3.11 $R(I)$ is regular iff every neutrosophic finitely generated right (left) ideal in $R(I)$ can be generated by an idempotent element.

Proof.

(⇒): Suppose $J + KI$ is a neutrosophic right ideal generated by n elements. We will use the principle of mathematical induction.

If $n = 1$, then using theorem 3.9, the desired result is achieved.

If $n = 2$, then $J + K = (r_1 + r_2I)(R + RI) + (r_3 + r_4I)(R + RI) = r_1R + [(r_1 + r_2)R]I + r_3R + [(r_3 + r_4)R]I = (r_1 + r_3)R + [(r_1 + r_2) + (r_3 + r_4)]RI \Rightarrow J = (r_1 + r_3)R \quad and \quad K = [(r_1 + r_2) + (r_3 + r_4)]R \Rightarrow J \; and \; K$ are finitely generated. Because R is regular, $J \; and \; K$ are generated by an idempotent element.

$$\Rightarrow J = (r_1 + r_3)R = e_1R \quad and \quad K = [(r_1 + r_2) + (r_3 + r_4)]R = e_2R, where, e_1, e_2 \in Id_R$$

Finally, $J + KI = e_1R + e_2RI = e_1R + e_2RI - e_1RI + e_1RI = e_1R + [e_1R + (e_2 - e_1)R]I$

$$= e_1R + [e_1R + (e_2 - e_1)R + (e_2 - e_1)R]I = [e_1 + (e_2 - e_1)I][R + RI]$$

It can easily be verified that $e_1 + (e_2 - e_1)I \in NId_{R(I)}$. Therefore, the desired result is achieved.

(⇐): This results directly from theorem 3.9.

In the same way, we prove it in the case of the left ideal.

Theorem 3.12 $R(I)$ is a regular iff every neutrosophic principal right (left) ideal is a direct actor in $R(I)$.

Proof.

(⇒): Suppose $K_1 + K_2I$ is a neutrosophic principal right ideal. Therefore, $K_1 + K_2I = (e_1 + e_2I)R(I)$, where $e_1 + e_2I \in NId_{R(I)}$, according to theorem 3.9.

We have $K_3 + K_4I = [(1 - e_1) - e_2I)]R(I)$ is a right ideal is generated by $(1 - e_1) - e_2I$, where $(1 - e_1) - e_2I \in NId_{R(I)}$. Now let us prove that $R(I) = (K_1 + K_2I) \oplus (K_3 + K_4I)$.

First, we have $(K_1 + K_2I) + (K_3 + K_4I) \subseteq R(I)$. At other hand, if $r_1 + r_2I \in$ R(I), then

$$r_1 + r_2I = (e_1 + e_2I)(r_1 + r_2I) + [(r_1 + r_2I) - (e_1 + e_2I)(r_1 + r_2I)]$$

$= (e_1 + e_2I)(r_1 + r_2I) + [1 - (e_1 + e_2I)](r_1 + r_2I) \in (e_1 + e_2I)R(I) + [(1 - e_1) - e_2I]R(I)$.

Thus $R(I) = (K_1 + K_2I) + (K_3 + K_4I)$.

Finally, $\forall s+tI \in (K_1+K_2I)\cap(K_3+K_4I) \Rightarrow s+tI \in (K_1+K_2I)\ and\ s+tI \in (K_3+K_4I)$

Therefore, $s+tI=(e_1+e_2I)(r_1+r_2I)\ and\ s+tI=[(1-e_1)-e_2I](r_3+r_4I)$, where $r_1+r_2I\ and\ r_3+r_4I \in R(I)$.

At other hand, we note $s+tI=(e_1+e_2I)(r_1+r_2I) \Rightarrow (e_1+e_2I)(s+tI)=s+tI$

Subsequently, $s+tI=(e_1+e_2I)(s+tI)=(e_1+e_2I)[(1-e_1)-e_2I](r_3+r_4I)$

$$=[e_1(1-e_1)+(-e_1e_2+e_2(1-e_1)-{e_2}^2)I](r_3+r_4I)$$
$$=[(e_1-{e_1}^2)+(-e_1e_2+e_2-e_2e_1-{e_2}^2+{e_1}^2-{e_1}^2)I](r_3+r_4I)$$
$$=[(e_1-{e_1}^2)+(-({e_1}^2+e_1e_2+e_2e_1+{e_2}^2)+e_2+{e_1}^2)I](r_3+r_4I)$$
$$=[(e_1-{e_1}^2)+(-(e_1+e_2)^2+e_2+{e_1}^2)I](r_3+r_4I)$$

Since $e_1+e_2I \in NId_{R(I)}$, so $e_1\ and\ e_1+e_2 \in Id_R$, According to theorem 2.4.

Therefore, $s+tI=[(e_1-e_1)+(-(e_1+e_2)+e_2+e_1)I](r_3+r_4I)=0$

So,$(K_1+K_2I)\cap(K_3+K_4I)=\{0\}$. Therefore, $R(I)=(K_1+K_2I)\oplus(K_3+K_4I)$.

Thus K_1+K_2I is a direct actor in $R(I)$.

($\Leftarrow$): $\forall r_1+r_2I \in R(I)$, then $(r_1+r_2I)R(I)$ is a principal right ideal in $R(I)$.

Therefore, according to this assumption, there is a neutrosophic right ideal K_1+K_2I, where, $R(I)=(r_1+r_2I)R(I)\ \oplus (K_1+K_2I)$. Thus $1=(r_1+r_2I)(x+yI)+(j+kI)$, where $x+yI \in R(I)\ and\ j+kI \in K_1+K_2I$.

And from it we find $r_1+r_2I=(r_1+r_2I)(x+yI)(r_1+r_2I)+(j+kI)(r_1+r_2I)$ ….$(*)$

Since K_1+K_2I is a neutrosophic right ideal, so $(j+kI)(r_1+r_2I) \in K_1+K_2I$.

At other hand, we note $(j+kI)(r_1+r_2I)=r_1+r_2I-(r_1+r_2I)(x+yI)(r_1+r_2I)=(r_1+r_2I)[1-(x+yI)(r_1+r_2I)] \in (r_1+r_2I)R(I)$. Thus $(j+kI)(r_1+r_2I) \in (r_1+r_2I)R(I)\cap\ (K_1+K_2I\ )=\{0\}$.

Subsequently, $(j+kI)(r_1+r_2I)=0$. Returning to $(*)$, we find that $r_1+r_2I=(r_1+r_2I)(x+yI)(r_1+r_2I)$. Therefore, $r_1+r_2I \in NReg_{R(I)} \Rightarrow R(I)=NReg_{R(I)}$.

In the same way, we prove it in the case of the left ideal.

Example 3.13 In any neutrosophic field $R(I)$, we have RI a neutrosophic principal ideal is generated by a neutrosophic idempotent I. At other hand, we have $(1-I)R(I)$ is a neutrosophic principal ideal is generated by a neutrosophic idempotent $1-I$.

Now , we note $(1-I)R(I)+RI=R(I)$. Also $\forall x \in (1-I)R(I)\cap RI \Rightarrow\ x \in (1-I)R(I) and\ x \in RI$.

So $x=(1-I)(r_1+r_2I)=r_1+r_2I-r_1I-r_2I=r_1-r_1I$. Since $x \in RI$, so $r_1=0$. Therefore, $x=(1-I)r_2I=0$. Subsequently, $(1-I)R(I)\oplus RI=R(I)$. Therefore, RI is a direct actor in $R(I)$.

Theorem 3.14 $R(I)$ is a regular $\Leftrightarrow$ each neutrosophic finitely generated right (left) ideal in $R(I)$ is a direct actor in $R(I)$.

Proof.

($\Rightarrow$): If K_1+K_2I is a finitely generated right ideal, then $K_1+K_2I=(e_1+e_2I)R(I)$, where, $(e_1+e_2I)^2=e_1+e_2I$ according to theorem.3.11, and from it we find that

$$R(I)=(e_1+e_2I)R(I)\oplus[(1-e_1)-e_2I)]R(I)$$

($\Leftarrow$): It results from theorem.3.11 and theorem.3.12.

In the same way, we prove it in the case of the left ideal.

Remark 3.15 Figure 2 shows the resulting relationship between the neutrosophic regular ring and their ideals, as follows:

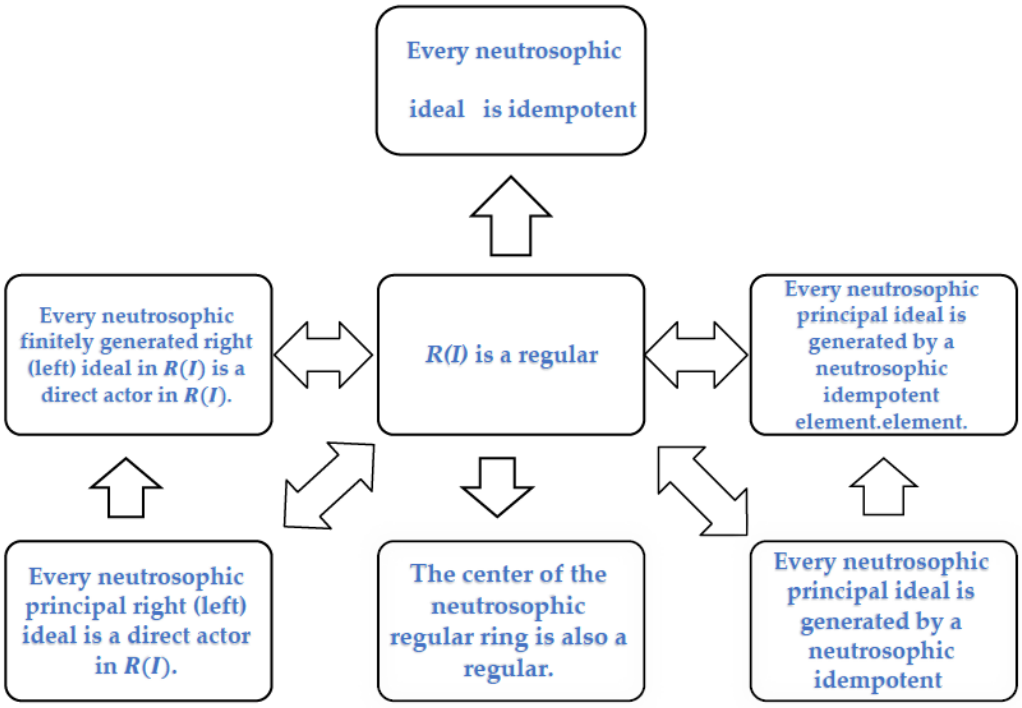
Every neutrosophic ideal is idempotent
Every neutrosophic finitely generated right (left) ideal in R(I) is a direct actor in R(I).
R(I) is a regular
Every neutrosophic principal ideal is generated by a neutrosophic idempotent element.element.
Every neutrosophic principal right (left) ideal is a direct actor in R(I).
The center of the neutrosophic regular ring is also a regular.
Every neutrosophic principal ideal is generated by a neutrosophic idempotent

Figure 2. Relationship between the neutrosophic regular ring and their ideals.

Theorem 3.16 $R(I) = NReg_{R(I)}$ iff $(K_1 + K_2I) \cap (K_3 + K_4I) = (K_1 + K_2I)(K_3 + K_4I) = K_1K_3 + K_2K_4I$ for every neutrosophic right ideal $K_1 + K_2I$ and every neutrosophic left ideal $K_3 + K_4I$.

Proof.

($\Rightarrow$): We have $(K_1 + K_2I) \cap (K_3 + K_4I) = (K_1 \cap K_3) + (K_2 \cap K_4)I$ and also K_1 *and* K_2 (K_3 *and* K_4) are right (left) ideals in R. Since $R(I)$ is regular, so R is regular according to theorem.3.4. Therefore, $K_1 \cap K_3 = K_1K_3$ *and* $K_2 \cap K_4 = K_2K_4$. Thus, $(K_1 + K_2I) \cap (K_3 + K_4I) = (K_1 \cap K_3) + (K_2 \cap K_4)I = K_1K_3 + K_2K_4I$.

At other hand, we have $(K_1 + K_2I)(K_3 + K_4I) \subseteq (K_1 + K_2I) \cap (K_3 + K_4I) = K_1K_3 + K_2K_4I \subseteq K_1K_3 + K_2K_4I + [K_1K_4 + K_2K_3]I = (K_1 + K_2I)(K_3 + K_4I)$.

Thus $(K_1 + K_2I) \cap (K_3 + K_4I) = (K_1 + K_2I)(K_3 + K_4I) = K_1K_3 + K_2K_4I$

($\Leftarrow$): $\forall\, r_1 + r_2I \in R(I)$, then $(r_1 + r_2I)R(I)$ is a right ideal and $R(I)(r_1 + r_2I)$ is a left ideal. At other hand, $r_1 + r_2I \in (r_1 + r_2I)R(I) \cap R(I)(r_1 + r_2I)$

$$= (r_1 + r_2I)R(I)R(I)(r_1 + r_2I)$$

Since $R(I)$ is unity, so $R(I) = R(I)^2$. Subsequently, $r_1 + r_2I = (r_1 + r_2I)R(I)(r_1 + r_2I)$

$\Rightarrow\ \exists\, r_3 + r_4I \in R(I)$ such that $r_1 + r_2I = (r_1 + r_2I)(r_3 + r_4I)(r_1 + r_2I) \Rightarrow r_1 + r_2I \in NReg_{R(I)}$

$\Rightarrow R(I) = NReg_{R(I)}$.

Example 3.17 In $\mathbb{Z}_5(I)$, we have $\{0\}$, *and* $\mathbb{Z}_5I$ are neutrosophic ideals, and it comes true $\{0\} \cap \mathbb{Z}_5I = \{0\} = \{0\}.\mathbb{Z}_5$

Theorem 3.18 If $R(I) = NReg_{R(I)}$, then $\forall\, K_1 + K_2I \in N\mathfrak{T}_{R(I)}$ is idempotent.

Proof. $\forall\, K_1 + K_2I \in N\mathfrak{T}_{R(I)}$. First, we have $(K_1 + K_2I)^2 \subseteq K_1 + K_2I$.

At other hand, $\forall\, r_1 + r_2I \in K_1 + K_2I \Rightarrow r_1 + r_2I \in R(I) \xrightarrow[R(I)=NReg_{R(I)}]{} \exists r_3 + r_4I \in R(I)$;

$$r_1 + r_2 I = \underbrace{(r_1 + r_2 I)}_{\in K_1 + K_2 I} \underbrace{(r_3 + r_4 I)(r_1 + r_2 I)}_{\in K_1 + K_2 I} \in (K_1 + K_2 I)^2$$

Thus $K_1 + K_2 I$ is idempotent.

Example 3.19 In a neutrosophic field $\mathbb{Z}_5$, we have only $\{0\}, \mathbb{Z}_5 I, and\ \mathbb{Z}_5(I)$ are neutrosophic ideals, and it comes true $\{0\}^2 = \{0\}$, $(\mathbb{Z}_5 I)^2 = \mathbb{Z}_5 I, and\ [\mathbb{Z}_5(I)]^2 = \mathbb{Z}_5(I)$.

4. Conclusion and Future Works

During our search, we created the essential and sufficient conditions for the neutrosophic ring to become a regular. Furthermore, several instances were created to demonstrate the search's reliability. We intend to generalize the regular refined neutrosophic rings and their attributes in future studies.

Funding: "This research received no external funding"

Conflicts of Interest

The authors declare that the research has no conflicts of interest.

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Received: March 17, 2024. Accepted: July 25, 2024

Neutrosophic Sets and Systems, Vol. 70, 2024
University of New Mexico

Developing Neutrosophic Cubic Spherical Fuzzy Sets along with their Exponential Aggregation Operators for Decision-Making Problems

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Abstract:

In this article, we have devised the notion of neutrosophic cubic spherical fuzzy sets (NCSFSs) for the first time and discussed their basic binary operations along with important properties. This proposition has been framed superimposing the existing notions of spherical neutrosophic set (SNS) as well as the interval-valued neutrosophic spherical fuzzy set (IVNSFS). This proposition computationally helps in handling incompatible situations where each element has been addressed by truth, indeterminacy, and false membership values. Next, some important exponential operational laws have been relationally established for NCSFSs, highlighting their significant properties. Furthermore, to address decision-making challenges in the NCSFS environment, we have developed exponential weighted aggregate operators based on the proposed & defined operational laws with important results. Finally, an algorithm for solving a decision-making problem has been presented by using the proposed exponential weighted aggregate operators, where the best site for waste material has been identified with the help of a numerical example by keeping the environmental factors into account. In the numerical example, we created a decision matrix based on experts' opinions and then applied the suggested NCSFWEA operators to a waste disposal site selection issue that has five alternatives and four attributes. At last, by utilizing the scoring function, the alternatives have been ranked.

Keywords: Spherical fuzzy set; Neutrosophic spherical fuzzy set; Aggregation operators; Multi-criteria decision making (MCDM); Solid waste disposal.

1. Introduction

As our society faces an ever-increasing volume and variety of solid wastes, addressing the obstacles of responsible management and sustainable disposal has become critical to shaping a cleaner, environmentally conscious future. Solid waste includes everyday objects like packaging for goods, green waste, textiles apparel, containers, debris from food, daily papers, equipment, color, and fuel cells [1]. If these materials are not removed, they may seriously harm the environment and the populace, particularly if they end up in the sewage system, water supply, or areas used for gathering and storing trash. Proper disposal is essential for a clean and sustainable environment, benefiting both community well-being and the earth's preservation. The process of choosing a waste material disposal technology is complicated and involves a multitude of qualitative and quantitative factors.

The primary cause is that managing the disposal of solid waste is not only a societal issue but also a multi-criteria decision problem that takes into account financial, technical, political, sociocultural, and environmental factors [2]. The population is rapidly growing, which is having a significant impact on the environment and threatening the globe's viability. Among the most visible effects of population growth is urban development. The growth of cities is affecting the natural balance of the environment. Every urban center produces a massive amount of solid matter [3]. According to Barzehkar et al. [4] and Kamdar et al. [5], landfills are an important component of the garbage management chain, which includes garbage bargaining, decomposition, recycling, reusing and disposal facilities. The GM (1, 1) model for pythagorean fuzzy numbers (PFNs) was presented by Li et al. [6], who also used their suggested model for solid waste management (SWM) site selection. Ren et al. [7] identified a novel multi-attribute decision- making (MADM) procedure for determining an appropriate location for SWM by applying the power Muirhead mean operator to a q-rung ortho-pair probabilistic hesitant fuzzy set. The REGIME method for Pythagorean fuzzy set (PFS) was first presented by Basar et al. [8] and employed to the selection of waste disposal sites. In order to have an overview of the generalizations/extensions of fuzzy sets, we may refer to the following Figure 1.

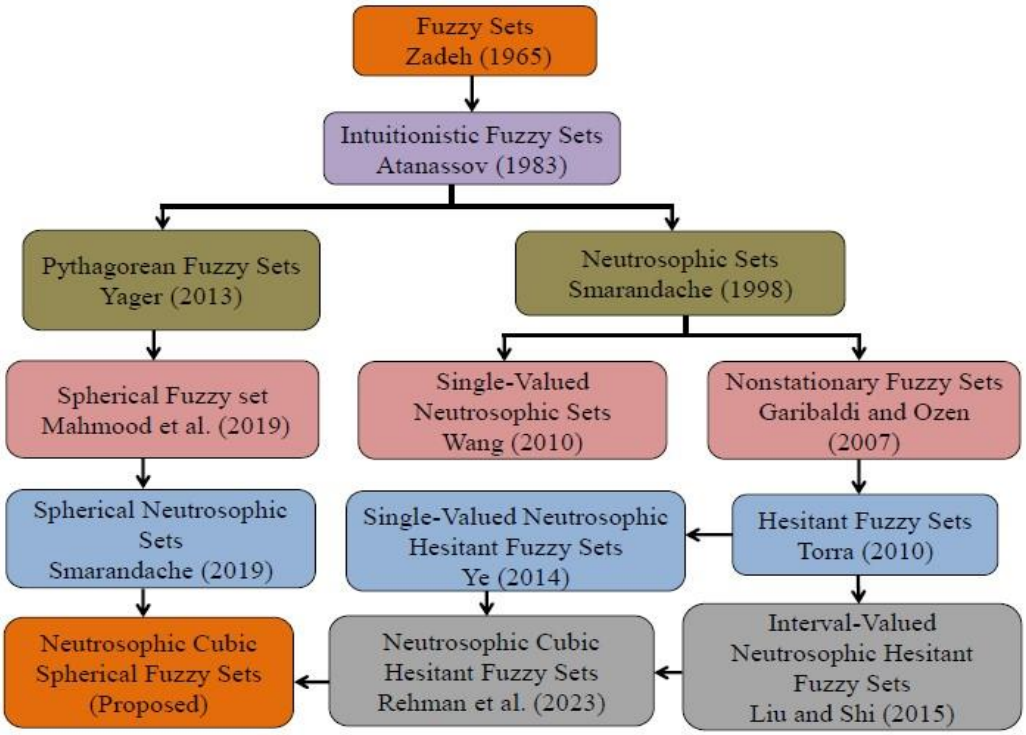

Fig 1: Overview on Generalized Extensions of Fuzzy Sets.

The most effective method for selecting a suitable alternative from all of the available options is the MCDM problem. Maximum evaluations were conducted in situations where goals and constraints are frequently ambiguous or unclear. In 1965, Zadeh [9] introduced a new way of dealing with vagueness and imperfectness called fuzzy set (FS) theory as a modified form of conventional theory. This theory is based on the concept of membership degree, which falls between zero and one. After that, he extended this theory to an interval-valued fuzzy set (IVFS). Since there are some limitations in this theory, many researchers have expanded it, which is discussed here. Attanasov [10] suggested intuitionistic fuzzy set (IFS) theory, a new form of the idea of fuzzy sets that includes both types of degrees: membership degrees and non-membership degrees. IFSs have been effectively implemented

in numerous areas of science, including decision-making processes. Moreover, Smarandache [11] presented the first theory on the neutrosophic set (NS) in 1998, which one is a broadening of intuitionistic fuzzy sets and also fuzzy sets theory. This set presents the following kinds of membership functions, such as truth, indeterminacy, and falsity membership functions, which range from 0 to 1. Then in 1999, Molodtsov [12] introduced a new concept known as the soft set (SS) which is a generalization of fuzzy theory. One particular kind of the NS is the single-valued neutrosophic set (SVNS) created via Smarandache et al. [13] in 2010. Values of the membership function for a neutrosophic set can exceed 1. The truth membership value of a given scenario could be bigger than one, if one element of the neutrosophic set gets valued greater. A single-valued neutrosophic set has membership values ranging from 0 to 1, as well as the sum of membership values ranging from 0 to 3. Another fuzzy set extension that can be used as a practical tool is the hesitant fuzzy set (HFS) [14], which permits the membership degree to be a component of the fuzzy set and considers multiple membership grade values ranging from 0 to 1. These sets have been used to solve MCDM in uncertain environments. Many researchers [15, 16] investigated algebraic concepts and developed a potential theory in cubic sets. In 2012, Jun [17] utilized the IVFS and the FS to form the cubic set. The intuitionistic fuzzy set and the interval intuitionistic fuzzy sets are generalized into the cubic set. Cubic sets have become essential tools for dealing with ambiguous data. Then, neutrosophic soft set theory was developed by Maji [18] as a new notion with the help of SS and also provided basic operations and important properties. After that, Yager [19] extended this concept to Pythagorean fuzzy set (PFS) with the condition that the total squares of the membership and non-membership degree is equal to or lesser than 1. Further, there are many extensions of this theory, including picture fuzzy set (PFS) [20], generalized ortho-pair fuzzy set (OPFS) [21], spherical fuzzy set (SFS) [22], etc. Next, Ye [23] introduced the single-valued neutrosophic hesitant fuzzy set (SVNHFS), combining SVNS and HFS to handle challenges involving multi-attribute decision-making (MADM). Moreover, based on an interval neutrosophic hesitant set, Liu and Shi [24] proposed hybrid weighted average aggregation operators as an application of MADM challenges. Exponential operations along with aggregation operators for interval neutrosophic set (INS) are given by Ye [25], along with their application in decision-making obstacles. In 2016, Mehmood et al. [26] recommended a theory of cubic hesitant fuzzy sets. Smarandache et al. [27] expanded the cubic set into the neutrosophic cubic fuzzy set, which includes both the NS and the INS. After that for single-valued neutrosophic numbers, Lu and Ye [28] suggested the exponential laws. To convert the single-valued function into a multiple-valued one, Smarandache [29] introduced a hybridized version of the soft set called hypersoft sets (HSS). Next, in the environment of probabilistic single–valued neutrosophic hesitant fuzzy (PSVNHF) Shao et al. [30] developed the probabilistic interval neutrosophic hesitant fuzzy weighted averaging (PINHFWA) and geometric (PINHFWG) operators for dealing with decision–making challenges.

Next, to create the TOPSIS extension subject to fuzziness, Gundogdu and Kahraman [31] introduced new interval-valued spherical fuzzy set (IVSFS) with aggregate operators, which include IVSF weighted arithmetic as well as geometric mean operators. Further, Smarandache [32] also provided the spherical neutrosophic set (SNS), which integrates the spherical and Pythagorean fuzzy sets. Some

new neutrosophic hesitant fuzzy set (NHFS) aggregation operators for MADM problems were developed by Liu and Luo [33]. Further, the stability evaluation with a neutrosophic cubic set for certain applications has been addressed by Al-Shumrani et al. [34]. In a neutrosophic cubic structures, Khan et al. [35] constructed exponential aggregation operators, which they then applied to MADM problems. By considering a spherical neutrosophic set (SNS), Molla and Giri [36] introduced the MCDM problem using the aggregation operator of SNS. Further, Dhumaras and Bajaj [37] proposed a new type of Hellinger divergence measure for symptomatic COVID-19 detection in the context of a single-valued neutrosophic hypersoft set. To promote sustainable soil practices in construction foundation projects, Ahmed et al. [38] introduced neutrosophic multi-criteria model as an application of life cycle assessment (LCA). Zenat et al. [39] presented a framework for sustainable supplier selection as a crucial part of supply chain management, encouraging ethical and environmentally friendly practices, using a neutrosophic multi-criteria decision-making methodology. Addressing vagueness and impreciseness in decision-making processes Dhumars and Bajaj [40] presented a bi-parametric discriminant measure for neutrosophic sets with application to symptomatic detection in medical diagnosis. After that, Kamran et al. [41] developed averaging and geometric aggregation operators coupled with confidence level within a probabilistic neutrosophic hesitant fuzzy rough environment to enhance the strategy for managing multi-criteria decision-making problems. Further, different types of aggregation operators proposed by Monika et al. [42] under the circumstances of T-spherical fuzzy hypersoft sets (TSFHSSs) also utilized these operators for choosing the best site in the context of renewable energy sources. After that, Muthukumaran [43] proposed n-hyperspherical neutrosophic matrices (n-HSNMs), expanding upon spherical neutrosophic matrices along with their algebraic properties and proof. To address environmental protection issues, Rehaman et al. [44] established exponential aggregate operators under neutrosophic cubic hesitant fuzzy sets. In 2024, Wang et al. [45] established a complex spherical fuzzy information-based selection framework for evaluating the food waste treatment system that includes mutual support and regularity decision information. Furthermore, Grag [46] proposed an exponential-logarithm-based SVNS that incorporates EL (exponential-logarithm) operations into the regular neutrosophic set. This integration aims to improve their effectiveness in solving decision-making problems. Next, for identifying the best alternative supplier type for strategic and non-strategic products categories, Onden et al. [47] provided micro mobility service using a type-2 neutrosophic number based decision-making approach.

From the literature, we observe that there are no theories or methods available that address cubic set theory in the context of a NCSFS, which is a hybrid of SNS and IVNSFS. This theory efficiently handles complex information in decision-making problems. Also, the decision-makers have more flexibility due to releasing the experts from the constraint, i.e., the sum of all uncertainty elements is less than 3. NCSFS and its aggregation operators are primarily intended to be used in real-life decision-making scenarios. This approach makes it possible to prepare ahead and minimize unfavorable effects in real-world situations like risk management, industrial operations, and artificial intelligence. Additionally, aggregation operators and exponential operational laws are in demand with urbanization growing rapidly and exponentially.

First, we have presented a new theory called the neutrosophic cubic spherical fuzzy set with some basic operations, this is a mixture of SNS and IVNSFS. Defined the accuracy function, the score function and the certainty function for NCSFS to compare the results. Next, we established exponential operational laws in NCSFS and presented important results. Moreover, we introduced a new aggregation operator known as the neutrosophic cubic spherical exponential weighted aggregation operator with the help of exponential laws and some theorems provided with proof. We then developed an algorithm based on proposed exponential aggregation operators and applied this in the application of decision-making challenges.

The remaining part of this paper is structured as follows. Some fundamental definitions and basic notions are presented in Section 2. We introduced a new notion/proposition coined as neutrosophic cubic spherical fuzzy sets (NCSFSs) in Section 3 and performed some fundamental operations on them. Then we presented some important exponential operational laws for NCSFSs along with some useful results in Section 4. Furthermore, we developed some exponential weighted aggregation operators for NCSFSs with important results in Section 5 for dealing with complex decision-making problems. Next, in Section 6, we presented an MCDM approach based on the proposed neutrosophic cubic spherical fuzzy weighted exponential aggregation operator (NCSFWEAO) and utilized this algorithm for the decision-making site selection issue of waste material disposal. Further, numerical simulations and computations for solid waste disposal site selection were conducted to better understand the proposed algorithms. Finally, the paper has been concluded in Section 7 outlining the scope for future work.

2. Basic Notions

This section has introduced few basic ideas which are helpful in the proposed work, such as such as spherical fuzzy set (SFS), interval-valued spherical fuzzy set (IVSFS), neutrosophic set (NS), spherical neutrosophic set (SNS) and neutrosophic cubic hesitant fuzzy set (NCHFS).

Definition 1. *[22] "Let $\mathfrak{N}$ represent the universal set. A spherical fuzzy set (SFS) ∇ is defined as:*

$$\nabla = \{< r, \mu_\nabla(r), \pi_\nabla(r), \vartheta_\nabla(r) > | r \in \mathfrak{N}\}$$

where $u_\nabla(r) : \mathfrak{N} \to [0,1]$ is the membership degree, $\pi_\nabla(r) : \to [0,1]$ is the indeterminacy degree and $\vartheta_\nabla(r) : \mathfrak{N} \to [0,1]$ is the non-membership degree and satisfies the condition s.t.

$$0 \leq \left(\mu_\nabla(r)\right)^2 + \left(\pi_\nabla(r)\right)^2 + \left(\vartheta_\nabla(r)\right)^2 \leq 1."$$

Example: Suppose the universal set $\mathfrak{N} = \{r_1, r_2, r_3, r_4\}$ represents the 4 car company's brand.

Let the criteria be: $a_1 =$ average, $a_2 =$automatic and $a_3 =$ safety. Then, the spherical fuzzy set (∇), for determining the best car among the alternatives based on these criteria is defined as:

$$\begin{aligned}\nabla =< r_1\{average\ (0.7, 0.1, 0.2), automatic(0.2, 0.6, 0.5), saftey(0.8, 0.2, 0.3)\},\\ r_2\{average\ (0.5, 0.5, 0.6), automatic(0.7, 0.6, 0.3), saftey(0.7, 0.4, 0.2)\},\\ r_3\{average\ (0.9, 0.1, 0.1), automatic(0.1, 0.2, 0.8), saftey(0.8, 0.2, 0.1)\},\\ r_4\{average\ (0.8, 0.1, 0.2), automatic(0.9, 0.1, 0.2), saftey(0.9, 0.2, 0.2)\} >.\end{aligned}$$

In this set each element is characterized by three parameters such as membership degree $\{\mu(r)\}$, indeterminancy degree $\{\pi(r)\}$ and non-membership degree $\{\nu(r)\}$ with condition s.t.

$$\mu^2(r) + \pi^2(r) + \nu^2(r) \leq 1.$$

Definition 2. [31] *"Let $\mathfrak{N}$ represent the universal set and I' is defined as an interval-valued spherical fuzzy set (IVSFS):*

$$I' = \{< r, ([\mu_{I'}(r)^L, \mu_{I'}(r)^U], [\pi_{I'}(r)^L, \pi_{I'}(r)^U], [\vartheta_{I'}(r)^L, \vartheta_{I'}(r)^U) | \ r \in \mathfrak{N} >]\}$$

where $1 \geq \mu_{I'}(r)^U \geq \mu_{I'}(r)^L \geq 0$, $1 \geq \pi_{I'}(r)^U \geq \pi_{I'}(r)^L \geq 0$, $1 \geq \vartheta_{I'}(r)^U \geq \vartheta_{I'}(r)^L \geq 0$ *and*

$$0 \leq (\mu_{I'}(r)^U)^2 + (\pi_{I'}(r)^U)^2 + (\vartheta_{I'}(r)^U)^2 \leq 1$$

For each $r \in \mathfrak{N}$, $\mu_{I'}(r)^U$, $\pi_{I'}(r)^U$ and $\vartheta_{I'}(r)^U$ are the upper membership degree, hesitancy degree and non-mebership degree of r to A respectively."

Example: Now based on the above example the interval-valued spherical fuzzy set is defined as;

$$I' = < r_1\{average\ ([0.7, 0.5], [0.3, 0.1], [0.4, 0.1]), automatic([0.3, 0.2], [0.7, 0.5], [0.5, 0.3]), saftey([0.8, 0.5], [0.2, 0.1], [0.3, 0.1])\},$$
$$r_2\{average\ ([0.5, 0.2], [0.6, 0.3], [0.5, 0.4]), automatic([0.7, 0.5], [0.6, 0.3], [0.4, 0.2]), saftey([0.6, 0.4], [0.4, 0.3], [0.2, 0.1])\}$$
$$r_3\{average\ ([0.8, 0.7], [0.2, 0.1], [0.2, 0.1]), automatic([0.1\ \ 0.1], [0.2, 0.1], [0.8, 0.6]), saftey([0.9, 0.7], [0.2, 0.1], [0.2, 0.1])\}$$
$$r_4\{average\ ([0.8, 0.7], [0.1, 0.1], [0.2, 0.1]), automatic([0.9, 0.7], [0.2, 0.2], [0.2, 0.1]), saftey([0.8, 0.7], [0.2, 0.1], [0.2, 0.1])\} >.$$

Definition 3. *[11] "Suppose $\mathfrak{N}$ represent the universal set. Then a neutrosopic set (NS) is expressed as:*

$$\mathcal{N} = \{< r, \mu_{\mathcal{N}}(r), \pi_{\mathcal{N}}(r), \vartheta_{\mathcal{N}}(r) > | r \in \mathfrak{N}, u_{\mathcal{N}}(r), \ \pi_{\mathcal{N}}(r), \ \vartheta_{\mathcal{N}}(r) \in \ [\ 0^-, 1^+\ []\}$$

where $u_{\mathcal{N}}(r)$, $\pi_{\mathcal{N}}(r)$ and $\vartheta_{\mathcal{N}}(r)$ are real standard element mapping from $\mathfrak{N}$ to $[0,1]$ called truth membership degree, indeterminacy membership degree and falsity membership degree respectively, with condition s.t.

$$0^- \leq \mu_{\mathcal{N}}(r) + \pi_{\mathcal{N}}(r) + \vartheta_{\mathcal{N}}(r) \leq 3^+."$$

Example: Suppose there is a need to evaluate the customer satisfaction with a newly launched smartphone model. Consider the set of customers $< X, Y, Z >$. Customer Feedback is collected and analyzed according to three criteria: Satisfaction, Indeterminancy, Dissatisfaction. Then the neutrosophic set is defined as:

$$\mathcal{N} = \{< X(0.8, 0.3, 0.2) >, < \ Y(0.5, 0.7, 0.7) >, < Z(0.5, 0.7, 0.5) >\};$$

where, all values of degree of truth, indeterminancy and falsity lies in [0, 1], and the sum of each components is less than or equal to 3. This example shows how neutrosophic sets provide a more detailed and nuanced approach to controlling uncertainty and variability in data, making them useful in complex decision-making scenarios due to their 3-*D* dimensional space.

Definition 4. *[32] "Let $\mathfrak{N}$ represent the universal set. The spherical neutrosophic set (SNS) A is defined as:*

$$A = \{< r, \mu_A(r), \pi_A(r), \vartheta_A(r) > | r \in \mathfrak{N}\}$$

where $u_A(r)$ *mapping from* $\mathfrak{N}$ *to* $[0,1]$ *called truth membership degree,* $\pi_A(r)$ *mapping from* $\mathfrak{N}$ *to* $[0,1]$ *called indeterminant membership degree and* $\vartheta_A(r)$ *mapping from* $\mathfrak{N}$ *to* $[0,1]$ *called falsity membership degree and satisfies the following condition s.t.*

$$0 \leq \left(\mu_A(r)\right)^2 + \left(\pi_A(r)\right)^2 + \left(\vartheta_A(r)\right)^2 \leq 3.''$$

Example: Based on the above example defined for neutosophic set, the spherical neutrosophic set is defined as:

$$A = \{< X(0.88, 0.78, 0.88) >, < Y(0.55, 0.29, 0.77) >, < Z(0.99, 0.59, 0.69) >\};$$

with the condition that sum of the square of each component is less than or equal to 3, i.e.

For customer X: $0.88^2 + 0.78^2 + 0.88^2 = 2.1572$.

For customer Y: $0.55^2 + 0.29^2 + 0.77^2 = 0.9795$.

For customer Z: $0.99^2 + 0.59^2 + 0.69^2 = 1.8043$.

Here, it can be noticed that the set meets the standard criteria for a spherical neutrosophic set.

Definition 5. *"Let* $\mathfrak{N}$ *represent the universal set. The object I=* $\{< r, \mu_I(r), \pi_I(r), \vartheta_I(r) > | r \in \mathfrak{N}\}$ *known interval-valued neutrosophic spherical fuzzy set (IVNSFS) on universal set, where* $\mu_I(r)$, $\pi_I(r)$ *and* $\vartheta_I(r)$ *are interval-valued spherical fuzzy set has the following expression:*

$$0 \leq (\mu_I(r)^U)^2 + (\pi_I(r)^U)^2 + (\vartheta_I(r)^U)^2 \leq 3.''$$

Example: Based on the above example, the IVNSFS is defined as:

$$I = \begin{cases} < X([0.98, 0.77\ ], [0.79, 0.67\ ], [0.59, 0.49]) >, \\ < Y([0.44, 0.39], [0.78, 0.66], [0.88, 0.67]) >, \\ < Z([0.99, 0.79], [0.68, 0.55], [0.88, 0.82]\ ) >. \end{cases}$$

3. Neutrosophic Cubic Spherical Fuzzy Sets (NCSFSs)

This section of the article presents the new notion of neutrosophic cubic spherical fuzzy sets (NCSFS) with some important basic operations. The definition of NCSFS is represented as follows:

Definition 6. *Consider* $\mathfrak{N}$ *represent the universal set. A NCSFS in* $\mathfrak{N}$ *is a set of values* $\alpha =< I, A >$, *where* $I = \{< r, \mu_I(r), \pi_I(r), \vartheta_I(r) > | r \in \mathfrak{N}$ *represents an interval valued neutrosophic spherical fuzzy set in* $\mathfrak{N}$ *and* $A = \{< r, \mu_A(x), \pi_A(r), \vartheta_A(r) > | r \in \mathfrak{N}\}$ *represent spherical neutrosophic set in* $\mathfrak{N}$.

Example 1. *Suppose* $\mathfrak{N} = \{r, t\}$. *The set of values* $\alpha =< I, A >$ *then,*

$\mu_I(r) = \{[0.2, 0.5]\}$, $\mu_A(r) = \{0.6\}$, $\pi_I(r) = \{[0.4, 0.7]\}$, $\pi_A(r) = \{0.7\}$, $\vartheta_I(r) = \{[0.2, 0.4]\}$, $\vartheta_A(r) = \{0.3\}$, *then we have* α *corresponding to* r *is* $\{< \{[0.2, 0.5]\}, \{[0.4, 0.7]\}, \{[0.1, 0.4]\}, \{(0.6, 0.7, 0.3)\} >\}$.

$\mu_I(t) = \{[0.1, 0.5]\}$, $\mu_A(y) = \{0.7\}$, $\pi_I(t) = \{[0.3, 0.6]\}$, $\pi_A(y) = \{0.9\}$, $\vartheta_I(t) = \{[0.7, 0.9]\}$, $\vartheta_A(t) = \{0.3\}$, *then we have* α *corresponding to* t *is* $\{< \{[0.1, 0.5]\}, \{[0.3, 0.6]\}, \{[0.7, 0.9]\}, \{(0.7, 0.9, 0.3)\} >\}$.

Example 2 Assume there is a requirement for staff in the company. Four candidates have applied for the available position. The company's HR division assigns a three-person specialist team to handle hiring decisions. Let $\mathfrak{R} = \{r_1, r_2, r_3, r_4\}$ be a set of four candidates with the following attributes as: E_1 = Education, E_2 = Behavioral Interviews, and E_3 = Experience. Now, based on the expert's rating of the applicant corresponding to the attributes defined in the NCSFS.

Ratings by expert's.

- For, $< r_1$ {Education([0.25, 0.59], [0.54, 0.65], [0.66, 0.78], (0.35,0.72, 0.61))}
  {Behavioral Interviews([0.29, 0.47], [0.65, 0.75], [0.67, 0.86], (0.45,0.73, 0.54))},
  {Experience([0.69, 0.75], [0.42, 0.56], [0.55, 0.67], (0.67,0.53, 0.45))}>

- For, $< r_2$ {Education([0.85, 0.92], [0.25, 0.37], [0.12, 0.45], (0.86,0.51, 0.33))}
  {Behavioral Interviews([0.75, 0.89], [0.34, 0.47], [0.41, 0.54], (0.76,0.37, 0.53))},
  {Experience([0.78, 0.89], [0.24, 0.47], [0.31, 0.44], (0.70,0.47, 0.33))}>

- For, $< r_3$ {Education([0.52, 0.59], [0.54, 0.67], [0.51, 0.64], (0.56,0.57, 0.63))}
  {Behavioral Interviews([0.65, 0.79], [0.54, 0.67], [0.61, 0.74], (0.56,0.55, 0.53))},
  {Experience([0.68, 0.79], [0.34, 04.7], [0.51, 0.64], (0.56,0.57, 0.63)>

- For, $< r_4$ {Education([0.72, 0.85], [0.44, 0.57], [0.41, 0.54], (0.86,0.57, 0.43))}
  {Behavioral Interviews([0.65, 0.79], [0.44, 0.57], [0.41, 0.54], (0.76,0.57, 0.43))},
  {Experience([0.75, 0.89], [0.34, 0.56], [0.41, 0.54], (0.76,0.47, 0.53))}>.

Some Basic Operation on Neutrosophic Cubic Spherical Fuzzy Sets:

We formally define some fundamental operations for understanding and improved readability, considering the previously proposed term, the neutrosophic cubic spherical fuzzy set.

If there are two NCSFSs $\Phi =< I, A >$ and $\Psi =< J, B >$ defined on $\mathfrak{R}$, then we have

$$\Phi \oplus \Psi =$$

$$\left\langle r, \left\{\left[\sqrt{(\mu_I(r)^L)^2 + \left(\mu_J(r)^L\right)^2 - (\mu_I(r)^L)^2\left(\mu_J(r)^L\right)^2}, \sqrt{(\mu_I(r)^U)^2 + \left(\mu_J(r)^U\right)^2 - (\mu_I(r)^U)^2\left(\mu_J(r)^U\right)^2}\right]\right\},\right.$$

$$\left\{\left[\sqrt{(\pi_I(r)^L)^2 + \left(\pi_J(r)^L\right)^2 - (\pi_I(r)^L)^2\left(\pi_J(r)^L\right)^2}, \sqrt{(\pi_I(r)^U)^2 + \left(\pi_J(r)^U\right)^2 - (\pi_I(r)^U)^2\left(\pi_J(r)^U\right)^2}\right]\right\},$$

$$\{[\vartheta_I(r)^L\vartheta_I(r)^L, \vartheta_I(r)^U\vartheta_I(r)^U]\}, \left\{\left(\sqrt{\left(\mu_A(r)\right)^2 + \left(\mu_B(r)\right)^2 - \left(\mu_A(r)\right)^2\left(\mu_B(r)\right)^2}\right),\right.$$

$$\left.\left.\left(\sqrt{\left(\pi_A(r)\right)^2 + \left(\pi_B(r)\right)^2 - \left(\pi_A(r)\right)^2\left(\pi_B(r)\right)^2}\right), \left(\vartheta_A(r)\vartheta_B(r)\right)\right\}\right\rangle.$$

$$\boldsymbol{\Phi}\otimes\boldsymbol{\Psi}=\left\langle r,\{[\mu_I(r)^L\mu_J(r)^L,\mu_I(r)^U\mu_J(r)^U]\},\{[\pi_I(r)^L\pi_J(r)^L,\pi_I(r)^U\pi_J(r)^U]\},\right.$$

$$\left\{\left[\sqrt{(\mu_I(r)^L)^2+\left(\mu_J(r)^L\right)^2-(\mu_I(r)^L)^2\left(\mu_J(r)^L\right)^2},\sqrt{(\mu_I(r)^U)^2+\left(\mu_J(r)^U\right)^2-(\mu_I(r)^U)^2\left(\mu_J(r)^U\right)^2}\right]\right\},\left\{\left(\mu_A(r)\mu_B(r)\right),\right.$$

$$\left.\left.\left(\pi_A(r)\pi_B(r)\right),\left(\sqrt{\left(\vartheta_A(r)\right)^2+\left(\vartheta_B(r)\right)^2-\left(\vartheta_A(r)\right)^2\left(\vartheta_B(r)\right)^2}\right)\right\}\right\rangle.$$

$$\boldsymbol{k\Phi}$$

$$=\left\langle\left\{\left[\sqrt{1-(1-(\mu_I(r)^L)^{2(k)})},\sqrt{1-(1-(\mu_I(r)^U)^{2(k)})},\right]\right\},\left\{\left[\sqrt{1-(1-(\pi_I(r)^L)^{2(k)})},\sqrt{1-(1-(\pi_I(r)^U)^{2(k)})},\right]\right\},\right.$$

$$\left.\{[\vartheta_I(r)^{L(k)},\vartheta_I(r)^{U(k)}]\},\left\{\left(\sqrt{1-\left(1-\left(\mu_A(r)\right)^{2(k)}\right)}\right),\left(\sqrt{1-\left(1-\left(\pi_A(r)\right)^{2(k)}\right)}\right),\left(\vartheta_A(r)\right)^{(k)}\right\}\right\rangle.$$

$$\boldsymbol{\Phi^k}=$$

$$\left\langle r,\{[\mu_I(r)^{L(k)},\mu_I(r)^{U(k)}]\},\{[\pi_I(r)^{L(k)},\pi_I(r)^{U(k)}]\},\left\{\left[\sqrt{1-(1-(\vartheta_I(r)^L)^{2(k)})},\sqrt{1-(1-(\vartheta_I(r)^U)^{2(k)})},\right]\right\},\right.$$

$$\left.\left\{\left(\left(\mu_A(r)\right)^{(k)}\right),\left(\left(\pi_A(r)\right)^{(k)}\right),\left(\sqrt{1-\left(1-\left(\vartheta_A(r)\right)^{2(k)}\right)}\right)\right\}\right\rangle.$$

where $\boldsymbol{\Phi^k=\Phi\otimes\Phi\otimes\ldots\otimes\Phi(k-\text{time})}$. For each positive value of *k*, this represents a neutrosophic cubic spherical fuzzy value.

To simplify calculations, consider NCSFS as a neutrosophic cubic spherical fuzzy Number (NCSFN), s.t.

$$\boldsymbol{\Phi}=\langle\{[\mu_I(r_i)^L,\mu_I(r_i)^U],[\pi_I(r_i)^L,\pi_I(r_i)^U],[\vartheta_I(r_i)^L,\vartheta_I(r_i)^U]\},\{(\mu_A(r_i),\pi_A(r_i),\vartheta_A(r_i))\},r_i\in\ \mathfrak{N}\rangle$$

This structure is referred to as neutrosophic cubic spherical fuzzy Number (NCSFN). To create a new algorithm for rating the options based on the recommended aggregation operators, the score, accuracy, and certainty functions over NCSFSs are suitably reframed as follows:

Consider $\boldsymbol{\Phi}=<\boldsymbol{I},\boldsymbol{A}>$, where $\boldsymbol{I}=\{<r,\mu_I(r),\pi_I(r),\vartheta_I(r)>\}$ and $\mu_I(r)=[\mu_I(r)^L,\mu_I(r)^U]$,

$\pi_I(r)=[\pi_I(r)^L,\pi_I(r)^U]$, $\vartheta_I(x)=[\vartheta_I(r)^L,\vartheta_I(r)^U]$ and $\boldsymbol{A}=<r,\mu_A(x),\pi_A(r),\vartheta_A(r)>$, then

$$\textbf{Score function }\boldsymbol{\delta(\Phi)}=\frac{1}{2}\left\langle\begin{matrix}\frac{1}{6}\left(\sqrt{\left(\left(\mu_I(r)\right)^{2^L}+\left(\mu_I(r)\right)^{2^U}\right)+\left(\left(\pi_I(r)\right)^{2^L}+\left(\pi_I(r)\right)^{2^U}\right)+\left(1-\left(\vartheta_I(r)\right)^{2^L}+1-\left(\vartheta_I(r)\right)^{2^U}\right)}\right)\\ +\frac{1}{3}\left(\sqrt{\left(\mu_A(r)\right)^2+\left(\pi_A(r)\right)^2+1-\left(\vartheta_A(r)\right)^2}\right)\end{matrix}\right\rangle.$$

$$\textbf{Accuracy function }\aleph(\boldsymbol{\Phi})=\frac{1}{3}\{(\mu_I(r)^L+\mu_I(r)^U)-(\vartheta_I(r)^L+\vartheta_I(r)^U)+\mu_A(r)-\vartheta_A(r)\}.$$

$$\textbf{Certainty function }\beth(\boldsymbol{\Phi})=\frac{1}{3}\{(\mu_I(r)^L+\mu_I(r)^U)+\mu_A(r)\}.$$

Let $\boldsymbol{\Phi}=<\boldsymbol{I},\boldsymbol{A}>$, and $\boldsymbol{\Psi}=<\boldsymbol{J},\boldsymbol{B}>$, are two neutrosophic cubic spherical fuzzy sets (NCSFSs) , we say that

If $\boldsymbol{\delta(\Phi)>\delta(\Psi)}$ the $\boldsymbol{\Phi>\Psi}$.

If $\boldsymbol{\delta(\Phi) = \delta(\Psi)}$ and $\boldsymbol{\aleph(\Phi) > \aleph(\Psi)}$ then $\boldsymbol{\Phi > \Psi}$.

If $\boldsymbol{\delta(\Phi) = \delta(\Psi)}$, $\boldsymbol{\aleph(\Phi) = \aleph(\Psi)}$ and $\boldsymbol{\beth(\Phi) > \beth(\Psi)}$ then $\boldsymbol{\Phi > \Psi}$.

If $\boldsymbol{\delta(\Phi) = \delta(\Psi)}$, $\boldsymbol{\aleph(\Phi) = \aleph(\Psi)}$ and $\boldsymbol{\beth(\Phi) = \beth(\Psi)}$ then $\boldsymbol{\Phi = \Psi}$.

The value of score function ranges from 0 and 1, i.e., $\boldsymbol{\delta(\Phi) \in [0,1]}$, the value the accuracy function ranges from -1 and 1, i.e., $\boldsymbol{\aleph(\Phi) \in [-1,1]}$ and the value of the certainty function ranges from 0 and 1, i.e., $\boldsymbol{\beth(\Phi) \in [0,1]}$.

4. Exponential Operational Laws in NCSFSs

For NCSFSs $\Phi =< I, A >$ and a scalar $e > 0$, the exponential operation of Φ is expressed as:

$$e^{\Phi} = \begin{cases} \left\langle \begin{array}{c} r, \left\{\left[(e)^{\sqrt{1-(\mu_I(r)^L)^2}}, (e)^{\sqrt{1-(\mu_I(r)^U)^2}}\right]\right\}, \left\{\left[(e)^{\sqrt{1-(\pi_I(r)^L)^2}}, (e)^{\sqrt{1-(\pi_I(r)^U)^2}}\right]\right\}, \\ \left\{\left[\sqrt{1-(e)^{2(\vartheta_{I(r)}{}^L)}}, \sqrt{1-(e)^{2(\vartheta_{I(r)}{}^U)}}\right]\right\}, \left\{(e)^{\sqrt{1-(\mu_{A(r)})^2}}\right\}, \left\{(e)^{\sqrt{1-(\pi_A(r))^2}}\right\}, \left\{\sqrt{1-(e)^{2(\vartheta_{A(r)})}}\right\} \end{array} \right\rangle; e \in (0,1) \\ \left\langle \begin{array}{c} r, \left\{\left[\left(\frac{1}{e}\right)^{\sqrt{(\mu_I(r)^L)^2}}, \left(\frac{1}{e}\right)^{\sqrt{(\mu_I(r)^U)^2}}\right]\right\}, \left\{\left[\left(\frac{1}{e}\right)^{\sqrt{(\pi_I(r)^L)^2}}, \left(\frac{1}{e}\right)^{\sqrt{(\pi_I(r)^U)^2}}\right]\right\}, \\ \left\{\left[\sqrt{1-\left(\frac{1}{e}\right)^{2(\vartheta_I(r)^L)}}, \sqrt{1-\left(\frac{1}{e}\right)^{2(\vartheta_I(r)^U)}}\right]\right\}, \left\{\left(\frac{1}{e}\right)^{\sqrt{1-(\mu_{A(r)})^2}}\right\}, \left\{\left(\frac{1}{e}\right)^{\sqrt{1-(\pi_A(r))^2}}\right\}, \left\{\sqrt{1-\left(\frac{1}{e}\right)^{2(\vartheta_A(r))}}\right\} \end{array} \right\rangle; e \geq 1 \end{cases} \quad (1)$$

If $\Phi > \beta$, then $e^{\Phi} > e^{\Psi}$.

Theorem 1. Consider $\Phi =< I, A >$, $\Psi =< J, B >$ and $\Upsilon =< K, C >$ be three NCSFSs, a saclar $e, e_1 \in (0,1)$, and $\lambda > 0$, then

i. $e^{\Phi} \oplus e^{\Psi} = e^{\Psi} \oplus e^{\Phi}$
ii. $e^{\Phi} \otimes e^{\Psi} = e^{\Psi} \otimes e^{\Phi}$
iii. $(e^{\Phi} \oplus e^{\Psi}) \oplus e^{\Upsilon} = e^{\Phi} (\oplus e^{\Psi} \oplus e^{\Upsilon})$
iv. $(e^{\Phi} \otimes e^{\beta}) \otimes e^{\Upsilon} = e^{\Phi} \otimes (e^{\Psi} \otimes e^{\Upsilon})$
v. $\lambda(e^{\Phi} \oplus e^{\Psi}) = \lambda e^{\Phi} \oplus \lambda e^{\Psi}$
vi. $(e^{\Phi} \otimes e^{\Psi})^{\lambda} = (e^{\Phi})^{\lambda} \otimes (e^{\Psi})^{\lambda}$
vii. $e^{\Phi} \otimes e_1{}^{\Phi} = (e e_1)^{\Phi}$.

Proof.1. $e^{\Phi} \oplus e^{\Psi} =$

$$\left\langle \begin{array}{c} \left\{\left[(e)^{\sqrt{1-(\mu_I(r)^L)^2}}, (e)^{\sqrt{1-(\mu_I(r)^U)^2}}\right]\right\}, \left\{\left[(e)^{\sqrt{1-(\pi_I(r)^L)^2}}, (e)^{\sqrt{1-(\pi_I(r)^U)^2}}\right]\right\}, \\ \left\{\left[\sqrt{1-(e)^{2(\vartheta_{I(r)}{}^L)}}, \sqrt{1-(e)^{2(\vartheta_I(r)^U)}}\right]\right\}, \left\{(e)^{\sqrt{1-(\mu_{A(r)})^2}}\right\}, \left\{(e)^{\sqrt{1-(\pi_A(r))^2}}\right\}, \left\{\sqrt{1-(e)^{2(\vartheta_A(r))}}\right\} \end{array} \right\rangle$$

$$\oplus \left\langle \begin{array}{c} \left\{\left[(e)^{\sqrt{1-(\mu_J(r)^L)^2}}, (e)^{\sqrt{1-(\mu_J(r)^U)^2}}\right]\right\}, \left\{\left[(e)^{\sqrt{1-(\pi_J(r)^L)^2}}, (e)^{\sqrt{1-(\pi_J(r)^U)^2}}\right]\right\}, \\ \left\{\left[\sqrt{1-(e)^{2(\vartheta_{J(r)}{}^L)}}, \sqrt{1-(e)^{2(\vartheta_J(r)^U)}}\right]\right\}, \left\{(e)^{\sqrt{1-(\mu_{B(r)})^2}}\right\}, \left\{(e)^{\sqrt{1-(\pi_{B(r)})^2}}\right\}, \left\{\sqrt{1-(e)^{2(\vartheta_B(r))}}\right\} \end{array} \right\rangle$$

$$= \left\langle \left\{ \left[(e)^{\sqrt{1-(\mu_I(r)^L)^2}} + (e)^{\sqrt{1-(\mu_J(r)^L)^2}} - (e)^{\sqrt{1-(\mu_I(r)^L)^2}} (e)^{\sqrt{1-(\mu_J(r)^L)^2}}, \ (e)^{\sqrt{1-(\mu_I(r)^U)^2}} + (e)^{\sqrt{1-(\mu_J(r)^U)^2}} - \right.\right.$$

$$\left.\left. (e)^{\sqrt{1-(\mu_I(r)^U)^2}} (e)^{\sqrt{1-(\mu_J(r)^U)^2}} \right] \right\}, \left\{ \left[(e)^{\sqrt{1-(\pi_I(r)^L)^2}} + (e)^{\sqrt{1-(\pi_J(r)^L)^2}} - (e)^{\sqrt{1-(\pi_I(r)^L)^2}} (e)^{\sqrt{1-(\pi_J(r)^L)^2}}, \right.\right.$$

$$\left.\left. (e)^{\sqrt{1-(\pi_I(r)^U)^2}} + (e)^{\sqrt{1-(\pi_J(r)^U)^2}} - \right.\right.$$

$$\left.\left. (e)^{\sqrt{1-(\pi_I(r)^U)^2}} (e)^{\sqrt{1-(\pi_J(r)^U)^2}} \right] \right\}, \left\{ \left[\left(\sqrt{1-(e)^{2(\vartheta_I(r)^L)}} \right) \left(\sqrt{1-(e)^{2(\vartheta_J(r)^L)}} \right), \left(\sqrt{1-(e)^{2(\vartheta_I(r)^U)}} \right) \left(\sqrt{1-(e)^{2(\vartheta_J(r)^U)}} \right) \right] \right\},$$

$$\left\{ (e)^{\sqrt{1-(\mu_{A(r)})^2}} + (e)^{\sqrt{1-(\mu_{B(r)})^2}} - (e)^{\sqrt{1-(\mu_{A(r)})^2}} (e)^{\sqrt{1-(\mu_{B(r)})^2}} \right\}, \left\{ (e)^{\sqrt{1-(\pi_A(r))^2}} + (e)^{\sqrt{1-(\pi_{B(r)})^2}} - \right.$$

$$\left. (e)^{\sqrt{1-(\pi_A(r))^2}} (e)^{\sqrt{1-(\pi_{B(r)})^2}} \right\}, \left\{ \sqrt{1-(e)^{2(\vartheta_{A(r)})}} \sqrt{1-(e)^{2(\vartheta_{B(r)})}} \right\} \Bigg\rangle$$

$$= \left\langle \left\{ \left[(e)^{\sqrt{1-(\mu_J(r)^L)^2}} + (e)^{\sqrt{1-(\mu_I(r)^L)^2}} - (e)^{\sqrt{1-(\mu_J(r)^L)^2}} (e)^{\sqrt{1-(\mu_I(r)^L)^2}}, (e)^{\sqrt{1-(\mu_J(r)^U)^2}} + \ (e)^{\sqrt{1-(\mu_I(r)^U)^2}} - \right.\right.$$

$$\left.\left. (e)^{\sqrt{1-(\mu_J(r)^U)^2}} (e)^{\sqrt{1-(\mu_I(r)^U)^2}} \right] \right\}, \left\{ \left[\left(\sqrt{1-(e)^{2(\vartheta_{J(r)^L})}} \right) \left(\sqrt{1-(e)^{2(\vartheta_{I(r)^L})}} \right), \left(\sqrt{1-(e)^{2(\vartheta_J(r)^U)}} \right) \left(\sqrt{1-(e)^{2(\vartheta_{I(r)^U})}} \right) \right] \right\},$$

$$\left\{ (e)^{\sqrt{1-(\mu_{B(r)})^2}} + (e)^{\sqrt{1-(\mu_{A(r)})^2}} - (e)^{\sqrt{1-(\mu_B(r))^2}} (e)^{\sqrt{1-(\mu_A(r))^2}} \right\}, \left\{ (e)^{\sqrt{1-(\pi_{B(r)})^2}} + (e)^{\sqrt{1-(\pi_{A(r)})^2}} - \right.$$

$$\left. (e)^{\sqrt{1-(\pi_{B(r)})^2}} (e)^{\sqrt{1-(\pi_{A(r)})^2}} \right\}, \left\{ \sqrt{1-(e)^{2(\vartheta_{B(r)})}} \sqrt{1-(e)^{2(\vartheta_{A(r)})}} \right\} \Bigg\rangle = e^{\Psi} \oplus e^{\Phi}.$$

Similarly, the properties (ii), (iii) and (iv) can be easily proved with the help to defined basic operations on NCSFS.

Proof. v. $\lambda(e^{\Phi} \oplus e^{\Psi}) = \lambda \left\langle \left\{ \left[(e)^{\sqrt{1-(\mu_I(r)^L)^2}} + (e)^{\sqrt{1-(\mu_J(r)^L)^2}} - \right.\right.\right.$

$$\left.\left.\left. (e)^{\sqrt{1-(\mu_I(r)^L)^2}} (e)^{\sqrt{1-(\mu_J(r)^L)^2}}, \ (e)^{\sqrt{1-(\mu_I(r)^U)^2}} + \ (e)^{\sqrt{1-(\mu_J(r)^U)^2}} - \right.\right.$$

$$\left.\left.\left. (e)^{\sqrt{1-(\mu_I(r)^U)^2}} (e)^{\sqrt{1-(\mu_J(r)^U)^2}} \right] \right\}, \left\{ \left[(e)^{\sqrt{1-(\pi_I(r)^L)^2}} + (e)^{\sqrt{1-(\pi_J(r)^L)^2}} - (e)^{\sqrt{1-(\pi_I(r)^L)^2}} (e)^{\sqrt{1-(\pi_J(r)^L)^2}}, \right.\right.$$

$$\left.\left. (e)^{\sqrt{1-(\pi_I(r)^U)^2}} + (e)^{\sqrt{1-(\pi_J(r)^U)^2}} - \right.\right.$$

$$\left.\left. (e)^{\sqrt{1-(\pi_I(r)^U)^2}} (e)^{\sqrt{1-(\pi_J(r)^U)^2}} \right] \right\}, \left\{ \left[\left(\sqrt{1-(e)^{2(\vartheta_{I(r)^L})}} \right) \left(\sqrt{1-(e)^{2(\vartheta_{J(r)^L})}} \right), \left(\sqrt{1-(e)^{2(\vartheta_I(r)^U)}} \right) \left(\sqrt{1-(e)^{2(\vartheta_J(r)^U)}} \right) \right] \right\},$$

$$\left\{ (e)^{\sqrt{1-\mu_{A(r)}{}^2}} + (e)^{\sqrt{1-\mu_B(r)^2}} - (e)^{\sqrt{1-\mu_A(r)^2}} (e)^{\sqrt{1-\mu_{B(r)}{}^2}} \right\}, \left\{ (e)^{\sqrt{1-\pi_{A(r)}{}^2}} + (e)^{\sqrt{1-\pi_{B(r)}{}^2}} - \right.$$

$$\left. (e)^{\sqrt{1-\pi_{A(r)}{}^2}} (e)^{\sqrt{1-\pi_{B(r)}{}^2}} \right\}, \left\{ \sqrt{1-(e)^{2(\vartheta_{A(r)})}} \sqrt{1-(e)^{2(\vartheta_{B(r)})}} \right\} \Bigg\rangle$$

$$= \left\langle \left\{ \left[1 - \left(1 - \left((e)^{\sqrt{1-(\mu_I(r)^L)^2}} + (e)^{\sqrt{1-(\mu_J(r)^L)^2}} - (e)^{\sqrt{1-(\mu_I(r)^L)^2}} (e)^{\sqrt{1-(\mu_J(r)^L)^2}} \right) \right)^{\lambda}, 1 - \left(1 \right. \right. \right. \right.$$

$$\left. \left. \left. \left. - \left((e)^{\sqrt{1-(\mu_I(r)^U)^2}} + (e)^{\sqrt{1-(\mu_J(r)^U)^2}} - (e)^{\sqrt{1-(\mu_I(r)^U)^2}} (e)^{\sqrt{1-(\mu_J(r)^U)^2}} \right)^{\lambda} \right] \right\}, \left\{ \left[1 - \left(1 \right. \right. \right. \right.$$

$$\left. \left. \left. \left. - \left((e)^{\sqrt{1-(\pi_I(r)^L)^2}} + (e)^{\sqrt{1-(\pi_J(r)^L)^2}} - (e)^{\sqrt{1-(\pi_I(r)^L)^2}} (e)^{\sqrt{1-(\pi_J(r)^L)^2}} \right)^{\lambda}, 1 \right. \right. \right. \right.$$

$$\left. \left. \left. \left. - \left(1 - \left((e)^{\sqrt{1-(\pi_I(r)^U)^2}} + (e)^{\sqrt{1-(\pi_J(r)^U)^2}} - (e)^{\sqrt{1-(\pi_I(r)^U)^2}} (e)^{\sqrt{1-(\pi_J(r)^U)^2}} \right)^{\lambda} \right] \right\}, \left\{ \left[\left(\left(1 \right. \right. \right. \right. \right.$$

$$\left. \left. \left. \left. \left. - \sqrt{1-(e)^{2(\vartheta_{I(r)}{}^L)}} \right) \left(1 - \sqrt{1-(e)^{2(\vartheta_J(r)^L)}} \right) \right)^{\lambda}, \left(\left(1 - \sqrt{1-(e)^{2(\vartheta_I(r)^U)}} \right) \left(1 \right. \right. \right. \right. \right.$$

$$\left. \left. \left. \left. \left. - \sqrt{1-(e)^{2(\vartheta_J(r)^U)}} \right) \right)^{\lambda} \right] \right\}, \left\{ 1 \right.$$

$$\left. - \left(1 - \left((e)^{\sqrt{1-(\mu_{A(r)})^2}} + (e)^{\sqrt{1-(\mu_{B(r)})^2}} - (e)^{\sqrt{1-(\mu_A(r))^2}} (e)^{\sqrt{1-(\mu_{B(r)})^2}} \right) \right)^{\lambda} \right\}, \left\{ 1 - \left(1 \right. \right.$$

$$\left. \left. - \left((e)^{\sqrt{1-(\pi_A(r))^2}} + (e)^{\sqrt{1-(\pi_{B(r)})^2}} - (e)^{\sqrt{1-(\pi_A(r))^2}} (e)^{\sqrt{1-(\pi_{B(r)})^2}} \right)^{\lambda} \right\}, \left\{ \left(\left(1 - \sqrt{1-(e)^{2(\vartheta_{A(r)})}} \right) \left(1 \right. \right. \right.$$

$$\left. \left. \left. - \sqrt{1-(e)^{2(\vartheta_B(r))}} \right) \right)^{\lambda} \right\} \right\rangle.$$

$$\lambda e^{\Phi} \oplus \lambda e^{\Psi} = \left\langle \left\{ \left[1 - \left(1 - (e)^{\sqrt{1-(\mu_I(r)^L)^2}} \right)^{\lambda}, 1 - \left(1 - (e)^{\sqrt{1-(\mu_I(r)^U)^2}} \right)^{\lambda} \right] \right\}, \left\{ \left[1 - \left(1 - \right. \right. \right. \right.$$

$$\left. \left. \left. \left. (e)^{\sqrt{1-(\pi_I(r)^L)^2}} \right)^{\lambda}, 1 - \left(1 - (e)^{\sqrt{1-(\pi_I(r)^U)^2}} \right)^{\lambda} \right] \right\}, \left\{ \left[\left(1 - \sqrt{1-(e)^{2(\vartheta_I(r)^L)}} \right)^{\lambda}, \left(1 - \right. \right. \right. \right.$$

$$\left. \left. \left. \left. \sqrt{1-(e)^{2(\vartheta_I(r)^U)}} \right)^{\lambda} \right] \right\}, \left\{ 1 - \left(1 - (e)^{\sqrt{1-(\mu_{A(r)})^2}} \right)^{\lambda} \right\}, \left\{ 1 - \left(1 - (e)^{\sqrt{1-(\pi_A(r))^2}} \right)^{\lambda} \right\}, \left\{ (1 - \right.$$

$$\left. \sqrt{1-(e)^{2(\vartheta_A(r))}} \right)^{\lambda} \right\} \left. \right\rangle \oplus \left\langle \left\{ \left[1 - \left(1 - (e)^{\sqrt{1-(\mu_J(r)^L)^2}} \right)^{\lambda}, 1 - \left(1 - (e)^{\sqrt{1-(\mu_J(r)^U)^2}} \right)^{\lambda} \right] \right\}, \left\{ \left[1 - \right. \right. \right.$$

$$\left. \left. \left. \left(1 - (e)^{\sqrt{1-(\pi_J(r)^L)^2}} \right)^{\lambda}, 1 - \left(1 - (e)^{\sqrt{1-(\pi_J(r)^U)^2}} \right)^{\lambda} \right] \right\}, \left\{ \left[\left(1 - \sqrt{1-(e)^{2(\vartheta_J(r)^L)}} \right)^{\lambda}, \left(1 - \right. \right. \right.$$

$$\left. \left. \left. \sqrt{1-(e)^{2(\vartheta_J(r)^U)}} \right)^{\lambda} \right] \right\}, \left\{ 1 - \left(1 - (e)^{\sqrt{1-(\mu_{B(r)})^2}} \right)^{\lambda} \right\}, \left\{ 1 - \left(1 - (e)^{\sqrt{1-(\pi_{B(r)})^2}} \right)^{\lambda} \right\}, \left\{ \left(1 - \right. \right.$$

$$\left. \left. \sqrt{1-(e)^{2(\vartheta_{B(r)})}} \right)^{\lambda} \right\} \left. \right\rangle$$

$$= \left\langle \left\{ \left[1 - \left(1 - \left((e)^{\sqrt{1-(\mu_I(r)^L)^2}} + (e)^{\sqrt{1-(\mu_J(r)^L)^2}} - (e)^{\sqrt{1-(\mu_I(r)^L)^2}} (e)^{\sqrt{1-(\mu_J(r)^L)^2}} \right) \right)^{\lambda}, 1 - \left(1 \right. \right. \right. \right.$$

$$\left. \left. \left. \left. - \left((e)^{\sqrt{1-(\mu_I(r)^U)^2}} + (e)^{\sqrt{1-(\mu_J(r)^U)^2}} - (e)^{\sqrt{1-(\mu_I(r)^U)^2}} (e)^{\sqrt{1-(\mu_J(r)^U)^2}} \right) \right)^{\lambda} \right] \right\}, \left\{ \left[1 - \left(1 \right. \right. \right. \right.$$

$$\left. \left. \left. \left. - \left((e)^{\sqrt{1-(\pi_I(r)^L)^2}} + (e)^{\sqrt{1-(\pi_J(r)^L)^2}} - (e)^{\sqrt{1-(\pi_I(r)^L)^2}} (e)^{\sqrt{1-(\pi_J(r)^L)^2}} \right) \right)^{\lambda}, 1 \right. \right. \right. \right.$$

$$\left. \left. \left. \left. - \left(1 - \left((e)^{\sqrt{1-(\pi_I(r)^U)^2}} + (e)^{\sqrt{1-(\pi_J(r)^U)^2}} - (e)^{\sqrt{1-(\pi_I(r)^U)^2}} (e)^{\sqrt{1-(\pi_J(r)^U)^2}} \right) \right)^{\lambda} \right] \right\}, \left\{ s \left[\left(\left(1 \right. \right. \right. \right. \right.$$

$$\left. \left. \left. \left. \left. - \sqrt{1-(e)^{2(\vartheta_I(r)^L)}} \right) \left(1 - \sqrt{1-(e)^{2(\vartheta_J(r)^L)}} \right) \right)^{\lambda}, \left(\left(1 - \sqrt{1-(e)^{2(\vartheta_I(r)^U)}} \right) \left(1 - \sqrt{1-(e)^{2(\vartheta_J(r)^U)}} \right) \right)^{\lambda} \right] \right\}, \left\{ 1 \right. \right. \right. \right. \right.$$

$$\left. \left. \left. \left. \left. - \left(1 - \left((e)^{\sqrt{1-(\mu_{A(r)})^2}} + (e)^{\sqrt{1-(\mu_{B(r)})^2}} - (e)^{\sqrt{1-(\mu_{A(r)})^2}} (e)^{\sqrt{1-(\mu_{B(r)})^2}} \right) \right)^{\lambda} \right\}, \left\{ 1 - \left(1 \right. \right. \right. \right. \right.$$

$$\left. \left. \left. \left. \left. - \left((e)^{\sqrt{1-(\pi_{A(r)})^2}} + (e)^{\sqrt{1-(\pi_{B(r)})^2}} - (e)^{\sqrt{1-(\pi_{A(r)})^2}} (e)^{\sqrt{1-(\pi_{B(r)})^2}} \right) \right)^{\lambda} \right\}, \left\{ \left(\left(1 - \sqrt{1-(e)^{2(\vartheta_{A(r)})}} \right) \left(1 \right. \right. \right. \right. \right.$$

$$\left. \left. \left. \left. \left. \left. - \sqrt{1-(e)^{2(\vartheta_B(r))}} \right) \right)^{\lambda} \right\} \right\rangle.$$

From equation (3) and (4), we get $\lambda(e^{\Phi} \oplus e^{\Psi}) = \lambda e^{\Phi} \oplus \lambda e^{\Psi}$.

Similarly, the properties (vi) and (vii) can be easily proved with the help of defined basic operations on NCSFSs.

5. Exponential Aggregation Operators for NCSFSs

Definition 7. Consider $\{\Phi_j = \langle I_j, A_j \rangle\}$ be a accumulation of NCSFSs with $e_j \in (0,1)$ be the real number. The neutrosophic cubic spherical fuzzy weighted exponential aggregation (NCSFWEA) operator is expressed as follows:

$$\text{NCSFWEA}(\Phi_1, \Phi_2 \cdots \Phi_n) = \bigotimes_{j=1}^{n} (e_j)^{\Phi_j}; \tag{5}$$

where $\{\Phi_j = \langle I_j, A_j \rangle\}$ are the exponential weighted vector of attribute values $e_j \in (0,1)$.

Theorem 2. Suppose $\{\Phi_j = \langle I_j, A_j \rangle\}$ be a accumulation of NCSFSs and $e_j \in (0,1)$ be the real number, the NCSF weighted exponential aggregation operator is

$$\text{NCSFWEA}(\Phi_1, \Phi_2 \cdots \Phi_n) =$$

$$\left\langle \begin{array}{c} \left\{ \left[\prod_{j=1}^{n} (e_j)^{\sqrt{1-(\mu_{I(j)}{}^L)^2}}, \prod_{j=1}^{n} (e_j)^{\sqrt{1-(\mu_{I(j)}{}^U)^2}} \right] \right\}, \left\{ \left[\prod_{j=1}^{n} (e_j)^{\sqrt{1-(\pi_{I(j)}{}^L)^2}}, \prod_{j=1}^{n} (e_j)^{\sqrt{1-(\pi_{I(j)}{}^U)^2}} \right] \right\}, \\ \left\{ \left[\sqrt{1-\prod_{j=1}^{n} (e)^{2(\vartheta_{I(j)}{}^L)}}, \sqrt{1-\prod_{j=1}^{n} (e)^{2(\vartheta_I{}^U)}} \right] \right\}, \left\{ \prod_{j=1}^{n} (e)^{\sqrt{1-\mu_{A(j)}{}^2}} \right\}, \left\{ \prod_{j=1}^{n} (e)^{\sqrt{1-\pi_{A(j)}{}^2}} \right\}, \left\{ \sqrt{1-\prod_{j=1}^{n} (e)^{2(\vartheta_{A(j)})}} \right\} \end{array} \right\rangle; \tag{6}$$

where, $\{\Phi_j = \langle I_j, A_j \rangle\}$ are the exponential weighted vector of attributes value $e_j \in (0,1)$. Moreover $\text{NCSFWEA}(\Phi_1, \Phi_2 \cdots \Phi_n)$ is also a NCSFSs.

Proof. Theorem proof is based on the mathematical induction principle.

6. Methodology for MCDM problem under NCSFSs

This section presents a methodology for solving the MCDM problem based on proposed exponential aggregation operators. The process is then demonstrated using a numerical example. The steps listed below are:

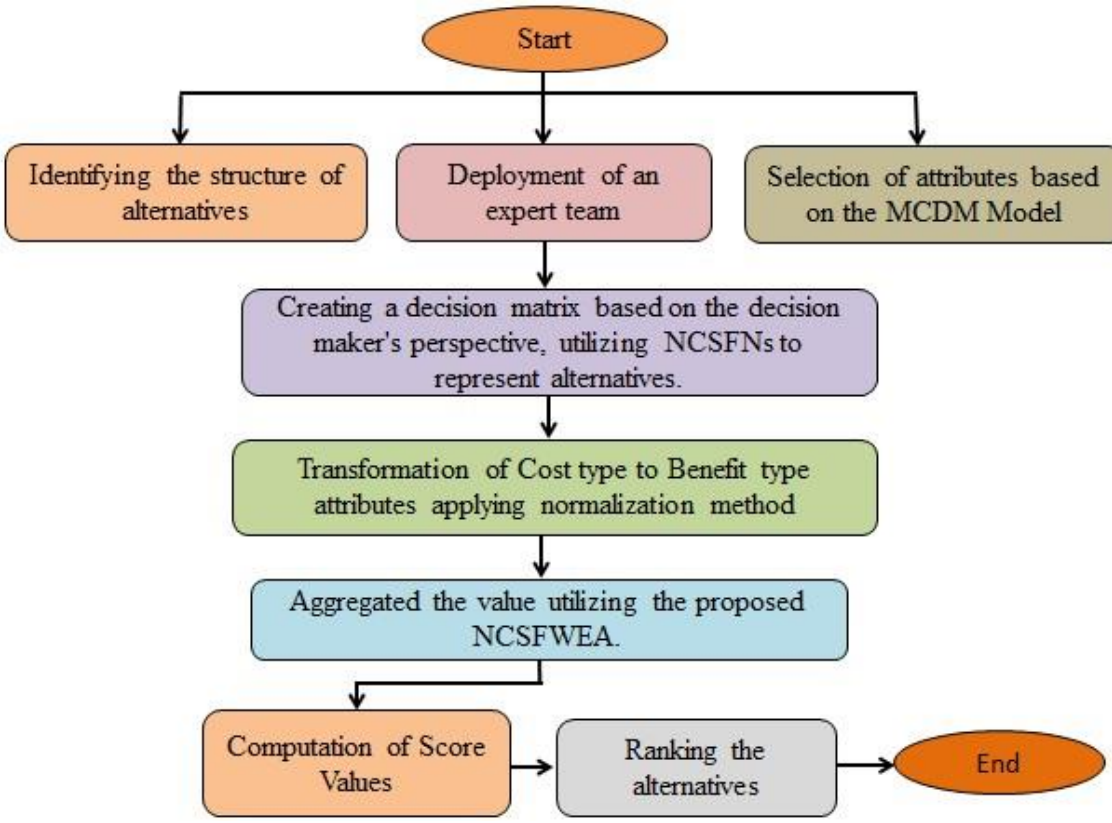
Start
Identifying the structure of alternatives
Deployment of an expert team
Selection of attributes based on the MCDM Model
Creating a decision matrix based on the decision maker's perspective, utilizing NCSFNs to represent alternatives.
Transformation of Cost type to Benefit type attributes applying normalization method
Aggregated the value utilizing the proposed NCSFWEA.
Computation of Score Values
Ranking the alternatives
End

Fig 2: Methodology procedure for addressing MCDM problem under NCSFSs.

Step1. Consider an issue of decision-making challenge with n alternatives, i.e. $\Im_1, \Im_2, \cdots \Im_n$ and m attributes, i.e. $E_1, E_2, \cdots E_n$. The NCSFSs Φ_j is used as a weight vector to the attributes E_j $(j = 1,2, \cdots m)$. Then experts evaluate the presented alternatives according to a set of attributes and give their choice of values based on the spherical fuzzy information indicated by $e_{ij} < i = 1,2, \cdots n >$ and $< j = 1,2, \cdots n >$ and $0 \leq e_{ij} \leq 1$. Next, depending on the choice values, a decision matrix $D = (e_{ij})$, $< i = 1,2, \cdots n >$ and $< j = 1,2, \cdots n >$ is presented, where e_{ij} indicates the choice of alternatives $\Im_i$ that correspond to attribute E_j.

$$D = \begin{bmatrix} e_{11} & e_{12} & \cdots & e_{1m} \\ e_{21} & e_{22} & \ldots & e_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ e_{n1} & e_{n2} & \cdots & e_{nm} \end{bmatrix}$$

Step 2. The attributes are usually divided into two groups: benefit type and cost type. If the MCDM attributes are of the same kind, choice values do not require normalization. If the MCDM attributes

are of distinct kinds, then the following formula can be utilized to convey choice values from the benefit type to the cost type

$$e_{ij} = \begin{cases} e_{ij} \ ; & j \in \text{ benifit type} \\ {e_{ij}}^c; & j \in \text{ cost type} \end{cases}.$$

Step 3. Now aggregated the value by using the proposed neutrosophic cubic spherical fuzzy weighted exponential aggregation operators (NCSFWEA) for each alternative $\mathfrak{J}_1, \mathfrak{J}_2, \cdots \mathfrak{J}_n$.

Step 4. We compute the scoring value $\delta(\varepsilon_i)$; $i = 1,2, \cdots n$ for each alternative $\mathfrak{J}_i; i = 1,2, \cdots n$ by utilizing the scoring function formula.

Step 5. And at last, ranking the alternative with the most favorable score value. If the score values of the two different options are identical, apply the accuracy function formula for ranking; otherwise, use the certainty function formula.

Selection of solid waste disposal sites under the proposed NCSFWEA with numerical illustration. We implement the suggested NCSFWEA to a waste disposal site selection task that has five alternatives i.e. sites $\mathfrak{J}_1, \mathfrak{J}_2, \mathfrak{J}_3, \mathfrak{J}_4, \mathfrak{J}_5$ and four attributes E_1 = Distance to residential areas; E_2 = savings in cost; E_3 = ground depth; E_4 =slope . Expert opinions are expressed in the structure of a decision matrix $\mathrm{D} = (\mathrm{e_{ij}})_{5\times 3}$, whose elements show the choice of alternative $\mathfrak{J}_i$ (i $= 1, \ldots, 5$) over the attribute E_j (j $= 1, 2, 3, 4$)

$$D = \begin{bmatrix} 0.3 & 0.4 & 0.2 & 0.5 \\ 0.3 & 0.6 & 0.7 & 0.4 \\ 0.8 & 0.2 & 0.2 & 0.3 \\ 0.7 & 0.2 & 0.2 & 0.4 \\ 0.8 & 0.4 & 0.5 & 0.6 \end{bmatrix}.$$

For each attribute, the exponential neutrosophic cubic spherical fuzzy weights are given as follows:

$\omega_1 = \langle\{[0.4, 0.8]\}, \{[0.2, 0.5]\}, \{[0.5,0.9]\}, \{(0.9,0.8,0.3)\}\rangle$,

$\omega_2 = \langle\{[0.2, 0.6]\}, \{[0.3, 0.7]\}, \{[0.4,0.8]\}, \{(0.7,0.5,0.2)\}\rangle$,

$\omega_3 = \langle\{[0.4, 0.6]\}, \{[0.2, 0.4]\}, \{[0.5,0.9]\}, \{(0.8,0.9,0.1)\}\rangle$,

$\omega_4 = \langle\{[0.3, 0.5]\}, \{[0.4, 0.7]\}, \{[0.2,0.6]\}, \{(0.5,0.7,0.3)\}\rangle$.

The following results are obtained by aggregating the various choice values of each aspect using the aggregation operator NCSFWEA given in equation (6):

$$\varepsilon_1 = \left\langle \begin{matrix} \{[0.015962,0.035322]\}, \{[0.014039,0.02555]\}, \{[0.989016,0.999682]\}, \\ \{(0.064241,0.066373,0.915998)\} \end{matrix} \right\rangle$$

$$\varepsilon_2 = \left\langle \begin{matrix} \{[0.06051,0.10971]\}, \{[0.057486,0.091746]\}, \{[0.950405,0.995559]\}, \\ \{(0.14997,0.138816,0.887293)\} \end{matrix} \right\rangle$$

$$\varepsilon_3 = \left\langle \begin{matrix} \{[0.012216,0.023479]\}, \{[0.011863,0.023286]\}, \{[0.986267,0.999668]\}, \\ \{(0.038582,0.045545,0.91558)\} \end{matrix} \right\rangle$$

$$\varepsilon_4 = \left\langle \begin{matrix} \{[0.014222,0.027802]\}, \{[0.013548,0.027662]\}, \{[0.986252,0.999632]\}, \\ \{(0.046699,0.051627,0.906982)\} \end{matrix} \right\rangle$$

$$\varepsilon_5 = \left\langle \begin{matrix} \{[0.108082,0.155078]\}, \{[0.106454,0.1576]\}, \{[0.918333,0.98791]\}, \\ \{(0.199907,0.203039,0.782001)\} \end{matrix} \right\rangle.$$

Now, use the scoring function to calculate the scoring value; $\delta(\varepsilon_1) = 0.063832$, $\delta(\varepsilon_2) = 0.095898$, $\delta(\varepsilon_3) = 0.063736$, $\delta(\varepsilon_4) = 0.066565$, $\delta(\varepsilon_5) = 0.133046$.

Lastly, depending on the outcomes of the score function, we rank the alternatives with the highest value

$$\delta(\varepsilon_5) > \delta(\varepsilon_2) > \delta(\varepsilon_4) > \delta(\varepsilon_1) > \delta(\varepsilon_3).$$

As a result, alternative $\delta(\varepsilon_5)$ is the most suitable.

7. Results & Discussions

- Finally, we can state that the suggested neutrosophic cubic spherical fuzzy set is a new and supportable combination of SNS and IVSNS. By enhancing the area of uncertainty compoment this gives more felxiblity to experts to give their opinion without any restiction.
- In the existing theory of neutrosophic sets, including the SVNSS [13], SVNHFS [23] and IVNHS [24] have limitations due to the restriction on the constraint.
- Exponential aggregation operators under NCSFSs have shown effectiveness in addressing industrial issues, reducing infectious diseases, and disposing of garbage properly. This improves disposal efficiency and decreases contamination risks by enhancing waste management systems.
- As a result of developing NCSFS and their aggregation operator, this research contributes significantly to the theoretical advancement of fuzzy set theory.
- NCSFS and its exponential aggregation operators need complex mathematical formulations and computations, resulting in significant computational draws in large-scale decision-making. Future studies should concentrate on creating algorithms or approximation techniques that are more effective to minimize computational time and resource use.
- The performance of the proposed aggregate operators may be sensitive to parameter selection. A detailed sensitivity analysis will be required to understand this impact and establish guidelines for practical applications.
- The study focuses primarily on the mathematical and theoretical aspects of NCSFS. Integrating these concepts with other interdisciplinary approaches such as machine learning, artificial intelligence, and big data analytics may enhance their applicability and effectiveness in complex decision-making scenarios.

8. Conclusions & Scope for Future Work

This article introduces a new notion of neutrosophic cubic spherical fuzzy sets (NCSFSs), which completely combines SNSs and IVSNSs using fundamental algebraic operations. The development of exponential operational laws under NCSFSs provided vital information, which has been subsequently expanded to establish exponential aggregation operators with distinguishing characteristics. Using these advancements, a methodology based on aggregation operators has been developed to solve Multiple Criteria Decision Making (MCDM) problems in a neutrosophic spherical environment while considering expert opinions. Finally, the proposed theory's practical application has been demonstrated by its effective use in determining the best site for solid waste disposal, as illustrated by a numerical example. This thorough examination demonstrates the flexibility and

suitability of neutrosophic cubic spherical fuzzy sets in addressing actual-world decision-making obstacles. Future developments of this theory could include hybrid weighted aggregating operators, which are helpful in multiple types of decision-making processes, including TOPSIS, VIKOR, WASPAS, and AHP, among others. It could also be applied to other real-world decision-making problems using the newly developed operators.

List of Symbols and Notations Used in this paper

Symbols\Notations	Description
$\mathfrak{N}$	Universal set
$\in$	Belongs to
∇	Spherical Fuzzy set
$\mu(r)$	Membership degree
$\pi(r)$	Indeterminancy/hesitation degree
$\nu(r)$	Non-membership degree
$\mu(r)^U$	Upper membership degree
$\pi(r)^U$	Upper indeterminancy degree
$\nu(r)^U$	Upper non-membership degree
I'	Interval-valued spherical fuzzy set
$\mathcal{N}$	Neutrosophic set
A	Spherical neutrosophic set
I	Interval-valued neutrosophic spherical fuzzy set
NCSFSs	Neutrosophic cubic spherical fuzzy sets
SNS	Spherical neutrosophic set
IVNSFS	Interval-valued neutrosophic spherical fuzzy set
MCDM	Multi-criteria decision making
PFNs	Pythagorean fuzzy numbers
SWM	Solid waste management
FS	Fuzzy set
IVFS	Interval-valued fuzzy set
IFS	Intuitionistic fuzzy set
PFS	Pythagorean fuzzy set
NS	Neutrosophic set
SS	Soft set
SVNS	Single-valued neutrosophic set
HFS	Hesitant fuzzy set
PFS	Picture fuzzy set
OPFS	Ortho-pair fuzzy set
SFS	Spherical fuzzy set
SVNHFS	Single-valued neutrosophic hesitant fuzzy set
INS	Interval neutrosophic set

HSS	Hypersoft set
PSVNHF	Probabilistic single-valued neutrosophic hesitant fuzzy
PINHFWA	Probabilistic interval neutrosophic hesitant fuzzy weighted averaging
PINHFWG	Probabilistic interval neutrosophic hesitant fuzzy weighted geometric
NHFS	Neutrosophic hesitant fuzzy set
LCA	Life cycle assessment
TSFHSSs	*T*-sperical fuzzy hypersoft sets
n-HSNMs	*n*-hyperspherical neutrosophic matrices
EL	Exponential-logarithm
NCSFWEAO	Neutrosophic cubic spherical fuzzy weighted exponential aggregation operator

Source of Funding:

This work is supported by University Grant Commission (UGC), New Delhi under the scheme ID: UGCES-22-GE-HIM-F-SJSGC-14036.

Conflict of interest: The authors affirm no conflicts of interest.

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Received: March 3, 2024. Accepted: July 28, 2024

Neutrosophic Sets and Systems, Vol. 70, 2024
University of New Mexico

TrF-BWM-Neutrosophic-TOPSIS Strategy under SVNS Environment Approach and Its Application to Select the Most Effective Water Quality Parameter of Aquaponic System

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Abstract: The rapid emergence of aquaponics presents a promising solution for food production in arid regions, heavily reliant on maintaining optimal water quality parameters for overall success. However, monitoring and controlling these parameters can be complex and costly, involving numerous sensors and actuators. Focusing on the most significant parameter for fish growth could alleviate some of this complexity and promote sustainability by ensuring a balanced system. This study introduces a novel fuzzy-based Multiple Criteria Decision Making (MCDM) methodology, combining Trapezoidal Fuzzy with the Best Worst Method (BWM) and Neutrosophic-TOPSIS strategy, to identify the primary water quality parameters impacting fish growth. Through this innovative approach, essential indicators are identified to conduct efficiency analyses. Moreover, the study not only advances scientific understanding but also offers practical guidance for farmers and aquaponic enthusiasts, aiming to foster sustainability and effectiveness in aquaponic effective.

Keywords: BWM, TOPSIS, Neutrosophic set, Aquaponic system, Water quality parameter.

1. Introduction

A major global concern the world is facing is to fulfill the nutritional requirements of an expanding human population, which is expected to reach 10 billion by the year 2050 [1]. The global pandemic, coupled with conflict and climate crisis-related weather patterns has delayed the first Millennium Development Goal which is "eradicate extreme poverty and hunger" [2]. The State of Food Security and Nutrition in the World (SOFI) report estimates that since 2019, an additional 122 million people have experienced hunger. The number of people who are hungry worldwide has plateaued between 2021 and 2022, but there are still numerous places where food crises are getting worse, which has prompted demands for an international effort to address the underlying causes of food insecurity [3]. To meet the additional food demands imposed by the almost 30% population expansion, global food production must rise by as much as 50% [4]. However, the rate of food production has been drastically lowered as a result of natural disasters, climate change, land grabs, declining soil quality, rapid urbanization, unfair trade rules, and other issues [5]. As per the

predictions made by [6], current trends in food production will not be able to meet the projected global food demand by 2050, even with the development of high-yielding crop varieties and improved food production techniques. According to [7], by the end of the twenty-first century, climate change alone may be responsible for up to 18% of losses of arable land, which will have a severe impact on the regions that are already facing food insecurity situation. The challenges associated with food production are driving up the demand for innovative practices, systems, and methods in the food production industry.

As a practical response to the food and environmental challenges, aquaponics farming is becoming more widely acknowledged as a way to increase food production rapidly without compromising the environment. Aquaponics is an environment-friendly sustainable method of food production that utilizes the concepts of circular economy and a biological natural system to maximize food production with minimized input and waste. Aquaponics is an integrated approach that uses two fundamental production systems: Aquaculture and Hydroponics. Aquaculture is the breeding of aquatic animals primarily Fish and hydroponics is the cultivation of plants without the use of soil (see **Figure 1**) [8]. In an aquaponic system, the nutrient-rich waste from aquatic animals is transformed into organic fertilizers through microbial activity steered by microorganisms, whereas the hydroponics plants detoxify the water by extracting the nutrients and then allow the fresh, oxygenated water in the fish tank to be recycled [9]. To put it briefly, aquaponics can be defined as an ecosystem in which fish, plants, and microbes coexist as symbionts and contribute to the production of sustainable foods. To support the vital bacteria involved in the nutrient cycle and to maintain the system's integrity, the aquaponic system does not allow any chemical additives and is free of antibiotics. As a result, crops are naturally healthy and essentially grown organically [10]. Being a closed circular system, aquaponics improves labor efficiency and can lead to sustainable output growth, which would boost food security and agricultural profitability [11]. In comparison with the conventional framing, the reduced land consumption and water use, make aquaponics a feasible food production solution for the regions with arid environments and help to develop the economic status with its rapid production. Thus, aquaponics can be seen as an innovative, low-carbon farming technique that is an intense, sustainable, circular, and highly productive farming method [12].

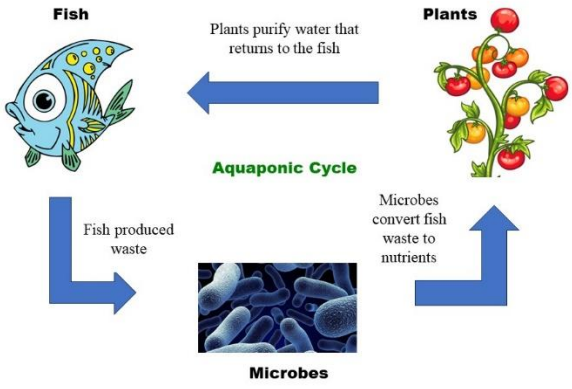

Figure 1: The aquaponic cycle

The aquaponic system operates on a closed-loop water medium, hosting fish, microorganisms, and plants. The physical, chemical, and biological features of the recirculating water play a vital role in the survival of all three living components of aquaponics. Thus, it's crucial to uphold optimal water quality parameters to ensure each component thrives independently. Fish growth rate in aquaponics is a significant trait with implications for ecology, evolution, and conservation. Key water quality parameters affecting fish growth include temperature (T), Dissolved Oxygen (DO) levels, water pH, ammonium concentration, nitrate levels, and many more [13]. Temperature directly influences the metabolic rate of fish which also has an impact on their energy balance and behavior of the body. Additionally, temperature impacts factors such as fish appetite, digestion, energy production, and nutrient absorption within the gastrointestinal tract [14]. According to findings from [15], inadequate water temperature can lead to significant fungal infections in both juvenile and adult fish populations, potentially resulting in the decay of eggs and larvae. Another crucial aspect of recirculating water quality is the level of DO. Fish rely on an adequate concentration of dissolved oxygen in the water for respiration, essential for their survival. The pH level of water also impacts the growth rate of fish, with slightly acidic environments potentially diminishing fish reproduction rates [16]. Ammonia serves as a significant parameter in aquaponics systems too. Even small amounts of ammonia can be highly toxic to fish, and its presence increases significantly in water solutions that become strongly acidic or alkaline [17]. The nitrate level of the water also plays a role in contributing to fish growth. High concentrations of nitrate can be particularly detrimental to fry and juvenile fish, significantly impairing their growth [18]. However, due to system dynamics and environmental factors, these parameters often deviate from ideal values, directly leading to stress, disease, and even death in fish and ultimately affecting the productivity and sustainability of the aquaponics system as a whole [19]. Additionally, the water quality parameters are interconnected to each other. Temperature influences the solubility of oxygen in water. Warmer water holds less dissolved oxygen compared to colder water. This means that as temperature increases, the amount of oxygen available for fish decreases [20]. pH plays a role in the toxicity of ammonia in water. Ammonia exists in two forms: un-ionized (NH_3) and ionized (NH_4^+). The proportion of these forms is influenced by pH, with un-ionized ammonia being more toxic to fish. In alkaline conditions (higher pH), more ammonia exists in its toxic form, while in acidic conditions (lower pH), more ammonia is present in its less harmful ionized form [17]. Understanding these interrelationships underscores the importance of holistic system management in aquaponics. The reasons mentioned above clearly indicate that maintaining the optimal values for water quality parameters in an aquaponics system is essential for promoting optimal fish growth by ensuring a stable, well-balanced environment that meets the physiological needs of the fish and supports nutrient cycling for both fish and plant health.

To maintain optimal water quality parameters, artificial control and monitoring systems are frequently deployed in aquaponics setups. These systems employ a network of sensors to gauge various water quality metrics [16]. Actuators are then employed to make necessary adjustments based on the sensor data. For instance, pH levels might require modulation through the addition of acids or bases [21], while oxygen levels can be regulated using aerators or pumps [22]. However, implementing such systems can be intricate due to the multitude of parameters needing supervision

and control. This complexity stems from the necessity for numerous sensors and actuators, as well as the integration of these components into a unified system. The hardware complexity presents several challenges. Procuring and installing a large array of sensors and actuators can be costly, particularly for larger aquaponics systems. Additionally, the ongoing expenses associated with maintaining and replacing these components further contribute to the financial burden. The setup and calibration of multiple sensors and actuators can be time-intensive, demanding meticulous adjustment and testing to ensure precision and dependability. Furthermore, the involvement of numerous hardware components necessitates ongoing maintenance to uphold sensor calibration and actuator functionality, potentially increasing the workload for system operators and risking downtime in case of component failures. Lastly, the environmental ramifications of producing and disposing of substantial quantities of sensors and actuators, along with the energy consumption required to power them, raise sustainability concerns. Selecting the most impactful water quality parameter for fish growth could serve as a solution to address these challenges and render aquaponics systems more sustainable. Determining the most influential parameter can lead to a reduction in the number of sensors, thereby simplifying the system, saving time, and lowering costs.

1.1 Literature review

In recent years, various control mechanisms for small-scale aquaponic systems have been proposed, leveraging strategies such as ON-OFF, PID (Proportional Integral Derivative), MPC (Model Predictive Control), and CAE (Controlled Environment Agriculture). Vernandhes et al. (2017) [23] introduced a smart aquaponics monitoring and control system utilizing a sensor network for water quality parameters, managed through a microcontroller. Dutta et al. (2018) [24] and Zamora-Izquierdo et al. (2019) [25] integrated Internet of Things (IoT) technology and sensor networks to regulate water quality parameters. Khaoula et al. (2021) [26] implemented an IoT-based solution for monitoring and controlling water quality and environmental parameters using sensors for water level, temperature, and CO_2, along with actuators. Li et al. (2022) [27] applied PID control to manage dissolved oxygen concentration in aquaponics recirculating water, and Wang et al. (2023) [28] proposed a Controlled Environment Agriculture (CEA) system discussing aspects related to aquaculture, aeroponics, and aquaponics. Channa et al. (2024) [29] explored the integration of Artificial Intelligence and IoT in a smart aquaponics system to monitor and control essential parameters. These studies primarily concentrate on control mechanisms tailored for small-scale aquaponic setups. Some literature also touches upon machine learning methods for yield estimation in aquaponic systems. Debroy and Seban (2022a) [30] and Debroy and Seban (2022a) [31] proposed two prediction methods for fish weight estimation using Artificial Neural Network (ANN) and its hybrid with fuzzy logic (ANFIS), as well as ANN models for predicting tomato biomass in aquaponic systems respectively. Eneh et al. (2023) [32] presented a yield prediction method for aquaponic systems employing various machine learning algorithms. A study on IoT integrated Machine learning based Indoor aquaponics farming is discussed by Rajendiran et al. (2024) [33].

1.2 Research gap

Previous research has predominantly focused on monitoring small-scale smart aquaponics systems, with some studies incorporating IoT-based machine learning methods for predicting yields and controlling system parameters. It's evident from existing literature that the integration of aquaculture and hydroponics in aquaponics systems significantly influences the dependence of fish growth and yield on water quality parameters. However, the complexity of implementing control and monitoring units to maintain optimal values for all these parameters presents challenges, including high costs, time consumption, and maintenance requirements, which can render the system unsustainable. A notable gap in the literature is the absence of a defined methodology for identifying the most significant water quality parameter affecting fish growth. This knowledge gap is crucial as it could help reduce system complexity while maximizing aquaponic system production. Therefore, the key research question arises: **Which water quality parameter holds the most significance for fish growth in aquaponic systems?**

Addressing this question is essential for optimizing aquaponic systems, enhancing sustainability, and improving overall efficiency in food production. By identifying the primary driver of fish growth, aquaponics practitioners can streamline their efforts and resources towards maintaining and optimizing that specific parameter. This not only reduces system complexity but also has the potential to increase fish yields, improve economic returns, and promote environmental sustainability in aquaponic food production.

1.3 Objective of the study

The study aims to explore the correlation between water quality parameters and fish production within aquaponics systems, with the primary objective of identifying the key factor influencing fish growth. Aquaponics is a sustainable farming technique that integrates aquaculture (fish farming) with hydroponics (soilless plant cultivation), creating a symbiotic ecosystem where fish waste serves as a nutrient source for plants, and the plants, in turn, purify the water for the fish. Understanding the water quality parameters is crucial in aquaponics because they directly influence the health and growth of both fish and plants. However, these parameters can fluctuate from their optimal values due to external disturbances. Additionally, these parameters are interrelated and can affect each other, so to maintain the right balance among them is essential. While using actuators to monitor and control these parameters is a potential solution, individually assigning actuators for each parameter would increase system complexity, duration, and costs. Hence, the study seeks to identify the most significant water quality parameter affecting fish growth within aquaponics systems. This involves statistical analyses and modeling techniques to determine which parameter has the strongest correlation or causative relationship with fish productivity. Understanding the primary driver of fish growth allows aquaponics practitioners to prioritize their efforts and resources towards maintaining and optimizing that particular parameter.

The findings of this study will not only contribute to the scientific understanding of aquaponics systems but also offer practical guidance to farmers and enthusiasts. Armed with knowledge of the most critical water quality parameters and their effects, farmers can refine their system management practices to achieve higher fish yields, improved economic returns, and enhanced sustainability in aquaponic food production. Ultimately, the research endeavors to facilitate the development of

efficient and environmentally friendly aquaponics systems capable of meeting the increasing demand for nutritious, locally produced seafood.

This study aims to pioneer the creation of a hybrid Multi-Criteria Decision-Making (MCDM) model designed to identify the most crucial water quality parameter influencing fish growth in aquaponic systems. The proposed approach, named TrF-BWM-Neutrosophic-TOPSIS under SVNS Environment, integrates various decision-making techniques into a unified framework, a novel concept not explored in existing literature. In this hybrid model, criteria weights are determined in a trapezoidal fuzzy environment through the BWM method. Subsequently, the ranking of alternatives is refined using TOPSIS within a Neutrosophic fuzzy environment. This comprehensive methodology offers a unique perspective on optimizing decision-making processes in aquaponic systems by leveraging diverse analytical tools synergistically.

Compared to AHP [34], BWM [35] offers several advantages by requiring fewer paired comparisons and maintaining higher consistency in its pairwise comparison matrix. This improved consistency significantly enhances problem-solving efficacy compared to AHP. However, despite these benefits, BWM also presents certain drawbacks such as potential ambiguity in human evaluations and inaccuracies in criterion-related facts [36]. To address these limitations, BWM can be adapted to handle fuzzy numbers instead of crisp values, leading to more robust ranking conclusions in real-world scenarios. Guo and Zhao (2017) [37] introduced a fuzzy BWM technique termed triangular fuzzy MCDM, which allows for estimating the prospective performance of decision-evaluation groups regarding each criterion using triangular or trapezoidal fuzzy numbers [38]. There exists a relationship between trapezoidal and triangular fuzzy numbers, with a triangular fuzzy number emerging when a trapezoidal fuzzy number equals one of the two most promising values at least once [39]. Researchers have determined that trapezoidal fuzzy numbers offer advantages over triangular fuzzy numbers in designing procedures [40]. The trapezoidal fuzzy BWM, introduced by Majumder et al. [40] in 2022, represents a significant advancement in this field, providing a more nuanced approach to decision-making under uncertainty.

In this study, the TOPSIS (Technique for Order of Preference by Similarity to Ideal Solution) method, as outlined in the primary reference [41], is employed to determine the optimal option. The process of assigning criteria weights is governed by TrF-FOCUM [42], facilitating this determination. The rationale for selecting TOPSIS lies in its user-friendly nature and its ability to meet both qualitative and quantitative requirements flexibly. By assessing each option based on its best and worst results, TOPSIS contributes to a more robust ranking outcome. Moreover, it can accommodate cost-benefit considerations, making it suitable for scenarios where performance-cost interaction is significant. While previous research has extensively explored fuzzy and intuitionistic fuzzy MCDM problems, the increasing recognition of ambiguity's role in MCDM complexities underscores the necessity of incorporating neutrosophic sets. Neutrosophic sets address environments characterized by uncertainty, indeterminacy, and inconsistency within the MCDM methodology. Despite the attention given to fuzzy and intuitionistic fuzzy MCDM challenges, incorporating indeterminacy into the realm of MCDM difficulties is considered essential. In 2018, Biswas et al. [43] developed TOPSIS strategy for trapezoidal neutrosophic number environment. In 2023, Neutrosophic BWM-TOPSIS

was developed by [44] to determine alternative rankings, marking a significant advancement in addressing the complexities of decision-making under uncertainty.

1.4 Advantage and novelty:

Numerous benefits come from the hybrid technique that combines Neutrosophic-TOPSIS strategy for alternative level evaluation in a Single Valued Neutrosophic (SVN) environment with Trapezoidal Fuzzy with the Best Worst Method (BWM) for criteria level evaluation.

- The hybrid method can handle various forms of uncertainty in SVN contexts by employing both Trapezoidal Fuzzy logic and Neutrosophic logic. While Neutrosophic logic addresses uncertainty at the alternative level, Trapezoidal Fuzzy logic addresses uncertainty at the criteria level, offering a thorough handling of ambiguity throughout the decision-making process.
- The Best Worst Method (BWM) enables decision-makers to accurately represent subjective preferences by methodically assessing criteria according to their relative significance. This guarantees that the evaluation process appropriately reflects the preferences of decision-makers, resulting in more trustworthy and significant conclusions.
- By offering a systematic approach to determining the significance of criteria, the BWM improves the clarity of the decision-making process. The weighting and ranking of the criteria are easily understood by decision-makers, which promotes intuitive decision-making.
- The neutrosophic-TOPSIS technique enhances the resilience of the decision-making process to ambiguity by explicitly considering degrees of truth, indeterminacy, and falsehood at the alternative level. This ensures the reliability of decision outcomes even in the presence of unclear or ambiguous facts.
- By merging the advantages of BWM, the TOPSIS method, and Trapezoidal Fuzzy BWM, the hybrid technique facilitates effective multi-criteria decision-making. Decision-makers can take uncertainties and personal preferences into account while methodically evaluating alternatives based on a variety of factors.
- In SVN contexts, the hybrid technique enhances the quality and reliability of decision outputs by integrating both Neutrosophic and Trapezoidal Fuzzy logic. By taking into account both objective standards and subjective preferences, decision-makers tend to make more robust and well-informed choices.

2. Initial preparations

2.1. TrFN (Trapezoidal fuzzy number)

Ever since Zadeh introduced fuzzy [45] sets in 1965, researchers have devised numerous types of fuzzy numbers, including triangular, trapezoidal, and Gaussian variants [46]. The subsequent summary offers a succinct outline of key definitions pertinent to fuzzy sets and TFNs (Triangular Fuzzy Numbers).

Definition 1: A fuzzy set, symbolized as $U=\{(u,\tau_u(u)):u\in\Re\}$, is defined by a set of ordered pairs, where $\tau_u(u)\in[0,1]$ indicates the membership mapping of object g within the fuzzy set.

Definition 2: TrFN is presented by $\tilde{O}=(\theta_1,\theta_2,\theta_3,\theta_4)$, where $\tilde{O}$'s membership mapping $\tau_{\tilde{O}}$ is presented by [47].

$$\tau_{\tilde{O}}=\begin{cases}0,\ u<\theta_1\\ \dfrac{u-\theta_1}{\theta_2-\theta_1},\ \theta_1\le u\le\theta_2\\ 1,\ \theta_2\le u\le\theta_3\\ \dfrac{\theta_3-u}{\theta_3-\theta_4},\ \theta_3\le u\le\theta_4\\ 0,\ u>\theta_4\end{cases}$$

Here, θ_1 and θ_4 denote parameters defining the range of potential assessments; $\tilde{O}$ represents the lower bound, θ_1 signifies the upper limit of θ_4, and $[\theta_1,\theta_4]$ corresponds to the interval of $\tilde{O}$.

When the two most favorable values of a Trapezoidal Fuzzy Number (TrFN) align, it transforms into a TFN, indicating that TFNs are a particular instance of TrFNs. TrFNs find utility in pessimistic or conservative contexts, where exclusive membership is attributed to a restricted subset of the discourse universe. On the other hand, in optimistic or tolerant scenarios, TrFNs assign full membership to a wider subset of the discourse universe. By fostering tolerance and optimism, TrFNs adeptly manage events [48]. Under the assumption that $\tilde{O}_1=(\theta_1^{/},\theta_2^{/},\theta_3^{/},\theta_4^{/})$ and $\tilde{O}_2=(\theta_1^{//},\theta_2^{//},\theta_3^{//},\theta_4^{//})$ [49] are the two positive TrFNs, the operational principles are as follows.

$$\tilde{O}_1\oplus\tilde{O}_2=(\theta_1^{/},\theta_2^{/},\theta_3^{/},\theta_4^{/})\oplus(\theta_1^{//},\theta_2^{//},\theta_3^{//},\theta_4^{//})=(\theta_1^{/}+\theta_1^{//},\theta_2^{/}+\theta_2^{//},\theta_3^{/}+\theta_3^{//},\theta_4^{/}+\theta_4^{//})$$

$$\tilde{O}_1\otimes\tilde{O}_2=(\theta_1^{/},\theta_2^{/},\theta_3^{/},\theta_4^{/})\otimes(\theta_1^{//},\theta_2^{//},\theta_3^{//},\theta_4^{//})=(\theta_1^{/}\theta_1^{//},\theta_2^{/}\theta_2^{//},\theta_3^{/}\theta_3^{//},\theta_4^{/}\theta_4^{//})$$

$$\tilde{O}_1/\tilde{O}_2=(\theta_1^{/},\theta_2^{/},\theta_3^{/},\theta_4^{/})/(\theta_1^{//},\theta_2^{//},\theta_3^{//},\theta_4^{//})=(\theta_1^{/}/\theta_4^{//},\theta_2^{/}/\theta_3^{//},\theta_3^{/}/\theta_2^{//},\theta_4^{/}/\theta_1^{//})$$

$$\mu\tilde{O}_1=(\mu\theta_1^{/},\mu\theta_2^{/},\mu\theta_3^{/},\mu\theta_4^{/});\mu\in\square^{+}$$

$$\tilde{O}_1^{-1}=(1/\theta_4^{/},1/\theta_3^{/},1/\theta_2^{/},1/\theta_1^{/})$$

Definition 3: Defuzzification refers to the method of converting a fuzzy number, which represents uncertainty or imprecision, into a single precise value within the crisp domain. This process is crucial for obtaining a concrete decision or action based on fuzzy logic outputs. It typically involves applying an inverse transformation to the fuzzy output, which could be a fuzzy set or a fuzzy number, to derive a precise value that best represents the fuzzy input. By doing so, defuzzification enables clearer and more actionable outcomes from fuzzy logic-based systems. Considering $\tilde{O}_1 = \left(\theta_1', \theta_2', \theta_3', \theta_4'\right)$ is a TrFN, the associated crisp value $\Re\left(\tilde{O}_1\right)$ can be computed using the equation given below:

$$\Re\left(\tilde{O}_1\right) = \frac{\theta_1' + 2\theta_2' + 2\theta_3' + \theta_4'}{6}$$

2.2. Preliminary of Single Valued Neutrosophic Set (SVNS)

The foundation of Neutrosophic Sets [50] was laid by Prof. Smarandache, who introduced the concept. Subsequently, [51] pioneered the idea of Single-Valued Neutrosophic Set (SVNS) to address situations marked by uncertainty and incomplete information. The overview of SVNS was presented by Pramanik [52] to address situations marked by uncertainty and incomplete information.

The subsequent definition characterizes an SVNS Δ over a predetermined set Y:

$$\Delta = \left\{\left(b, \mathrm{A}_Z(b), \mathrm{B}_Z(b), \mathrm{M}_Z(b)\right) : b \in Y\right\}$$

where $\mathrm{A}_Z : R \to \{0,1\} \cup (0,1)$, $\mathrm{B}_Z : R \to \{0,1\} \cup (0,1)$, $\mathrm{M}_Z : R \to \{0,1\} \cup (0,1)$ and so $0 \le \mathrm{A}_Z(b) + \mathrm{B}_Z(b) + \mathrm{M}_Z(b) \le 3$. If an SVNS Δ over a given set Y, we refer to the triplet $(\mathrm{A}_Z(b), \mathrm{B}_Z(b), \mathrm{M}_Z(b))$ as a Single-Valued Neutrosophic Number (SVNN).

Mandal & Basu (2019) [53] introduced a novel scoring function aimed to address MADM challenges within the SVNS framework, outlined as follows:
The scoring mechanism is established through the following steps:

(i) Let's assume Θ represents the origin in a three-dimensional space, and $\Omega = \left(h_\theta, d_\theta, \upsilon_\theta\right)$, denoted as an SVNN, signifies a point within this space. Perform a translation of this point into Ω to arrive at $\Lambda = \left(h_\varpi, d_\varpi, \upsilon_\varpi\right)$. Here $h_\varpi = h_\theta + \xi,\ d_\varpi = d_\theta + \xi,\ \upsilon_\varpi = \upsilon_\theta + \xi$, where $\xi > 0,$ each representing υ_ϖ, a real number that remains distinct and unchanging throughout the specific problem, play a crucial role. Now, let's contemplate another point, $\Lambda' = \left(h_\varpi, -d_\varpi, -\upsilon_\varpi\right)$, which is the result of reflecting $\Lambda = \left(h_\varpi, d_\varpi, \upsilon_\varpi\right)$ across the *x-axis*.

(ii) Locate the score function, which is $\mathrm{N}_1(\Lambda) = \cos\vartheta$, with ϑ representing the angle between $\Theta\Lambda$ and $\Theta\Lambda^{/}$, and Θ denoting the origin.

(iii) If the score values for two distinct SVNNs, $\Lambda_1 = \left(h_{\varpi_1}, d_{\varpi_1}, \upsilon_{\varpi_1}\right)$ and $\Lambda_2 = \left(h_{\varpi_2}, d_{\varpi_2}, \upsilon_{\varpi_2}\right)$, denoted as $\mathrm{N}_1(\Lambda_1)$ and $\mathrm{N}_1(\Lambda_2)$ respectively, are equal, determine $\Lambda_1^{**} = \left(h_{\varpi_1}, -d_{\varpi_1}, -\sqrt{\upsilon_{\varpi_1}}\right)$ and $\Lambda_2^{**} = \left(h_{\varpi_2}, -d_{\varpi_2}, -\sqrt{\upsilon_{\varpi_2}}\right)$ respectively for the corresponding translated points $\Lambda_1^{*} = \left(h_{\varpi_1^*}, d_{\varpi_1^*}, \upsilon_{\varpi_1^*}\right)$ and $\Lambda_2^{*} = \left(h_{\varpi_2^*}, d_{\varpi_2^*}, \upsilon_{\varpi_2^*}\right)$ where, $h_{\varpi_1^*} = h_{\varpi_1} + \xi$, $d_{\varpi_1^*} = d_{\varpi_1} + \xi$, $\upsilon_{\varpi_1^*} = \upsilon_{\varpi_1} + \xi$ and $h_{\varpi_2^*} = h_{\varpi_2} + \xi, d_{\varpi_2^*} = d_{\varpi_2} + \xi, \upsilon_{\varpi_2^*} = \upsilon_{\varpi_2} + \xi$.

(iv) Determine $\cos\phi$ and $\cos\lambda$, where ϕ represents the angle between $\Theta\Lambda_2^{*}$ and $\Theta\Lambda_2^{**}$, and λ signifies the angle between $\Theta\Lambda_2^{*}$ and $\Theta\Lambda_2^{**}$, with Θ denoting the origin.

(v) The score mapping $\mathrm{N}_2(\Lambda_1) = \cos\phi$ as well as $\mathrm{N}_2(\Lambda_2) = \cos\lambda$.

3. TrF-BWM-Neutrosophic-TOPSIS Strategy under SVNS Environment Approach

The TrF-BWM-Neutrosophic-TOPSIS Strategy under the SVNS Environment approach [54, 55] comprises two main phases. In the initial phase, the TrF-BWM method is employed to determine the weight or priority value (PV) of the criteria. Following this, in the second phase, the Neutrosophic-TOPSIS Strategy under the SVNS Environment is utilized to establish the ranking of alternatives. **Figure 2** provides a visual representation of the computational steps integral to this approach.

3.1 Phase I: Trapezoidal Fuzzy Best-Worst Method (TrF BWM)

The TrFBWM process entails five distinct steps:

Step 1: Selection of Criteria: MCDM strategies utilize selection criteria to identify the primary indicators. The letters $\Gamma = \left\{\Gamma_{\wp} : \wp = 1(1)\Psi\right\}$ and $\mathrm{M} = \left\{\mathrm{M}_{\sigma} : \sigma = 1(1)\Phi\right\}$ represent the criteria and indicators, respectively. The set of elements referred to as Γ influences the performance of each indicator.

Step 2: Selecting the Best and Worst Criteria: In this process, the Decision Maker (DM) identifies the optimal (best) and least favorable (worst) criteria. Through various surveys, such as expert opinions,

literature reviews, and media analyses, the highest and lowest-ranking criteria is identified by the DM. Γ_{ε} and Γ_{ω} represent the best and worst standards, respectively. This is followed by $\Gamma_{\varepsilon} \in \Gamma$ and $\Gamma_{\omega} \in \Gamma$.

Step 3: Relative Assessment: Developing a pair-wise comparison matrix is deemed superfluous. Step 3 provides further detail on the pair-wise comparisons undertaken in sub-steps 1 and 2. The criteria undergo assessment utilizing linguistic metrics of relevance, employing trapezoidal fuzzy pair-wise comparisons. **Table 1** showcases trapezoidal fuzzy measurements of importance for each item within the linguistic domain. During the pair-wise comparisons of criteria conducted by the Decision Maker (DM), "EI" signifies equally important, "UMS" denotes upper measure scale, and "Intermediate" indicates the intermediate scale.

Table 1: Assessments relying on linguistic measures of importance.

List of linguistic phrases defined	Essential fuzziness measures for the trapezoidal shape (χ, τ, ρ, ξ)	SI
UMS	$\left(8, \frac{17}{2}, 9, 9\right)$	9
Intermediate scale	$\left(\delta - 1, \delta - \frac{1}{2}, \delta + \frac{1}{2}, \delta + 1\right)$	$\delta = 2(1)8$
EI	$(1,1,1,1)$	1

Sub-Step 1: Comparison of Best Criteria (BC) to all criteria:

In this step, comparison of each criterion of Set Γ with the best criteria of Set $\Gamma_{\varepsilon} \in \Gamma$ is performed. This phase is detailed in **Table 2**. In this scenario, both $\tilde{\partial}_{\varepsilon\wp}\ (\wp = 1(1)\Psi)$ and $\tilde{\partial}_{\varepsilon\varepsilon} = \left(8, \frac{17}{2}, 9, 9\right)$ have the option to select one action from **Table 1**.

Table 2: The criterion that most closely aligns with each pair of criteria.

BC (Best Criteria)	Γ_1	Γ_2	$\cdots$	Γ_{Ψ}
Γ_{ε}	$\tilde{\partial}_{\varepsilon 1}$	$\tilde{\partial}_{\varepsilon 2}$	$\cdots$	$\tilde{\partial}_{\varepsilon\Psi}$

Sub-Step 2: Every criterion is evaluated in relation to the worst criteria (WC):

Table 3 presents the comparison of each criterion in Set Γ with the worst criterion in Set $\Gamma_{\omega} \in \Gamma$. Additionally, the measures of significance from **Table 1** are selected by $\tilde{\partial}_{\wp\omega}\ (\wp = 1(1)\Psi)$ as well as $\tilde{\partial}_{\omega\omega} = (1,1,1,1)$.

Table 3: Comparing every criterion with the most rigorous ones.

WC (Worst Criteria)	Γ_{ω}
Γ_1	$\tilde{\partial}_{1\omega}$
Γ_2	$\tilde{\partial}_{2\omega}$
$\vdots$	$\vdots$
Γ_{Ψ}	$\tilde{\partial}_{\wp\omega}$

Step 4: Evaluation of PV of trapezoidal fuzzy:

This step entails identifying the optimal fuzzy trapezoidal priority values. Let $\tilde{P}_1, \tilde{P}_2, ..., \tilde{P}_{\wp}$ represent the decision variables in optimization approaches, and let $\tilde{P}_1, \tilde{P}_2, ..., \tilde{P}_{\wp}$ denote the priority values for the corresponding $\Gamma_1, \Gamma_2, ..., \Gamma_{\wp}$ criteria. The respective priority values for criteria Γ_{ε} and Γ_{ω} are $\tilde{P}_{\varepsilon}$ and $\tilde{P}_{\omega}$.

As per equations (1) and (2), the optimal trapezoidal fuzzy priority value is characterized by two ratios, denoted as $\frac{\tilde{P}_{\varepsilon}}{\tilde{P}_{\wp}}$ and $\frac{\tilde{P}_{\wp}}{\tilde{P}_{\omega}} \left(\wp = 1(1)\Psi\right)$.

$$\frac{\tilde{P}_{\varepsilon}}{\tilde{P}_{\wp}} - \tilde{\partial}_{\varepsilon\wp} = 0 \qquad (1)$$

and $$\frac{\tilde{P}_{\wp}}{\tilde{P}_{\omega}} - \tilde{\partial}_{\wp\omega} = 0, \forall \wp = 1(1)\Psi \qquad (2)$$

For all *i*, the situation remains consistent if the distances between $\left|\frac{\tilde{P}_{\varepsilon}}{\tilde{P}_{\wp}} - \tilde{\partial}_{\varepsilon\wp}\right|$ and $\left|\frac{\tilde{P}_{\wp}}{\tilde{P}_{\omega}} - \tilde{\partial}_{\wp\omega}\right|$, $\forall \wp = 1(1)\Psi$ are minimized or maximized. A trapezoidal fuzzy set comprises each TrF priority value. $\tilde{P}_{\wp} = \left(\chi_{\tilde{P}_{\wp}}, \tau_{\tilde{P}_{\wp}}, \rho_{\tilde{P}_{\wp}}, \xi_{\tilde{P}_{\wp}}\right), \left(\wp = 1(1)\Psi\right)$ should be considered when selecting an optimal indicator. In this context, the problem can be formulated as follows:

$$\left.\begin{aligned}
&\min \max_{u}\left\{\left|\frac{\tilde{\mathrm{P}}_{\varepsilon}}{\tilde{\mathrm{P}}_{\wp}}-\tilde{\partial}_{\varepsilon\wp}\right|,\left|\frac{\tilde{\mathrm{P}}_{\wp}}{\tilde{\mathrm{P}}_{\omega}}-\tilde{\partial}_{\wp\omega}\right|\right\}\\
&\text{Subject to}\sum_{\wp=1}^{\Psi}\Re\left(\tilde{\mathrm{P}}_{\wp}\right)=1\\
&\chi_{\tilde{\mathrm{P}}_{\wp}}\le\tau_{\tilde{\mathrm{P}}_{\wp}}\le\rho_{\tilde{\mathrm{P}}_{\wp}}\le\xi_{\tilde{\mathrm{P}}_{\wp}},\forall\,\wp=1(1)\Psi\\
&\chi_{\tilde{\mathrm{P}}_{\wp}}\ge 0,\forall\,\wp=1(1)\Psi
\end{aligned}\right\}\qquad(3)$$

wher $\tilde{\mathrm{P}}_{\varepsilon}=\left(\chi_{\tilde{\mathrm{P}}_{\varepsilon}},\tau_{\tilde{\mathrm{P}}_{\varepsilon}},\rho_{\tilde{\mathrm{P}}_{\varepsilon}},\xi_{\tilde{\mathrm{P}}_{\varepsilon}}\right)$, $\tilde{\mathrm{P}}_{\wp}=\left(\chi_{\tilde{\mathrm{P}}_{\wp}},\tau_{\tilde{\mathrm{P}}_{\wp}},\rho_{\tilde{\mathrm{P}}_{\wp}},\xi_{\tilde{\mathrm{P}}_{\wp}}\right)$, $\tilde{\mathrm{P}}_{\omega}=\left(\chi_{\tilde{\mathrm{P}}_{\omega}},\tau_{\tilde{\mathrm{P}}_{\omega}},\rho_{\tilde{\mathrm{P}}_{\omega}},\xi_{\tilde{\mathrm{P}}_{\omega}}\right)$, $\tilde{\partial}_{\varepsilon\wp}=\left(\chi_{\tilde{\partial}_{\varepsilon\wp}},\tau_{\tilde{\partial}_{\varepsilon\wp}},\rho_{\tilde{\partial}_{\varepsilon\wp}},\xi_{\tilde{\partial}_{\varepsilon\wp}}\right)$,

and $\tilde{\partial}_{\omega\wp}=\left(\chi_{\tilde{\partial}_{\omega\wp}},\tau_{\tilde{\partial}_{\omega\wp}},\rho_{\tilde{\partial}_{\omega\wp}},\xi_{\tilde{\partial}_{\omega\wp}}\right)$.

The process of transforming problem (3) into a non-linear optimization problem is as follows. The non-linear adapted form of equation (3) is expressed as follows:

$$\left.\begin{aligned}
&\min\ \tilde{\Delta}\\
&\text{Subject to}\left|\frac{\tilde{\mathrm{P}}_{\varepsilon}}{\tilde{\mathrm{P}}_{\wp}}-\tilde{\partial}_{\varepsilon\wp}\right|\le\tilde{\Delta}\\
&\left|\frac{\tilde{\mathrm{P}}_{\wp}}{\tilde{\mathrm{P}}_{\omega}}-\tilde{\partial}_{\wp\omega}\right|\le\tilde{\Delta}\\
&\sum_{i=1}^{n}\Re\left(\hat{w}_{i}\right)=1\\
&\chi_{\tilde{\mathrm{P}}_{\wp}}\le\tau_{\tilde{\mathrm{P}}_{\wp}}\le\rho_{\tilde{\mathrm{P}}_{\wp}}\le\xi_{\tilde{\mathrm{P}}_{\wp}},\forall\,\wp=1(1)\Psi\\
&\chi_{\tilde{\mathrm{P}}_{\wp}}\ge 0,\forall\,\wp=1(1)\Psi
\end{aligned}\right\}\qquad(4)$$

where, $\tilde{\Delta}=\left(\chi_{\tilde{\Delta}},\tau_{\tilde{\Delta}},\rho_{\tilde{\Delta}},\xi_{\tilde{\Delta}}\right)$

Assume, $\Delta=\chi_{\tilde{\Delta}}$ becomes $\tilde{\Delta}=(\Delta,\Delta,\Delta,\Delta)$. Consequently, equation (4) can be reformulated as: a

$$\left.\begin{aligned}
&\min\ (\Delta,\Delta,\Delta,\Delta)\\
&\text{Subject to}\left|\frac{\left(\chi_{\tilde{\mathrm{P}}_{\varepsilon}},\tau_{\tilde{\mathrm{P}}_{\varepsilon}},\rho_{\tilde{\mathrm{P}}_{\varepsilon}},\xi_{\tilde{\mathrm{P}}_{\varepsilon}}\right)}{\left(\chi_{\tilde{\mathrm{P}}_{\wp}},\tau_{\tilde{\mathrm{P}}_{\wp}},\rho_{\tilde{\mathrm{P}}_{\wp}},\xi_{\tilde{\mathrm{P}}_{\wp}}\right)}-\left(\chi_{\tilde{\partial}_{\varepsilon\wp}},\tau_{\tilde{\partial}_{\varepsilon\wp}},\rho_{\tilde{\partial}_{\varepsilon\wp}},\xi_{\tilde{\partial}_{\varepsilon\wp}}\right)\right|\le(\Delta,\Delta,\Delta,\Delta)\\
&\left|\frac{\left(\chi_{\tilde{\mathrm{P}}_{\wp}},\tau_{\tilde{\mathrm{P}}_{\wp}},\rho_{\tilde{\mathrm{P}}_{\wp}},\xi_{\tilde{\mathrm{P}}_{\wp}}\right)}{\left(\chi_{\tilde{\mathrm{P}}_{\omega}},\tau_{\tilde{\mathrm{P}}_{\omega}},\rho_{\tilde{\mathrm{P}}_{\omega}},\xi_{\tilde{\mathrm{P}}_{\omega}}\right)}-\left(\chi_{\tilde{\partial}_{\omega\wp}},\tau_{\tilde{\partial}_{\omega\wp}},\rho_{\tilde{\partial}_{\omega\wp}},\xi_{\tilde{\partial}_{\omega\wp}}\right)\right|\le(\Delta,\Delta,\Delta,\Delta)\\
&\sum_{\wp=1}^{\Psi}\Re\left(\tilde{\mathrm{P}}_{\wp}\right)=1\\
&\chi_{\tilde{\mathrm{P}}_{\wp}}\le\tau_{\tilde{\mathrm{P}}_{\wp}}\le\rho_{\tilde{\mathrm{P}}_{\wp}}\le\xi_{\tilde{\mathrm{P}}_{\wp}},\forall\,\wp=1(1)\Psi\\
&\chi_{\tilde{\mathrm{P}}_{\wp}}\ge 0,\forall\,\wp=1(1)\Psi
\end{aligned}\right\}\qquad(5)$$

The solution to equation (5) yields the optimal trapezoidal fuzzy priority value $\left(\tilde{P}_1^*, \tilde{P}_2^*, ..., \tilde{P}_\wp^*\right)$, where $\tilde{P}_\wp^* = \left(\chi_{\tilde{P}_\wp^*}, \tau_{\tilde{P}_\wp^*}, \rho_{\tilde{P}_\wp^*}, \xi_{\tilde{P}_\wp^*}\right)$, $\forall \wp = 1(1)\Psi$ signifies the optimal trapezoidal fuzzy priority value.

Step 5: Verify TrF-BWM in CR:

Considering the Consistency Ratio (CR), pairwise comparisons hold the highest significance. Pairwise comparisons remain consistent if $\tilde{\partial}_{\varepsilon\wp} \times \tilde{\partial}_{\wp\omega} - \tilde{\partial}_{\varepsilon\omega} = 0$ is true. A pairwise comparison matrix with $\tilde{\partial}_{\varepsilon\wp} \times \tilde{\partial}_{\wp\omega} - \tilde{\partial}_{\varepsilon\omega} > or < 0$ indicates inconsistency. The maximum inequality in $\tilde{\Delta}$ arises when $\tilde{\partial}_{\varepsilon\omega}$ equals $\tilde{\partial}_{\varepsilon\wp}$ and $\tilde{\partial}_{\wp\omega}$ .As $\left(\frac{\tilde{P}_\varepsilon}{\tilde{P}_\wp}\right) \times \left(\frac{\tilde{P}_\wp}{\tilde{P}_\omega}\right) - \left(\frac{\tilde{P}_\varepsilon}{\tilde{P}_\omega}\right) = 0$ represents the largest inequality, it can be interpreted as follow:

$$\left(\tilde{\partial}_{\varepsilon\wp} - \tilde{\Delta}\right) \times \left(\tilde{\partial}_{\wp\omega} - \tilde{\Delta}\right) - \left(\tilde{\partial}_{\varepsilon\omega} - \tilde{\Delta}\right) = 0 \qquad (6)$$

To ascertain the maximum possible level of fuzzy consistency, utilize $\tilde{\partial}_{\varepsilon\wp} = \tilde{\partial}_{\wp\omega} = \tilde{\partial}_{\varepsilon\omega}$. As a result, the equation (6) is modified to

$$\tilde{\Delta}^2 - \left(1 + 2\tilde{\partial}_{\varepsilon\omega}\right)\tilde{\Delta} + \left(\tilde{\partial}_{\varepsilon\omega}^2 - \tilde{\partial}_{\varepsilon\omega}\right) = 0 \qquad (7)$$

where $\tilde{\Delta} = \left(\chi_{\tilde{\Delta}}, \tau_{\tilde{\Delta}}, \rho_{\tilde{\Delta}}, \zeta_{\tilde{\Delta}}\right)$ and $\tilde{P}_{\varepsilon\omega} = \left(\chi_{\varepsilon\omega}, \tau_{\varepsilon\omega}, \rho_{\varepsilon\omega}, \zeta_{\varepsilon\omega}\right)$ are both TrFS

In terms of trapezoidal fuzzy values, $\tilde{P}_{\varepsilon\omega} = \left(\chi_{\varepsilon\omega}, \tau_{\varepsilon\omega}, \rho_{\varepsilon\omega}, \zeta_{\varepsilon\omega}\right)$ can reach a maximum of $\left(8, \frac{17}{2}, 9, 9\right)$. The consistency index (CI) of TrFS is calculated as 13.77 according to equation (7). Consequently, $\max\left\{\chi_{\varepsilon\omega}, \tau_{\varepsilon\omega}, \rho_{\varepsilon\omega}, \zeta_{\varepsilon\omega}\right\}$ values exceeding 9 are prohibited. It is feasible to derive the CI for TrFS through a similar procedure. **Table 4** displays the CI values for TrFS.

Table 4: CI for TrFS.

(χ, τ, ρ, ξ)	**CI**
(1,1,1,1)	3
$\left(8, \frac{17}{2}, 9, 9\right)$	13.77
$\delta = 7$	12.58
$\delta = 5$	10
$\delta = 3$	7.37

Step 6: Crisp priority's value:

Finally, convert the ideal PV $\left(\tilde{\mathrm{P}}_1{}^*, \tilde{\mathrm{P}}_2{}^*, ..., \tilde{\mathrm{P}}_{\wp}{}^*\right)$ where $\tilde{\mathrm{P}}_{\wp}{}^* = \left(\chi_{\tilde{\mathrm{P}}_{\wp}{}^*}, \tau_{\tilde{\mathrm{P}}_{\wp}{}^*}, \rho_{\tilde{\mathrm{P}}_{\wp}{}^*}, \xi_{\tilde{\mathrm{P}}_{\wp}{}^*}\right)$, $\forall\, \wp = 1(1)\Psi$ into its crisp value using formula (8).

$$\Re(\tilde{\mathrm{P}}_{\wp}{}^*) = \frac{\chi_{\tilde{\mathrm{P}}_{\wp}{}^*} + 2\tau_{\tilde{\mathrm{P}}_{\wp}{}^*} + 2\rho_{\tilde{\mathrm{P}}_{\wp}{}^*} + \xi_{\tilde{\mathrm{P}}_{\wp}{}^*}}{6}, \forall\, \wp = 1(1)\Psi \qquad (8)$$

Step 7: PVs with normalized values: Equation (9) is employed to standardize the priority values of each criterion.

$$\tilde{\mathrm{P}}_{\wp}{}^* = \frac{\Re(\tilde{\mathrm{P}}_{\wp}{}^*)}{\sum_{\wp=1}^{\Psi} \Re(\tilde{\mathrm{P}}_{\wp}{}^*)}, \forall\, \wp = 1(1)\Psi \qquad (9)$$

3.2 Phase II: Neutrosophic-TOPSIS Strategy under SVNS Environment

Consider the set of alternative $\mathrm{E} = \left\{\alpha_\gamma : \gamma = 1(1)\vartheta\right\}$, $\gamma \geq 1$ and $\mathrm{H} = \left\{\sigma_\theta : \theta = 1(1)\eta\right\}$, $\theta \geq 2$ be the set of attributes with weights $\Re(\mu_\theta^*)$, $\theta = 1(1)\eta$ respectively.

Decision-makers assign ratings to the $\alpha_\gamma, \gamma = 1(1)\vartheta$ alternatives based on the attributes $\sigma_\theta, \theta = 1(1)\eta$, which are represented using an SVNN. Let's assume the rating for the γ^{th} attribute concerning the θ^{th} alternative is presented as follows:

$\alpha_\gamma^* = \left(\kappa_\theta, \mathrm{A}_{\alpha_\kappa}(\sigma_\theta), \mathrm{B}_{\alpha_\kappa}(\sigma_\theta), \mathrm{M}_{\alpha_\kappa}(\sigma_\theta)\right)$, $\gamma = 1(1)\vartheta$, where, $0 \leq \mathrm{A}_{\alpha_\kappa}(\mathrm{Z}_\varphi) + \mathrm{B}_{\alpha_\kappa}(\mathrm{Z}_\sigma) + \mathrm{M}_{\alpha_\kappa}(\mathrm{Z}_\varphi) \leq 3.$ Here $\left(\mathrm{A}_{\gamma\theta}, \mathrm{B}_{\gamma\theta}, \mathrm{M}_{\gamma\theta}\right)$ is denoted as an SVNN. $\alpha_{\gamma\theta}^*$, ($\gamma = 1(1)\vartheta$ and $\theta = 1(1)\eta$), where θ represents the number of attributes and γ represents the number of alternatives. Determine the decision matrix based on the ratings:

$\nabla^* = \left[\alpha_{\gamma\theta}^*\right]_{\vartheta\times\eta}$

Now, the TOPSIS method is encapsulated as follows:

Step 1: The score-matrix $\nabla = \left[\alpha_{\gamma\theta}\right]_{\vartheta\times\eta}$, ($\gamma = 1(1)\vartheta$ and $\theta = 1(1)\eta$)is acquired from the decision matrix $\nabla^* = \left[\alpha_{\gamma\theta}^*\right]_{\vartheta\times\eta}$ utilizing the following described in the preliminary section: i.e.,

$\alpha_{\gamma\theta} = \mathrm{N}_1\left(\left[\alpha^*_{\gamma\theta}\right]_{\vartheta\times\eta}\right)$.

Step 2: Determination of normalized decision matrix $\Upsilon = \left[\upsilon_{\gamma\theta}\right]_{\vartheta\times\eta}$

where, $\upsilon_{\gamma\theta} = \dfrac{\alpha_{\gamma\theta}}{\sqrt{\sum_{\theta=1}^{\eta}\alpha_{\gamma\theta}}}$,.....(10); $\gamma = 1(1)\vartheta$

Step 3: Calculate the weighted normalized decision matrix $\Xi = \left[\tau_{\gamma\theta}\right]_{\vartheta\times\eta}$

where $\tau_{\gamma\theta} = \Re(\tilde{\mu}^*_{\theta}) \otimes \upsilon_{\gamma\theta}$, $\gamma = 1(1)\vartheta$ and $\theta = 1(1)\eta$

Step 4: Identify the Neutrosophic Positive Ideal Solution (NPIS) and Neutrosophic Negative Ideal Solution (NNIS), denoted by ρ^+ as well as ρ^- respectively,

$\rho^+ = \left\{\tau^+_1, \tau^+_2, ..., \tau^+_\eta\right\}$, where, $\tau^+_\theta = \max_\theta \tau_{\gamma\theta}, \theta = 1(1)\eta$

$\rho^- = \left\{\tau^-_1, \tau^-_2, ..., \tau^-_\eta\right\}$, where, $\tau^-_\theta = \min_\theta \tau_{\gamma\theta}, \theta = 1(1)\eta$

Step 5: Calculate the distance of each alternative between NPIS as well as NNIS using the equations (11) and (12) which are given below:

$$\hbar^+_\gamma = \sqrt{\sum_{\theta=1}^{\eta}\left(\tau_{\gamma\theta} - \rho^+\right)^2}, \gamma = 1(1)\vartheta \qquad (11)$$

$$\hbar^-_\gamma = \sqrt{\sum_{\theta=1}^{\eta}\left(\tau_{\gamma\theta} - \rho^-\right)^2}, \gamma = 1(1)\vartheta \qquad (12)$$

Step 6: Evaluate the performance score for each alternative using the equation (13):

$$\aleph_\gamma = \hbar^-_\gamma \Big/ \left(\hbar^+_\gamma + \hbar^-_\gamma\right), \gamma = 1(1)\vartheta \qquad (13)$$

Step 7: Rank the alternatives according to their performance scores. The alternative with the highest performance score will be given the top ranking, while the alternative with the lowest performance score will be assigned the lowest ranking.

4. Methodology

This research endeavors to determine the principal indicators that are essential for carrying out an efficiency analysis through the application of a novel fuzzy-based MCDM methodology. The methodology comprises two fundamental stages: the implementation of the MCDM technique and the validation of the proposed model. The intricate process is shown schematically in **Figure 2**.

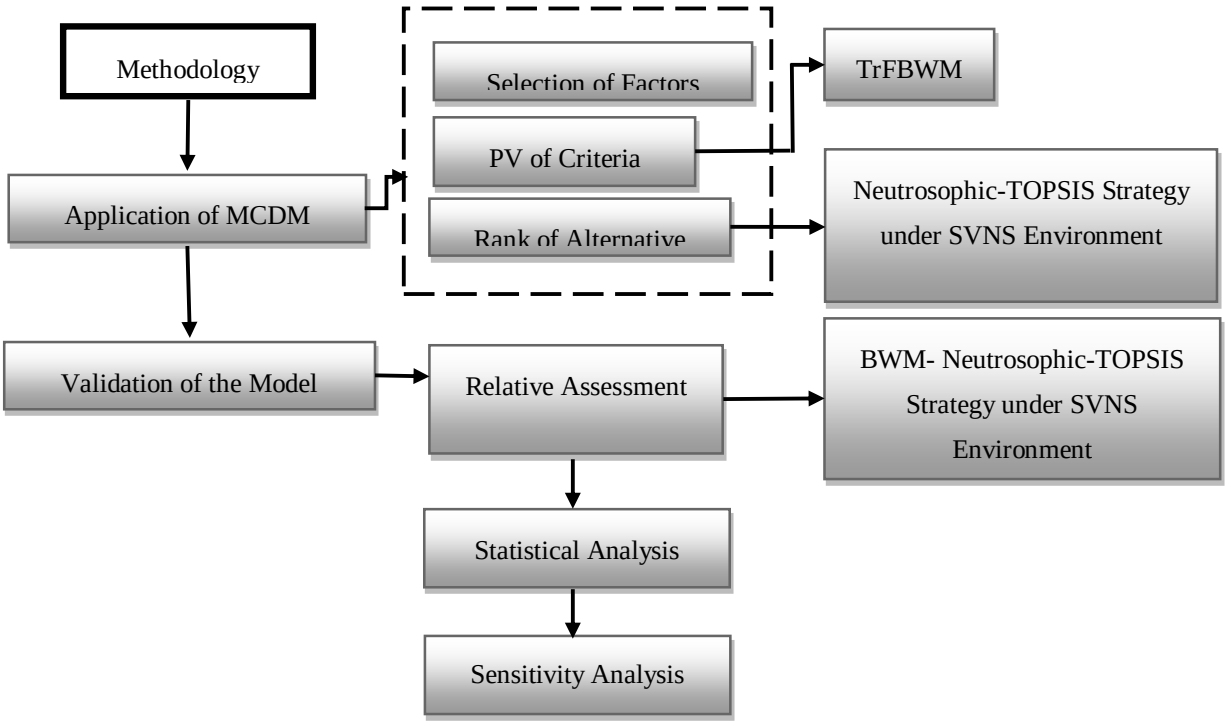

Figure 2: Schematic diagram of Methodology.

4.1 Application of MCDM

The purpose of the following section is to evaluate the PV of each criterion and alternative. Consequently, this phase encompasses three components: the identification of factors, the utilization of TrFBWM, and the application of Neutrosophic-TOPSIS strategy [54, 55] under SVNS environment. The decision hierarchy of the issue is depicted in **Figure 3**.

Step 1: Selection of Factors:

A thorough study of the relevant literature is reviewed for the selection of criteria and alternatives followed by the selection of a panel of specialists and stakeholders. **Table 5** displays all the chosen criteria and alternatives under examination.

Table 5: The selected alternatives and criteria for the consideration of this study.

Criteria	Alternative	Definition
Literature Review (Γ_1)	Nitrate (M_1)	Nitrate levels in aquaponic recirculating water can have several effects on fish growth. Elevated nitrate levels can lead to stress in fish and may impair their growth rates over time. High nitrate concentrations may affect water quality parameters such as pH and alkalinity, which can in turn impact fish metabolism. Changes in

		water chemistry can disrupt metabolic processes, leading to reduced growth rates in fish [18].
Aquaponics Experts (Γ_2)	Temperature (M_2)	The temperature in aquaponics refers to the measurement of the heat level within the aquaponic system, which encompasses both the water and the surrounding environment. Maintaining an optimal temperature is crucial for the overall health and productivity of the aquaponics living components, especially fish, within the system. Temperature directly influences the metabolic rate, oxygen demand, efficiency of feed conversion, digestive efficiency, of the fish. Additionally, it effects the susceptibility of the fish to disease. Overall, maintaining a stable and suitable temperature range within the aquaponics system is essential for promoting optimal fish growth [13-15, 20].
Aquaponics Field Workers (Γ_3)	pH (M_3)	Extreme pH levels can be harmful to fish. Very low or very high pH can cause stress to fish, leading to reduced appetite, impaired growth, and increased susceptibility to diseases. pH also effects the respiration process of fish. Optimal pH levels support efficient respiration, ensuring adequate oxygen supply for growth [16,17].
	Ammonia (M_4)	Ammonia is released into the water through fish waste and decomposing organic matter. High levels of ammonia are toxic to fish, causing stress, tissue damage, and even death. Ammonia toxicity can impair fish growth by affecting metabolic processes and reducing appetite [17].
	Turbidity (M_5)	Turbidity in aquaponics system is the haziness of the recirculating water due to Total Suspended Solids (TSS). High turbidity levels can interfere with fish feeding behaviour. Elevated levels of turbidity can disrupt fish feeding behavior and induce stress in fish. Furthermore, it can compromise the respiratory process and increase the likelihood of fish diseases [56].
	Dissolved Oxygen (M_6)	Fish rely on dissolved oxygen in the water for respiration. Adequate oxygen levels are essential to support the metabolic processes involved in growth. Insufficient oxygen can lead to stress, reduced growth rates, and even mortality in fish [16, 20].

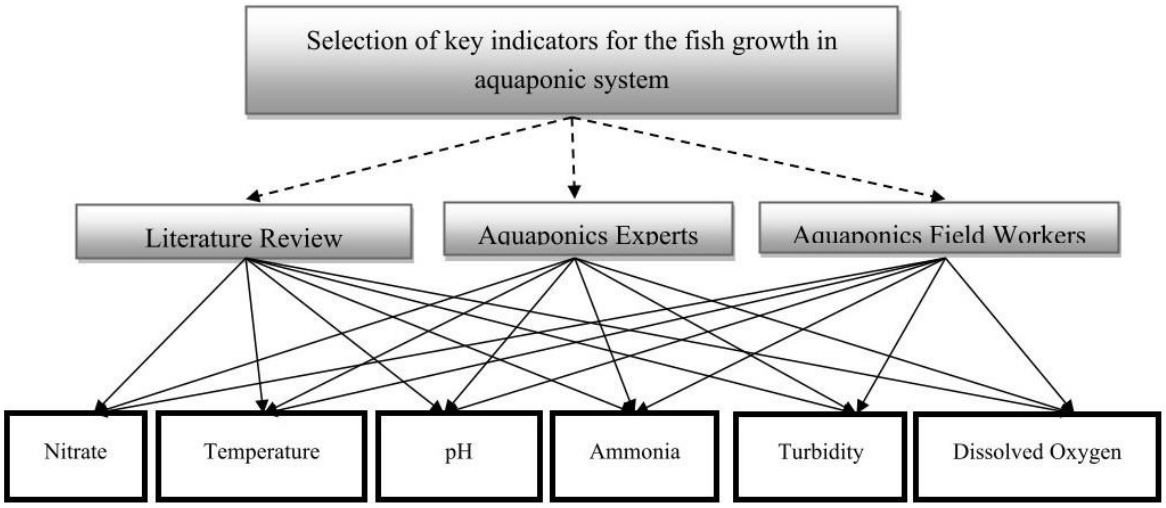

Figure 3: Hierarchical structure of the decision-making problem.

Step 2: Application of TrFBWM (Model I):

Initially, the process involves identifying the most favorable and least favorable criteria according to the consensus among the majority of specialists. Subsequently, pairwise evaluations for each criterion are conducted, as outlined in **Table 6**. Following this, additional pairwise comparisons are made utilizing the least favorable criteria, as demonstrated in **Table 7**. Equation (3) embodies this mathematical formulation after constructing the pairwise comparison matrix.

Table 6: Table illustrating the comparison of each criterion with the best criteria (BC).

	Γ_1	Γ_2	Γ_3
Γ_1 (Best Criteria)	$(1,1,1,1)$	$\left(2,\frac{5}{2},\frac{7}{2},4\right)$	$\left(6,\frac{13}{2},\frac{15}{2},8\right)$

Table 7: Table illustrating the comparison of the worst criteria (WC) with each set of criterion.

	Γ_3 (Worst Criteria)
Γ_1	$\left(6,\frac{13}{2},\frac{15}{2},8\right)$
Γ_2	$\left(4,\frac{9}{2},\frac{11}{2},6\right)$
Γ_3	$(1,1,1,1)$

$$
\begin{aligned}
&\min\ (\Delta,\Delta,\Delta,\Delta)\\
\text{Subject to}\ &\left|\frac{\left(\chi_{\tilde{P}_\varepsilon},\tau_{\tilde{P}_\varepsilon},\rho_{\tilde{P}_\varepsilon},\xi_{\tilde{P}_\varepsilon}\right)}{\left(\chi_{\tilde{P}_1},\tau_{\tilde{P}_1},\rho_{\tilde{P}_1},\xi_{\tilde{P}_1}\right)}-\left(\chi_{\tilde{\partial}_{\varepsilon 1}},\tau_{\tilde{\partial}_{\varepsilon 1}},\rho_{\tilde{\partial}_{\varepsilon 1}},\xi_{\tilde{\partial}_{\varepsilon 1}}\right)\right|\le(\Delta,\Delta,\Delta,\Delta)\\
&\left|\frac{\left(\chi_{\tilde{P}_\varepsilon},\tau_{\tilde{P}_\varepsilon},\rho_{\tilde{P}_\varepsilon},\xi_{\tilde{P}_\varepsilon}\right)}{\left(\chi_{\tilde{P}_2},\tau_{\tilde{P}_2},\rho_{\tilde{P}_2},\xi_{\tilde{P}_2}\right)}-\left(\chi_{\tilde{\partial}_{\varepsilon 2}},\tau_{\tilde{\partial}_{\varepsilon 2}},\rho_{\tilde{\partial}_{\varepsilon 2}},\xi_{\tilde{\partial}_{\varepsilon 2}}\right)\right|\le(\Delta,\Delta,\Delta,\Delta)\\
&\left|\frac{\left(\chi_{\tilde{P}_\varepsilon},\tau_{\tilde{P}_\varepsilon},\rho_{\tilde{P}_\varepsilon},\xi_{\tilde{P}_\varepsilon}\right)}{\left(\chi_{\tilde{P}_3},\tau_{\tilde{P}_3},\rho_{\tilde{P}_3},\xi_{\tilde{P}_3}\right)}-\left(\chi_{\tilde{\partial}_{\varepsilon 3}},\tau_{\tilde{\partial}_{\varepsilon 3}},\rho_{\tilde{\partial}_{\varepsilon 3}},\xi_{\tilde{\partial}_{\varepsilon 3}}\right)\right|\le(\Delta,\Delta,\Delta,\Delta)\\
&\left|\frac{\left(\chi_{\tilde{P}_1},\tau_{\tilde{P}_1},\rho_{\tilde{P}_1},\xi_{\tilde{P}_1}\right)}{\left(\chi_{\tilde{P}_\omega},\tau_{\tilde{P}_\omega},\rho_{\tilde{P}_\omega},\xi_{\tilde{P}_\omega}\right)}-\left(\chi_{\tilde{\partial}_{\omega 1}},\tau_{\tilde{\partial}_{\omega 1}},\rho_{\tilde{\partial}_{\omega 1}},\xi_{\tilde{\partial}_{\omega 1}}\right)\right|\le(\Delta,\Delta,\Delta,\Delta)\\
&\left|\frac{\left(\chi_{\tilde{P}_2},\tau_{\tilde{P}_2},\rho_{\tilde{P}_2},\xi_{\tilde{P}_2}\right)}{\left(\chi_{\tilde{P}_\omega},\tau_{\tilde{P}_\omega},\rho_{\tilde{P}_\omega},\xi_{\tilde{P}_\omega}\right)}-\left(\chi_{\tilde{\partial}_{\omega 2}},\tau_{\tilde{\partial}_{\omega 2}},\rho_{\tilde{\partial}_{\omega 2}},\xi_{\tilde{\partial}_{\omega 2}}\right)\right|\le(\Delta,\Delta,\Delta,\Delta)\\
&\left|\frac{\left(\chi_{\tilde{P}_3},\tau_{\tilde{P}_3},\rho_{\tilde{P}_3},\xi_{\tilde{P}_3}\right)}{\left(\chi_{\tilde{P}_\omega},\tau_{\tilde{P}_\omega},\rho_{\tilde{P}_\omega},\xi_{\tilde{P}_\omega}\right)}-\left(\chi_{\tilde{\partial}_{\omega 3}},\tau_{\tilde{\partial}_{\omega 3}},\rho_{\tilde{\partial}_{\omega 3}},\xi_{\tilde{\partial}_{\omega 3}}\right)\right|\le(\Delta,\Delta,\Delta,\Delta)\\
&\sum_{i=1}^{3}\Re\left(\tilde{P}_i\right)=1\\
&0\le\chi_{\tilde{P}_i}\le\tau_{\tilde{P}_i}\le\rho_{\tilde{P}_i}\le\xi_{\tilde{P}_i}\le 1,\ \ i=1,2,3\\
&\Delta\ge 0
\end{aligned}
\qquad (14)
$$

The equation (14) presents an optimization problem that is nonlinear in fuzzy form. The crisp Mathematical expression of equation (14) is represented by equation (15).

$$
\begin{aligned}
&\min \Delta \\
\text{Subject to } &\chi_{\tilde{P}_\varepsilon} - 2\xi_{\tilde{P}_2} \le \Delta\xi_{\tilde{P}_2}, \chi_{\tilde{P}_\varepsilon} - 82\xi_{\tilde{P}_2} \ge -\Delta\xi_{\tilde{P}_2} \\
&\tau_{\tilde{P}_\varepsilon} - \frac{5}{2}\rho_{\tilde{P}_2} \le \Delta\rho_{\tilde{P}_2}, \tau_{\tilde{P}_\varepsilon} - \frac{5}{2}\rho_{\tilde{P}_2} \ge -\Delta\rho_{\tilde{P}_2} \\
&\rho_{\tilde{P}_\varepsilon} - \frac{7}{2}\tau_{\tilde{P}_2} \le \Delta\tau_{\tilde{P}_2}, \ \rho_{\tilde{P}_\varepsilon} - \frac{7}{2}\tau_{\tilde{P}_2} \ge -\Delta\tau_{\tilde{P}_2} \\
&\xi_{\tilde{P}_\varepsilon} - 4\chi_{\tilde{P}_2} \le \Delta\chi_{\tilde{P}_2}, \ \xi_{\tilde{P}_\varepsilon} - 4\chi_{\tilde{P}_2} \ge -\Delta\chi_{\tilde{P}_2} \\
&\chi_{\tilde{P}_\varepsilon} - 6\xi_{\tilde{P}_3} \le \Delta\xi_{\tilde{P}_3}, \ \chi_{\tilde{P}_\varepsilon} - 6\xi_{\tilde{P}_3} \ge -\Delta\xi_{\tilde{P}_3} \\
&\tau_{\tilde{P}_\varepsilon} - \frac{13}{2}\rho_{\tilde{P}_3} \le \Delta\rho_{\tilde{P}_3}, \ \tau_{\tilde{P}_\varepsilon} - \frac{13}{2}\rho_{\tilde{P}_3} \ge -\Delta\rho_{\tilde{P}_3} \\
&\rho_{\tilde{P}_\varepsilon} - \frac{15}{2}\tau_{\tilde{P}_3} \le \Delta\tau_{\tilde{P}_3}, \ \rho_{\tilde{P}_\varepsilon} - \frac{15}{2}\tau_{\tilde{P}_3} \ge -\Delta\tau_{\tilde{P}_3} \\
&\xi_{\tilde{P}_\varepsilon} - 8\chi_{\tilde{P}_3} \le \Delta\chi_{\tilde{P}_3}, \ \xi_{\tilde{P}_\varepsilon} - 8\chi_{\tilde{P}_3} \ge -\Delta\chi_{\tilde{P}_3} \\
&\chi_{\tilde{P}_2} - 4\xi_{\tilde{P}_\omega} \le \Delta\xi_{\tilde{P}_\omega}, \chi_{\tilde{P}_2} - 4\xi_{\tilde{P}_\omega} \ge -\Delta\xi_{\tilde{P}_\omega} \\
&\tau_{\tilde{P}_2} - \frac{9}{2}\rho_{\tilde{P}_\omega} \le \Delta\rho_{\tilde{P}_\omega}, \quad \tau_{\tilde{P}_2} - \frac{9}{2}\rho_{\tilde{P}_\omega} \ge -\Delta\rho_{\tilde{P}_\omega} \\
&\rho_{\tilde{P}_2} - \frac{11}{2}\tau_{\tilde{P}_\omega} \le \Delta\tau_{\tilde{P}_\omega}, \ \rho_{\tilde{P}_2} - \frac{11}{2}\tau_{\tilde{P}_\omega} \ge -\Delta\tau_{\tilde{P}_\omega} \\
&b_{\tilde{P}_2} - 6\chi_{\tilde{P}_\omega} \le \Delta\chi_{\tilde{P}_\omega}, b_{\tilde{P}_2} - 6\chi_{\tilde{P}_\omega} \ge -\Delta\chi_{\tilde{P}_\omega} \\
&\sum_{i=1}^{3}\left(\frac{\chi_{\tilde{P}_i} + 2\tau_{\tilde{P}_i} + 2\rho_{\tilde{P}_i} + \xi_{\tilde{P}_i}}{6}\right) = 1 \\
&0 \le \chi_{\tilde{P}_i} \le \tau_{\tilde{P}_i} \le \rho_{\tilde{P}_i} \le \xi_{\tilde{P}_i} \le 1, \ i = 1,2,3 \\
&\Delta \ge 0
\end{aligned}
\qquad (15)
$$

The relative relevance of each criterion is determined through PVs and the TrF-BWM approach. The optimal trapezoidal fuzzy PVs for three criteria, denoted as Γ_1, Γ_2 and Γ_3, can be evaluated utilizing the solutions to equation (15), which are $\tilde{P}^*_1$ = (0.544, 0.582, 0.660, 0.679), $\tilde{P}^*_2$ = (0.229, 0.267, 0.345, 0.373), $\tilde{P}^*_3$ = (0.075, 0.077, 0.077, 0.077), and Δ = (1.036, 1.036, 1.036, 1.036).

Subsequently, employing equation (8), the precise PVs for $\Re(\tilde{P}^*_1)$ =0.618, $\Re(\tilde{P}^*_2)$ =0.304, and $\Re(\tilde{P}^*_3)$ = 0.076 are determined. The normalised PVs for each criterion are given in equation (9) and are $\tilde{P}^*_1$=0.059, $\tilde{P}^*_2$ =0.281, and $\tilde{P}^*_3$ = 0.660.Since $\tilde{P}_{\varepsilon\wp} = \tilde{P}_{\varepsilon 3} = \left(6, \frac{13}{2}, \frac{15}{2}, 8\right)$, so CI is 12.58 in this case that gives CR = 0.544/12.58 i.e., 0.0432. As a result, the model indicates that the most significant criterion is Γ_1.

Step 3: Neutrosophic-TOPSIS Strategy under SVNS Environment (Model II):

The decision maker employs SVNNs to evaluate alternatives based on their attributes, resulting in the creation of decision **Matrix-1**.

Matrix-1: Decision Matrix

$$\nabla = \begin{array}{c} \\ M_1 \\ M_2 \\ M_3 \\ M_4 \\ M_5 \\ M_6 \end{array} \begin{array}{c} \begin{array}{ccc} \Gamma_1 & \Gamma_2 & \Gamma_3 \end{array} \\ \begin{bmatrix} (0.72,0.25,0.23) & (0.72,0.3,0.13) & (0.74,0.17,0.02) \\ (0.99,0.13,0.3) & (1,0.24,0.04) & (0.73,0.17,0.11) \\ (0.77,0.16,0.27) & (0.81,0.35,0.21) & (1,0.24,0.1) \\ (0.71,0.16,0.2) & (0.7,0.31,0.14) & (0.73,0.42,0.05) \\ (0.77,0.42,0.18) & (0.97,0.14,0.11) & (0.91,0.11,0.03) \\ (0.95,0.15,0.26) & (0.26,0.89,0.22) & (0.79,0.34,0.07) \end{bmatrix} \end{array}$$

Matrix ∇ is obtained by translating the entries of each entry. Each entry in matrix ∇ is increased by 0.01 across all components, resulting in **Matrix-2**.

Matrix-2: Translation of ∇

$$\begin{array}{c} \\ M_1 \\ M_2 \\ M_3 \\ M_4 \\ M_5 \\ M_6 \end{array} \begin{array}{c} \begin{array}{ccc} \Gamma_1 & \Gamma_2 & \Gamma_3 \end{array} \\ \begin{bmatrix} (0.73,0.26,0.24) & (0.73,0.31,0.14) & (0.75,0.18,0.03) \\ (1,0.14,0.31) & (1.01,0.25,0.05) & (0.74,0.18,0.12) \\ (0.78,0.17,0.28) & (0.82,0.36,0.22) & (1.01,0.25,0.11) \\ (0.72,0.17,0.21) & (0.71,0.32,0.15) & (0.74,0.43,0.06) \\ (0.78,0.43,0.19) & (0.98,0.15,0.12) & (0.92,0.12,0.04) \\ (0.96,0.16,0.27) & (0.9,0.23,0.03) & (0.8,0.35,0.08) \end{bmatrix} \end{array}$$

The subsequent stage involves determining the score matrix through the application of the score function. **Matrix-3** represents the score matrix denoted as ∇^*. The score value of

$$\begin{aligned} \mathrm{N}_1(0.73,0.26,0.24) &= \frac{0.73\times 0.73+0.26\times(-0.26)+0.24\times(-0.24)}{\sqrt{0.73^2+0.26^2+0.24^2}\sqrt{0.73^2+(-0.26)^2+(-0.24)^2}} \\ &= 0.61951 \end{aligned}$$

Matrix-3: Score Matrix

$$\nabla^* = \begin{array}{c} \\ M_1 \\ M_2 \\ M_3 \\ M_4 \\ M_5 \\ M_6 \end{array} \begin{array}{c} \begin{array}{ccc} \Gamma_1 & \Gamma_2 & \Gamma_3 \end{array} \\ \begin{bmatrix} 0.61951 & 0.643231 & 0.888217 \\ 0.792596 & 0.880195 & 0.842530 \\ 0.700153 & 0.581373 & 0.863706 \\ 0.753128 & 0.602861 & 0.487841 \\ 0.467084 & 0.926 & 0.962894 \\ 0.806881 & 0.875434 & 0.664715 \end{bmatrix} \end{array}$$

The Normalized Decision Matrix is determined by using equation (10) on matrix ∇^*. As shown in **Matrix-4**, the Normalized Decision Matrix decision matrix is denoted by Υ.

$$\begin{aligned} \upsilon_{11} &= \frac{0.61951}{\sqrt{0.61951^2+0.792596^2+0.700153^2+0.753128^2+0.467084^2+0.806881^2}} \\ &= 0.361389 \end{aligned}$$

Matrix-4: Decision Matrix with Normalization

$$\Upsilon = \begin{array}{c} \\ M_1 \\ M_2 \\ M_3 \\ M_4 \\ M_5 \\ M_6 \end{array} \begin{array}{c} \begin{array}{ccc} \Gamma_1 & \Gamma_2 & \Gamma_3 \end{array} \\ \begin{bmatrix} 0.361389 & 0.343146 & 0.452573 \\ 0.462358 & 0.46956 & 0.429294 \\ 0.408431 & 0.310147 & 0.440084 \\ 0.439334 & 0.32161 & 0.24857 \\ 0.272472 & 0.493996 & 0.490623 \\ 0.470691 & 0.46702 & 0.338692 \end{bmatrix} \end{array}$$

After establishing the weights of the criteria, the weighted normalized decision matrix is calculated. This involves multiplying each criteria weight by the corresponding element in its respective row of the matrix Υ. Denoted as matrix Ξ, it is represented by **Matrix-5.**

Matrix-5: Weighted Normalized Decision Matrix

$$\Xi = \begin{array}{c} \\ M_1 \\ M_2 \\ M_3 \\ M_4 \\ M_5 \\ M_6 \end{array} \begin{array}{c} \begin{array}{ccc} \Gamma_1 & \Gamma_2 & \Gamma_3 \end{array} \\ \begin{bmatrix} 0.361389\times 0.61823 & 0.343146\times 0.304791 & 0.452573\times 0.076977 \\ 0.462358\times 0.61823 & 0.46956\times 0.304791 & 0.429294\times 0.076977 \\ 0.408431\times 0.61823 & 0.310147\times 0.304791 & 0.440084\times 0.076977 \\ 0.439334\times 0.61823 & 0.32161\times 0.304791 & 0.24857\times 0.076977 \\ 0.272472\times 0.61823 & 0.493996\times 0.304791 & 0.490623\times 0.076977 \\ 0.470691\times 0.61823 & 0.46702\times 0.304791 & 0.338692\times 0.076977 \end{bmatrix} \end{array}$$

$$= \begin{array}{c} \\ M_1 \\ M_2 \\ M_3 \\ M_4 \\ M_5 \\ M_6 \end{array} \begin{array}{c} \begin{array}{ccc} \Gamma_1 & \Gamma_2 & \Gamma_3 \end{array} \\ \begin{bmatrix} 0.223422 & 0.104588 & 0.034838 \\ 0.285844 & 0.143118 & 0.033046 \\ 0.252505 & 0.09453 & 0.033877 \\ 0.27161 & 0.098024 & 0.019134 \\ 0.16845 & 0.150566 & 0.037767 \\ 0.29099 & 0.142344 & 0.026072 \end{bmatrix} \end{array}$$

Subsequently, ascertain NPIS and NNIS using the formulas $\tau^{+}_{\theta} = \max_{\theta} \tau_{\gamma\theta}, \theta = 1(1)6$ and $\tau^{-}_{\theta} = \min_{\theta} \tau_{\gamma\theta}, \theta = 1(1)6$ values, respectively.

$$\tau^{-}_{1} = \min\{0.223422, 0.285844, 0.252505, 0.27161, 0.16845, 0.290996\} = 0.16845$$

So, $\rho^{+} = \{\tau^{+}_{1}, \tau^{+}_{2}, \tau^{+}_{3}\} = \{0.290996, 0.150565, 0.037767\}$

$$\rho^{-} = \{\tau^{-}_{1}, \tau^{-}_{2}, \tau^{-}_{3}\} = \{0.16845, 0.09453, 0.019134\}$$

$$\tau^{+}_{1} = \max\{0.223422, 0.285844, 0.252505, 0.27161, 0.16845, 0.290996\} = 0.290996$$

Subsequently, calculate the distance between each alternative using NPIS and NNIS formulas (11) and (12). **Table 8** presents the distances between alternatives computed via NPIS and NNIS.

$$\begin{aligned}\hbar_1^+ &= \sqrt{\left(\tau_{11}-\rho_1^+\right)^2+\left(\tau_{12}-\rho_2^+\right)^2+\left(\tau_{13}-\rho_3^+\right)^2}\\ &= \sqrt{\left(0.223422-0.290996\right)^2+\left(0.104588-0.150565\right)^2+\left(0.034838-0.037767\right)^2}\\ &= 0.081785\end{aligned}$$

$$\begin{aligned}\hbar_1^- &= \sqrt{\left(\tau_{11}-\rho_1^-\right)^2+\left(\tau_{12}-\rho_2^-\right)^2+\left(\tau_{13}-\rho_3^-\right)^2}\\ &= \sqrt{\left(0.223422-0.16845\right)^2+\left(0.104588-0.09453\right)^2+\left(0.034838-0.019134\right)^2}\\ &= 0.058048\end{aligned}$$

Table 8: NPIS and NNIS distances from each alternative.

	Value		**Value**
$\hbar_1^+$	0.081785	$\hbar_1^-$	0.058048
$\hbar_2^+$	0.010213	$\hbar_2^-$	0.12781
$\hbar_3^+$	0.068093	$\hbar_3^-$	0.085338
$\hbar_4^+$	0.059022	$\hbar_4^-$	0.103219
$\hbar_5^+$	0.122545	$\hbar_5^-$	0.059052
$\hbar_6^+$	0.014296	$\hbar_6^-$	0.131725

Compute the performance score for each alternative utilizing formula (13). **Table 9** illustrates the performance scores for all alternatives. Arrange the alternatives in increasing order based on their performance scores and assign ranks correspondingly.

$$\begin{aligned}\aleph_1 &= \hbar_1^- \Big/ \left(\hbar_1^+ + \hbar_1^-\right)\\ &= 0.058048 \Big/ \left(0.081785 + 0.058048\right)\\ &= 0.415126\end{aligned}$$

Table 9: Performance scores of alternative.

Alternatives	Performance scores	Rank
M_1	$\aleph_1 = 0.415126$	5
M_2	$\aleph_2 = 0.926008$	1
M_3	$\aleph_3 = 0.556197$	4
M_4	$\aleph_4 = 0.636206$	3
M_5	$\aleph_5 = 0.325182$	6
M_6	$\aleph_6 = 0.902095$	2

4.2 Result from comparative study

Experts have been consulted to ascertain the vectors suitable for employment in the BWM-Neutrosophic-TOPSIS Strategy under the SVNS Environment, as well as the most and least significant aspects. Through expert consensus, it has been determined that Γ_1 holds the highest significance, whereas, Γ_3 is regarded as the least significant criterion. The best-to-others vector is outlined in **Table 10** and the worst-to-others vector is delineated in **Table 11**.

Table 10: Comparison of best criteria with other criteria.

	Γ_1	Γ_2	Γ_3
Γ_1 (Best Criteria)	1	2	6

Table 11: To the worst criteria, there are other criteria.

	Γ_3 (Worst Criteria)
Γ_1	6
Γ_2	4
Γ_3	1

The nonlinear mathematical model (16) can also be used to determine the weight of each criterion.

$$\min \Delta$$

$$s.t. \left|\frac{P_1}{P_2} - 2\right| < \Delta$$

$$\left|\frac{P_1}{P_3} - 6\right| < \Delta$$

$$\left|\frac{P_2}{P_3} - 4\right| < \Delta \qquad (16)$$

$$\sum_{j=1}^{3} P_j = 1$$

$$P_j \geq 0, \text{ for all } j = 1(1)3$$

Utilizing the equation (10) on matrix ∇^*, the Normalized Decision Matrix using can be determined. As shown in **Matrix-6**, the Normalized Decision Matrix decision matrix is denoted by Υ.

Matrix-6: Normalized Decision Matrix

$$\Upsilon = \begin{array}{c} \\ M_1 \\ M_2 \\ M_3 \\ M_4 \\ M_5 \\ M_6 \end{array} \begin{array}{c} \begin{array}{ccc} \Gamma_1 & \Gamma_2 & \Gamma_3 \end{array} \\ \begin{bmatrix} 0.361389 & 0.343146 & 0.452573 \\ 0.462358 & 0.46956 & 0.429294 \\ 0.408431 & 0.310147 & 0.440084 \\ 0.439334 & 0.32161 & 0.24857 \\ 0.272472 & 0.493996 & 0.490623 \\ 0.470691 & 0.46702 & 0.338692 \end{bmatrix} \end{array}$$

After the criteria weights have been established, next step is to determine the weighted normalized decision matrix. This matrix is created by multiplying each criterion weight by the corresponding element in the associated matrix Υ row. This resulting weighted normalized decision matrix (**Matrix-7**) is denoted by matrix Ξ.

Matrix-7: Weighted Normalized Decision Matrix

$$\Xi = \begin{array}{c} \\ M_1 \\ M_2 \\ M_3 \\ M_4 \\ M_5 \\ M_6 \end{array} \begin{array}{c} \begin{array}{ccc} \Gamma_1 & \Gamma_2 & \Gamma_3 \end{array} \\ \begin{bmatrix} 0.213548 & 0.109183 & 0.041143 \\ 0.273212 & 0.149405 & 0.039027 \\ 0.241346 & 0.098683 & 0.040008 \\ 0.259607 & 0.102331 & 0.022597 \\ 0.161006 & 0.15718 & 0.044602 \\ 0.278136 & 0.148597 & 0.03079 \end{bmatrix} \end{array}$$

Next step is to identify NPIS and NNIS using the formulas $\tau^+_\theta = \max_\theta \tau_{\gamma\theta}, \theta = 1(1)6$ and $\tau^-_\theta = \min_\theta \tau_{\gamma\theta}, \theta = 1(1)6$ values, respectively.

So, $\rho^+ = \left\{\tau^+_1, \tau^+_2, \tau^+_3\right\} = \{0.278135, 0.15718, 0.044602\}$

$\rho^- = \left\{\tau^-_1, \tau^-_2, \tau^-_3\right\} = \{0.161006, 0.098683, 0.0225972\}$

Next, use formulae (11) and (12) to determine each alternative's distance from both NPIS and NNIS. **Table 12** shows the distances for the weights of each alternative from NPIS and NNIS.

Table 12: Distances of each alternative between NPIS as well as NNIS.

	Value		Value
$\hbar_1^-$	0.080544	$\hbar_1^-$	0.0567
$\hbar_2^-$	0.01076	$\hbar_2^-$	0.124229
$\hbar_3^-$	0.069257	$\hbar_3^-$	0.082205
$\hbar_4^-$	0.061936	$\hbar_4^-$	0.098668
$\hbar_5^-$	0.11713	$\hbar_5^-$	0.062499
$\hbar_6^-$	0.016262	$\hbar_6^-$	0.127585

Use formula (13) to determine the evaluation score for each option. The performance scores for each alternative by proposed method and current method are shown in **Figure 4**.

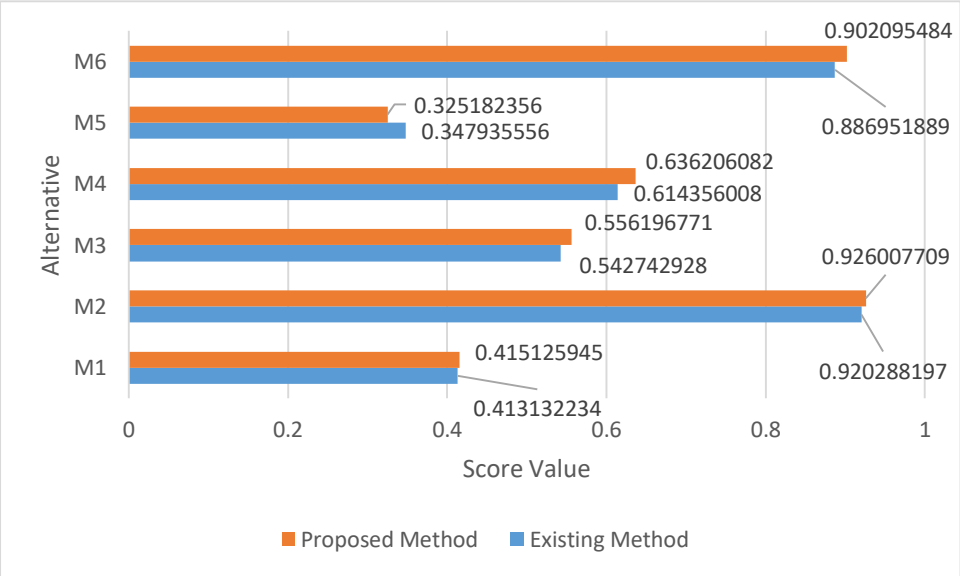

Figure 4: Comparative study.

4.3 Statistical Analysis

The rankings generated by the two methods can be compared using the Spearman correlation coefficient, which quantifies the linear association between two variables. This coefficient ranges from -1 to 1: -1 indicates no linear correlation, 0 signifies no linear correlation, and 1 denotes the complete absence of linear correlation. To assess the relationship between variables on interval scales, Pearson's correlation coefficient [56] can be utilized, as illustrated in equation (17).

$$\rho(\varpi,\iota)=\frac{\operatorname{cov}(\varpi,\iota)}{\sigma_{\varpi}\sigma_{\iota}} \qquad (17)$$

γ as well as ς are covariant in $\operatorname{cov}(\gamma,\varsigma)$. SD is represented by γ as well as ς in both σ_{γ} as well as σ_{ς}.

The Pearson correlation coefficient is important for assessing the absence of a perfect correlation between two variables when it deviates from a value of 1, as given in equation (18).

$$\left.\begin{array}{l} H_0 : -\infty < \rho \le 0 \\ H_1 : 0 < \rho < \infty \end{array}\right\} \qquad 18)$$

Pearson correlation coefficient alongside Student's *t-distribution* with degrees of freedom -1, is presented by equation (19).

$$t = \rho\left(\frac{-2}{1-\rho^2}\right)^{\frac{1}{2}} \qquad (19)$$

The null hypothesis should be rejected if t (equation (19)) suppresses $t_\alpha(-2)$. The Pearson correlation coefficient ρ falls within the range of .

Both methods produce rankings, which are evaluated using the Spearman correlation coefficient. If the rankings are identical, resulting a Spearman correlation coefficient of 1, there is no need for further hypothesis testing. However, if the rankings differ, a hypothesis test can be conducted to validate the Spearman correlation coefficients, as detailed in equation (17). To compare the proposed approach with BWM-TOPSIS strategy [44] under SVNS environment weights, the analysis will utilize the Pearson correlation coefficient, as outlined in **Table 13**. Notably, there is a discernible correlation between the proposed PV models and the currently established PV models, as supported by $\alpha = 0.05$ [58]. The analysis findings suggest that $t = \rho\left(\frac{-2}{1-\rho^2}\right)^{\frac{1}{2}}$ should outperform $t_{0.05}(-2)$. A hypothesis test is carried out to confirm a positive correlation between the attributes of the existing and proposed methods.

Table 13: t-Test: Paired Two Sample for Means.

	Proposed Method (Variable 1)	**Existing Method (Variable 2)**
Mean	0.626802	0.620901
Variance	0.061233	0.056849
Observations ρ	6	6
Pearson Correlation	0.9986	
Hypothesized Mean Difference α	0	
Df	5	
t Stat	0.920422	
P(T<=t) one-tail	0.199791	
t Critical one-tail	2.015048	
P(T<=t) two-tail	0.399583	
t Critical two-tail	2.570582	

4.4 Sensitivity Analysis

To explore the effect of altering weight coefficient values, different scenarios are examined, each consisting of a distinct set. Sensitivity analysis is employed to assess the predominant criterion using equation (20) and determine the sensitivity of the criterion PVs ($\Re(\tilde{P}_{\wp})$). Additionally, the progression of the dominant criterion is investigated.

$$\tilde{P}'_{\wp} = \tilde{P}_{\wp}\left(\frac{1-r\tilde{P}_{\varepsilon}}{1-\tilde{P}_{\varepsilon}}\right),\ \wp = 1(1)3 \qquad (20)$$

When $P'_{\wp}$ represents the initial value of the criterion, denoted, $\tilde{P}_{\wp}(\wp = 1(1)\Psi)$; $\tilde{P}_{\varepsilon}$, it signifies the criterion's starting point. $r_{\wp} \in (0,1) \cup \{0,1\}$, on the other hand, represents the adjusted value.

Equation (20) is employed to generate 25 distinct scenarios for this study. Across these scenarios, the variable r has the capacity to take on random values ranging from 0 to 1. **Figure 5** and **6** illustrate the outcomes of the sensitivity analysis conducted seperately for each alternative of $\hbar_{\gamma}^{+}$ and $\hbar_{\gamma}^{-}$. **Figure 7** shows the results of the sensitivity analysis. As shown in the **Figure 7**, "Temperature" is the most sensitive parameter in each case.

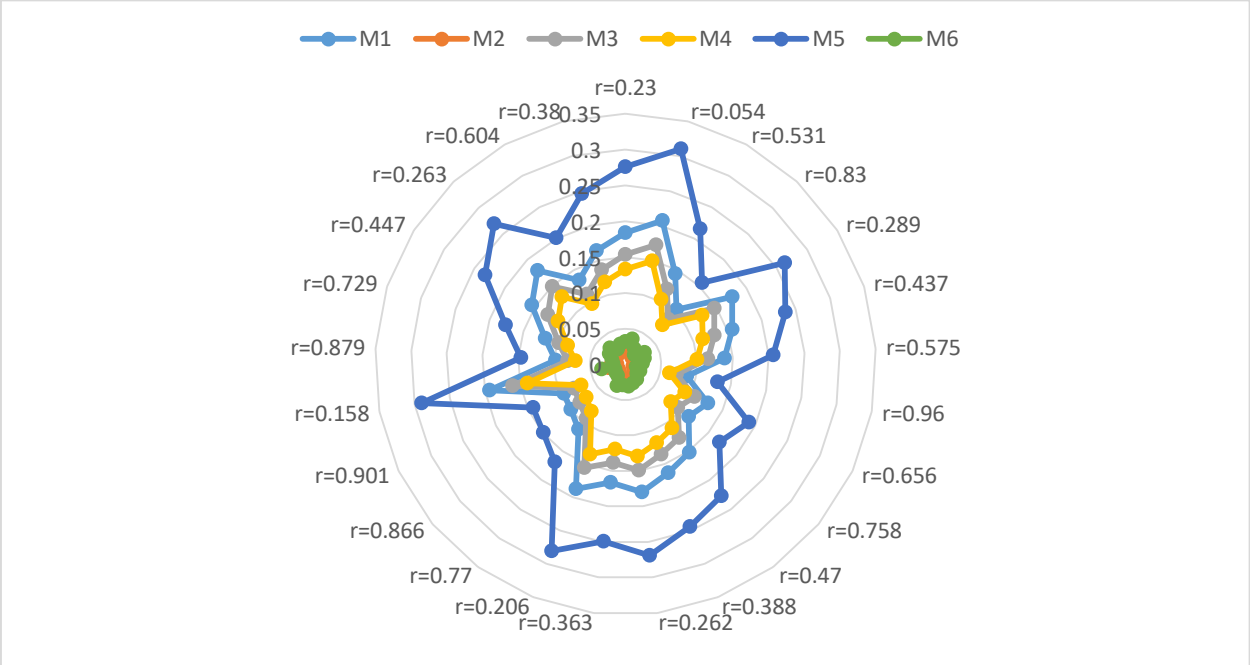

Figure 5: Results of the sensitivity analysis for $\hbar_{\gamma}^{+}$.

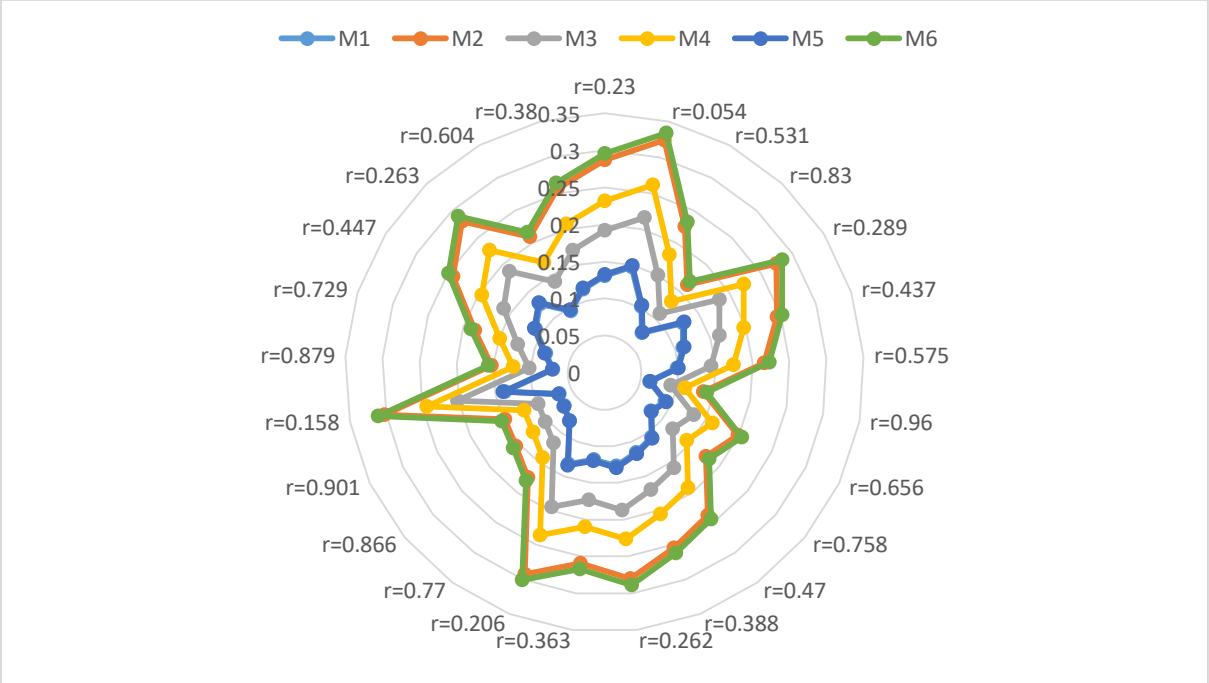

Figure 6: Results of the sensitivity analysis for $\hbar_{\gamma}^{-}$.

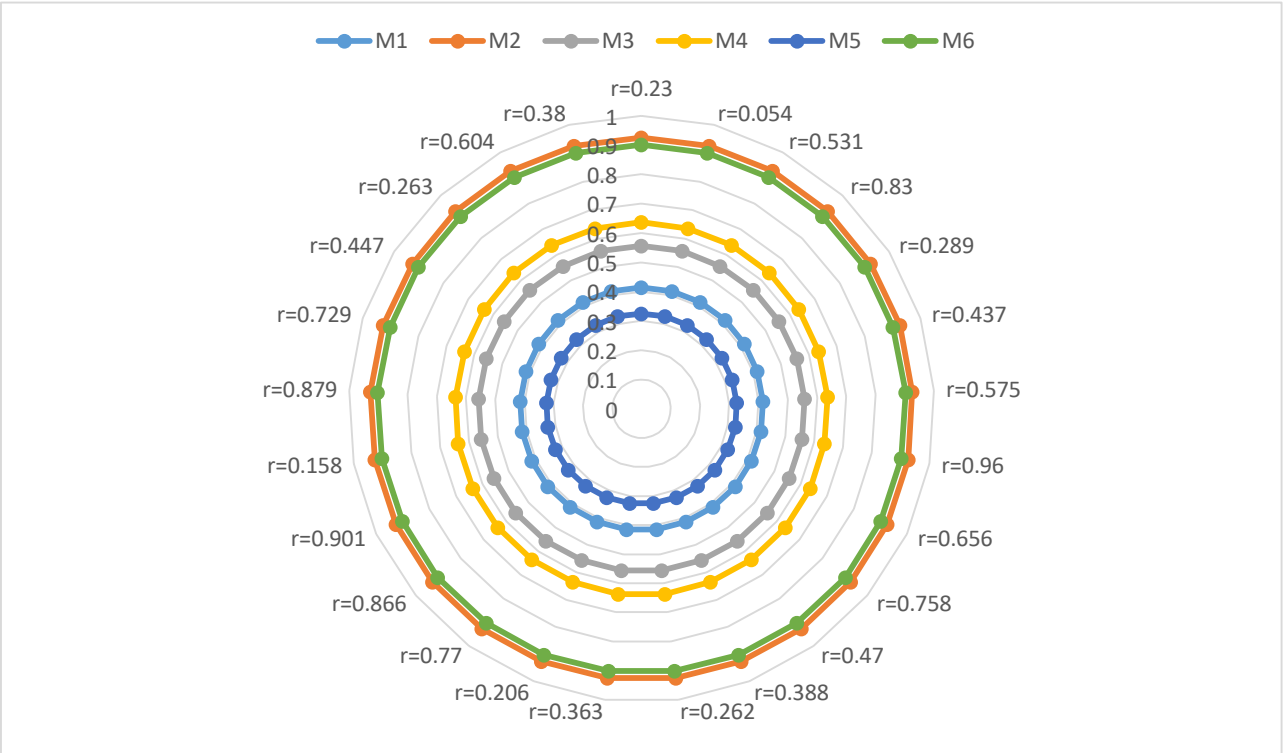

Figure 7: Results of the sensitivity analysis.

5. Conclusions

The objective of the present study was to enhance a hybrid MCDM method namely TrF-BWM-Neutrosophic-TOPSIS Strategy under SVNS Environment Approach. The method was applied to

identify the most significant water quality parameter for fish growth within the aquaponic system. For this, the study selected three criteria and six alternatives through literature reviews. According to the results of the proposed method "Temperature" is the most significant factor, followed by "Dissolved oxygen".

The result was validated by comparing the proposed method with the BWM- Neutrosophic-TOPSIS Strategy under SVNS Environment Approach. The results of the proposed method, supported by the existing MCDM technique, are for the majority of significant alternatives. Statistical support is also provided for the results of the proposed method. Also, the model was validated by using Sensitivity Analysis and Scenario Analysis.

The current paradigm's heavy dependence on the method of development, as well as its features, is one of its main drawbacks. The results of the mathematical framework can shift when fresh variables or different approaches are used. Moreover, this method can be extended to aid in the selection of the most suitable fish species for aquaponic system, aiming to maximize overall system production.

In a nutshell, by pinpointing the primary factor driving fish growth, aquaponics practitioners can focus their efforts and resources on maintaining and optimizing that particular parameter. This not only simplifies system management but also carries the potential to boost fish yields, enhance economic returns, and advance environmental sustainability in aquaponic food production.

Conflict of Interest: There authors have no conflicts of interest to declare.

Authors' Contribution: Each author has made an equal contribution in developing the paper.

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Received: March 6, 2024. Accepted: July 28, 2024

Neutrosophic Sets and Systems, Vol. 70, 2024
University of New Mexico

Harmonic Mean Operator Based MADM-Strategy under SVPN-Set Environment

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Abstract: By using single-valued pentapartitioned neutrosophic set (briefly SVPN-set), one can express uncertainty events having incomplete, indeterminate, and erroneous information in practical applications. This study's primary goal is to present a multi-attribute decision-making (briefly MADM) algorithm that operates in the SVPN-set environment using the notion of single-valued pentapartitioned neutrosophic harmonic mean (briefly SVPNHM) operator and single-valued pentapartitioned neutrosophic weighted harmonic mean (briefly SVPNWHM) operator. Further, we provide a numerical illustration here like, "Location Selection for Installation of Outdoor Air Purifier" to validate the effectiveness and rationality of the proposed MADM algorithm.

Keywords: SVPN-set; Indeterminacy; MADM; SVPNHM-operator; SVPNWHM-operator.

1. Introduction:

During 1998, Smarandache [28] presented the idea of neutrosophic set (briefly N-set) and neutrosophic number (briefly N-number), which address uncertain and deficient data in another manner. Two components make up an N-number: a determinate part and an indeterminate part. In this manner, the N-numbers are more effective to manage uncertain and deficient data in certifiable issues. Afterwards, Wang et al. [29] grounded and studied single-valued N-set (briefly SVN-set). MADM and multi-attribute group decision-making (briefly MAGDM) are two critical parts of choice speculations which have been ordinarily applied in numerous logical fields. Till now, many researchers around the globe used SVN-set in their theoretical research [10, 11, 14, 17] and also

proposed several MADM and MAGDM techniques [3, 9, 18, 20-23, 26-28] under the SVN-set environment. In the year 2009, Xu [30] presented the notion of harmonic mean operator under the fuzzy environment. Afterwards, Aydin et al. [2] developed a harmonic aggregation operator under trapezoidal Pythagorean fuzzy number environment. A MAGDM approach in fuzzy environment was suggested by Park et al. [25] using generalized fuzzy bonferroni harmonic mean operator. Later, in an intuitionistic fuzzy environment, Zhou et al. [32] presented a MADM method using the normalized weighted bonferroni harmonic mean operator. Thereafter, Mondal et al. [22] proposed a MAGDM strategy based on normalized neutrosophic harmonic mean operator under the SVN-set environment. The principles of SVPN-set and single-valued pentapartitioned neutrosophic number (briefly SVPN-number) were initially introduced in 2020 by Mallick and Pramanik [19]. In the context of the SVPN-set environment, Das et al. [6] came up with a tangent similarity based MADM approach. Afterwards, Das et al. [5] established a MADM algorithm using grey relational analysis method in the context of SVPN-set environment.

The main goal of this research is to demonstrate a novel MADM methodology using the on the SVPNWHM operator. Furthermore, we validate our MADM methodology by providing a real life example, namely "Location Selection for Installation of Outdoor Air Purifier".

The rest of this paper is organized in the following manner:

A few preliminary results on SVPN-set as well as score and accuracy function of SVPN-numbers are shown in section-2. In section-3, two aggregation operators namely, SVPNHM operator and SVPNWHM operator have been introduced. Besides, a unique MADM methodology based on the SVPNWHM operator is presented in section-4. The suggested MADM methodology is demonstrated using a numerical example in section-5, and the outcomes demonstrate their viability and efficacy. In section-6, we provide some closing thoughts and outline some potential areas for further investigation.

2. Preliminaries and Definitions:

The idea of SVPN-set theory was first grounded in 2020 by Mallick and Pramanik [19]. They achieved this by dividing the indeterminacy-membership into three distinct categories: contraction, ignorance and unknown-memberships. Below are the concepts of SVPN-set and SVPN-number.

Definition 2.1.[19] Consider Ü be a non-null set. Then *H*, a SVPN-set over Ü is defined by

$$Đ=\{(ể, Ť_Đ(ể), Č_Đ(ể), Ğ_Đ(ể), Ŭ_Đ(ể), Ň_Đ(ể)) : ể∈Ü\},$$

where $Ť_Đ(ể)$, $Č_Đ(ể)$, $Ğ_Đ(ể)$,$Ŭ_Đ(ể)$ and $Ň_Đ(ể)$ are the truth, contradiction, ignorance, unknown and falsity membership function lies in the interval [0,1] such that

$$0 \le Ť_Đ(ể) + Č_Đ(ể) + Ğ_Đ(ể) + Ŭ_Đ(ể) + Ň_Đ(ể) \le 5, \ \forall ể∈Ü.$$

For each $ể∈Ü$, $[Ť_Đ(ể), Č_Đ(ể), Ğ_Đ(ể), Ŭ_Đ(ể), Ň_Đ(ể)]$ is referred to as a SVPN-number.

Definition 2.2.[19] Assume that Ü be a non-empty set. Suppose that Đ={(ê̆, Ť$_Đ$(ê̆), Č$_Đ$(ê̆), Ğ$_Đ$(ê̆), Ŭ$_Đ$(ê̆), Ň$_Đ$(ê̆)): ê̆∈Ü} and Q={(ê̆, Ť$_Q$(ê̆), Č$_Q$(ê̆), Ğ$_Q$(ê̆), Ŭ$_Q$(ê̆), Ň$_Q$(ê̆)): ê̆∈Ü} be two SVPN-sets defined over Ü. Then, *Đ*⊆*Q* if and only if Ť$_Đ$(ê̆)≤ Ť$_Q$(ê̆), Č$_Đ$(ê̆)≤Č$_Q$(ê̆), Ğ$_Đ$(ê̆)≥Ğ$_Q$(ê̆), Ŭ$_Đ$(ê̆)≥Ŭ$_Q$(ê̆), Ň$_Đ$(ê̆)≥Ň$_Q$(ê̆), ∀ ê̆∈Ü.

Definition 2.3.[19] Assume that Ü be a non-empty set. Suppose that Đ={(ê̆, Ť$_Đ$(ê̆), Č$_Đ$(ê̆), Ğ$_Đ$(ê̆), Ŭ$_Đ$(ê̆), Ň$_Đ$(ê̆)): ê̆∈Ü} and Q={(ê̆, Ť$_Q$(ê̆), Č$_Q$(ê̆), Ğ$_Q$(ê̆), Ŭ$_Q$(ê̆), Ň$_Q$(ê̆)): ê̆∈Ü} be two SVPN-sets defined over Ü. Then, their intersection is defined by *Đ*∩*Q* = {(ê̆, min{Ť$_Đ$(ê̆),Ť$_Q$(ê̆)}, min{Č$_Đ$(ê̆),Č$_Q$(ê̆)}, max{Ğ$_Đ$(ê̆),Ğ$_Q$(ê̆)}, max{Ŭ$_Đ$(ê̆),Ŭ$_Q$(ê̆)}, max{Ň$_Đ$(ê̆),Ň$_Q$(ê̆)}): ê̆∈Ü}.

Definition 2.4.[19] Assume that Ü be a non-empty set. Suppose that Đ={(ê̆, Ť$_Đ$(ê̆), Č$_Đ$(ê̆), Ğ$_Đ$(ê̆), Ŭ$_Đ$(ê̆), Ň$_Đ$(ê̆)): ê̆∈Ü} and Q={(ê̆, Ť$_Q$(ê̆), Č$_Q$(ê̆), Ğ$_Q$(ê̆), Ŭ$_Q$(ê̆), Ň$_Q$(ê̆)): ê̆∈Ü} be two SVPN-sets defined over Ü. Then, their union is defined by *Đ*∪*Q* = {(ê̆, max{Ť$_Đ$(ê̆),Ť$_Q$(ê̆)}, max{Č$_Đ$(ê̆),Č$_Q$(ê̆)}, min{Ğ$_Đ$(ê̆),Ğ$_Q$(ê̆)}, min{Ŭ$_Đ$(ê̆),Ŭ$_Q$(ê̆)}, min{Ň$_Đ$(ê̆),Ň$_Q$(ê̆)}): ê̆∈Ü}.

Definition 2.5.[19] Assume that Ü be a non-empty set. Suppose that Đ={(ê̆, Ť$_Đ$(ê̆), Č$_Đ$(ê̆), Ğ$_Đ$(ê̆), Ŭ$_Đ$(ê̆), Ň$_Đ$(ê̆)): ê̆∈Ü} be a SVPN-set defined over Ü. Then, the complement of *Đ* is *Đ*^{c} = {(ê̆, Ň$_Đ$(ê̆), Ŭ$_Đ$(ê̆), 1-Ğ$_Đ$(ê̆), Č$_Đ$(ê̆), Ť$_Đ$(ê̆)): ê̆ ∈Ü}.

Definition 2.6.[28] Assume that ê̆=[Ť$_Đ$(ê̆), Č$_Đ$(ê̆),Ğ$_Đ$(ê̆),Ŭ$_Đ$(ê̆), Ň$_Đ$(ê̆)] and ή̆=[Ť$_Đ$(ή̆), Č$_Đ$(ή̆),Ğ$_Đ$(ή̆),Ŭ$_Đ$(ή̆), Ň$_Đ$(ή̆)] be two different SVPN-numbers over Ü. Then,

(*i*) ê̆⊆ή̆ if and only if Ť$_Đ$(ê̆) ≤Ť$_Đ$(ή̆), Č$_Đ$(ê̆) ≤Č$_Đ$(ή̆), Ğ$_Đ$(ê̆) ≥Ğ$_Đ$(ή̆), Ŭ$_Đ$(ê̆) ≥Ŭ$_Đ$(ή̆), Ň$_Đ$(ê̆) ≥Ň$_Đ$(ή̆);

(*ii*) ê̆ = ή̆ if and only if Ť$_Đ$(ê̆) = Ť$_Đ$(ή̆), Č$_Đ$(ê̆) = Č$_Đ$(ê̆), Ğ$_Đ$(ê̆) = Ğ$_Đ$(ή̆), Ŭ$_Đ$(ê̆) = Ŭ$_Đ$(ή̆), Ň$_Đ$(ê̆) = Ň$_Đ$(ή̆);

(*iii*) μ. ê̆ = [1-(1-Ť$_Đ$(ê̆))^μ, 1-(1-Č$_Đ$(ê̆))^μ, (Ğ$_Đ$(ê̆))^μ, (Ŭ$_Đ$(ê̆))^μ, (Ň$_Đ$(ê̆))^μ], where μ>0;

(i*v*) ê̆^{μ} = [(Ť$_Đ$(ê̆))^μ, (Č$_Đ$(ê̆))^μ, 1-(1-Ğ$_Đ$(ê̆))^μ, 1-(1-Ŭ$_Đ$(ê̆))^μ, 1-(1-Ň$_Đ$(ê̆))^μ], where μ>0;

(*v*) ê̆+ή̆ = [Ť$_Đ$(ê̆)+ Ť$_Đ$(ή̆)-Ť$_Đ$(ê̆).Ť$_Đ$(ή̆), Č$_Đ$(ê̆)+C$_Đ$(ή̆)-Č$_Đ$(ê̆).Č$_Đ$(ή̆), Ğ$_Đ$(ê̆).Ğ$_Đ$(ή̆), Ŭ$_Đ$(ê̆).Ŭ$_Đ$(ή̆), Ň$_Đ$(ê̆).Ň$_Đ$(ή̆)];

(*vi*) ê̆. ή̆ = [Ť$_Đ$(ê̆).Ť$_Đ$(ή̆), Č$_Đ$(ê̆).C$_Đ$(ή̆), Ğ$_Đ$(ê̆)+Ğ$_Đ$(ή̆)-Ğ$_Đ$(ê̆).Ğ$_Đ$(ή̆), Ŭ$_Đ$(ê̆)+Ŭ$_Đ$(ή̆)-Ŭ$_Đ$(ê̆).Ŭ$_Đ$(ή̆), Ň$_Đ$(ê̆)+ Ň$_Đ$(ή̆)-Ň$_Đ$(ê̆).Ň$_Đ$(ή̆)].

Definition 2.7. [28] Suppose that ê̆=[Ť$_Đ$(ê̆),Č$_Đ$(ê̆),Ğ$_Đ$(ê̆),Ŭ$_Đ$(ê̆),Ň$_Đ$(ê̆)] be a SVPN-number over Ü. Then,

(*i*) score function is defined as follows:

$$S(\breve{e}) = \frac{[\breve{T}_Đ(\breve{e})+\breve{C}_Đ(\breve{e})+1-\breve{G}_Đ(\breve{e})+1-\breve{U}_Đ(\breve{e})+1-\breve{N}_Đ(\breve{e})]}{5} \quad (1)$$

(*ii*) accuracy function is defined as follows:

$$A(\breve{e}) = \frac{[\breve{T}_Đ(\breve{e})+\breve{C}_Đ(\breve{e})-\breve{N}_Đ(\breve{e})]}{3} \quad (2)$$

Definition 2.8.[28] Let ê̆=[Ť$_Đ$(ê̆), Č$_Đ$(ê̆),Ğ$_Đ$(ê̆),Ŭ$_Đ$(ê̆), Ň$_Đ$(ê̆)] and ή̆=[Ť$_Đ$(ή̆), Č$_Đ$(ή̆),Ğ$_Đ$(ή̆),Ŭ$_Đ$(ή̆), Ň$_Đ$(ή̆)] be two SVPN-numbers defined over Ü. Then,

(*i*) *S*(ê̆) > *S*(ή̆) ⇒ ê̆ >ή̆;

(*ii*) *S*(ê̆) = *S*(ή̆), *A*(ê̆) > *A*(ή̆) ⇒ ê̆>ή̆;

(*iii*) *S*(ê̆) = *S*(ή̆), *A*(ê̆) = *A*(ή̆), T$_K$(ê̆)>T$_K$(ή̆) ⇒ ê̆>ή̆.

Proposition 2.1.[28] The accuracy and score functions of a SVPN-number are bounded.

Proposition 2.2.[28] The accuracy and score functions of a SVPN-number are monotone.

3. Single-Valued Pentapartitioned Neutrosophic Harmonic Mean Operator:

Here, we introduce the notions of SVPNHM operator and SVPNWHM operator under the SVPN-set environment.

Definition 3.1. Suppose that $\hat{e}_t=[\check{T}_K(\hat{e}_t),\check{C}_K(\hat{e}_t),\check{G}_K(\hat{e}_t),\check{U}_K(\hat{e}_t),\check{N}_K(\hat{e}_t)]$ (t=1, 2, 3, …., y) be the family of SVPN-numbers over Ü. Then, the SVPNHM operator is defined as follows:

$$\text{SVPNHM } (\hat{e}_1, \hat{e}_2,\ldots,\hat{e}_y) = \frac{y}{\sum_{t=1}^{y}\frac{1}{\hat{e}_t}}, \tag{3}$$

where $\frac{1}{v_t}$ = [1/(Ť$_R$($\hat{e}$)+1), 1/(Č$_R$($\hat{e}$)+1), 1/(Ğ$_R$($\hat{e}$)+1), 1/(Ŭ$_R$($\hat{e}$)+1), 1/(Ň$_R$($\hat{e}$)+1)] (4)

Example 3.1. Let u = (0.6, 0.2, 0.6, 0.2, 0.7) and v = (0.7, 0.2, 0.1, 0.4, 0.5) be two SVPN-numbers over Ü. Then, SVPNHM (u, v) = $2.\frac{1}{(\frac{1}{u}+\frac{1}{v})}$ = (0.86, 0.79, 0.27, 0.34, 0.36), which is also a SVPN-number.

Definition 3.2. Suppose that $\hat{e}_t=[\check{T}_K(\hat{e}_t),\check{C}_K(\hat{e}_t),\check{G}_K(\hat{e}_t),\check{U}_K(\hat{e}_t),\check{N}_K(\hat{e}_t)]$ (t=1, 2, 3, …., y) be the family of SVPN-numbers over Ü. Then, the SVPNWHM operator is defined as follows:

$$\text{SVPNWHM } (\hat{e}_1, \hat{e}_2,\ldots,\hat{e}_y) = \frac{y}{\sum_{t=1}^{y}\frac{\delta_t}{\hat{e}_t}}, \tag{5}$$

where δ_t is the corresponding weight of $\hat{e}_t$ (t=1,2,…, y), such that $\delta_t \in [0,1]$ and $\sum_{t=1}^{y}\delta_t$ =1.

Example 3.2. Let u=(0.8, 0.2, 0.7, 0.4, 0.3) and v=(0.6, 0.5, 0.4, 0.3, 0.2) be two SVPN-numbers over Ü. Let δ_1=0.5 and δ_2=0.4 be the corresponding weighted value of u and v.

Then, SVPNWHM (u, v) = $2.\frac{1}{(\frac{\delta_1}{u}+\frac{\delta_2}{v})}$ = $2.\frac{1}{(\delta_1.\ (\frac{1}{u})+\ \delta_2.\ (\frac{1}{v}))}$ = (0.32, 0.34, 0.81, 0.85, 0.80), which is also a SVPN-number.

4. SVPNWHM Operator Based MADM Strategy under the SVPN-set Environment:

This section developed a MADM technique using SVPNWHM operator under the SVPN-set environment.

In a MADM problem, suppose that, Θ be the set of n alternatives namely Θ$_1$, Θ$_2$,…, Θ$_n$, and Á be the set of m attributes namely Á$_1$, Á$_2$,…, Á$_m$. Assume that, the decision maker gives his/her evaluation information for each alternative Θ$_i$ (i=1, 2, …, n) corresponding to the attributes Á$_j$ (j=1, 2, ……, m) by using the SVPN-sets as follows:

$$\Theta_i=\{(\acute{A}_j,\check{T}_{\Theta_i}(\acute{A}_j),\check{C}_{\Theta_i}(\acute{A}_j),\check{G}_{\Theta_i}(\acute{A}_j),\check{U}_{\Theta_i}(\acute{A}_j),\check{N}_{\Theta_i}(\acute{A}_j)):\ \acute{A}_j\in\acute{A}\},$$

where $0\le\check{T}_{\Theta_i}(\acute{A}_j)+\ C_{\Theta_i}(\acute{A}_j)+\ \check{G}_{\Theta_i}(\acute{A}_j)+\ \check{U}_{\Theta_i}(\acute{A}_j)+\ \check{N}_{\Theta_i}(\acute{A}_j)\le 5$, for each i=1,2,…, n, and j=1,2,….., m.

For convenience,

$\Theta_{ij}=(\check{T}_{ij},\check{C}_{ij},\check{G}_{ij},\check{U}_{ij},\check{N}_{ij})$ denotes the SVPN-number $(\check{T}_{\Theta_i}(\acute{A}_j),\check{C}_{\Theta_i}(\acute{A}_j),\check{G}_{\Theta_i}(\acute{A}_j),\check{U}_{\Theta_i}(\acute{A}_j),\check{N}_{\Theta_i}(\acute{A}_j))$ in the SVPN-set $\Theta_i(i=1,2,\ldots,n)$.

Then, the single-valued pentapartitioned neutrosophic decision matrix is denoted by $D=\left(A_{ij}\right)_{n\times m}$.

Our suggested MADM methodology consists of the subsequent steps:

Step-1: ***Start***

Step-2: ***Formation of the Decision Matrix***

By utilizing the assessment data of each alternative against each attribute, the decision maker constructed the decision matrix in this step.

Step-3: ***Calculate the Weights of Each Attribute***

In this step, the decision maker can calculate the weight of each attribute by using the compromise function, which is described as follows:

<u>Compromise Function:</u>

For each attribute $\hat{w}_j(j=1, 2, \ldots,m)$, the compromise function is defined by:

$$\Omega_j=\sum_{i=1}^{n}(3+\check{T}_{\Theta_i}(\acute{A}_j)+\check{C}_{\Theta_i}(\acute{A}_j)-\check{G}_{\Theta_i}(\acute{A}_j)-\check{U}_{\Theta_i}(\acute{A}_j)-\check{N}_{\Theta_i}(\acute{A}_j))/5 \quad (6)$$

The j^{th} attribute's weights are then determined as follows:

$$\hat{w}_j=\frac{\Omega_j}{\sum_{j=1}^{m}\Omega_j} \quad (7)$$

Here, $\sum_{j=1}^{m}\hat{w}_j=1$.

Step-4. ***Calculation of Aggregate Value using the SVPNWHM Operator***

In this stage, the decision maker can use eq. (4) and eq. (5) to calculate the aggregate value of all the attributes with respect to the corresponding alternative.

Step-5. ***Determine the Score Value of Each Alternative with respect to the Aggregate Value***

Using equations (1) and (2), the decision maker calculates each alternative's score and accuracy value in relation to the aggregate values of all alternatives in this stage.

Step-6. ***Ranking of the Alternatives***

In this stage, the decision-maker uses Definition 2.8 to assign a rank to each choice.

Step-7. ***End***

The flow chart of our proposed MADM strategy was shown by the following flow chart:

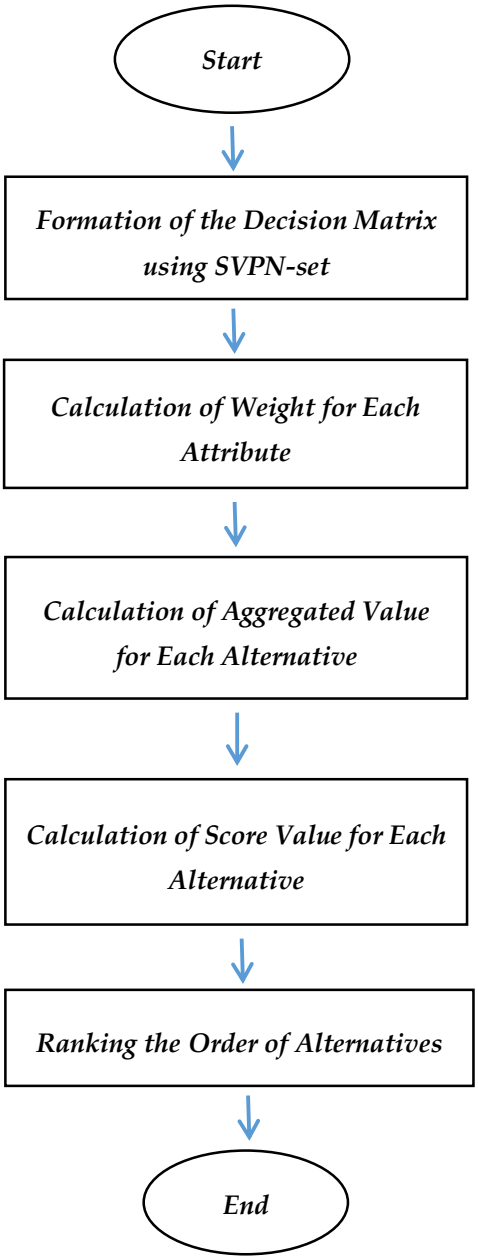
Start
Formation of the Decision Matrix using SVPN-set
Calculation of Weight for Each Attribute
Calculation of Aggregated Value for Each Alternative
Calculation of Score Value for Each Alternative
Ranking the Order of Alternatives
End

Figure-1

5. Validation of the Proposed MADM-Strategy:

Example 5.1. "***Location Selection for the Installation of Outdoor Air Purifier***".

Suppose that, the state government of any state (of India) wants to install an Air Purifier in a particular area to supply the fresh air in the environment to neutralize the air concentration. Let the government initiates to select the appropriate place for the installation of Air Purifier. After the initial screening, the decision makers select four alternatives (areas) $\breve{A}_1$, $\breve{A}_2$ and $\breve{A}_3$, for further evaluation.

The appropriate and scientific process that makes the fresh air from pollute air is done by air purifier. The process must improve the quality of air in the environment. Air pollution means where we meet the undesired concentration of impurities in air. After the filtering Air by purifier it remove dust, airborne bacteria, airborne particles, neutralize smoke, and many other particles.

The following are the stages of the air purifier treatment plants:

(*a*) Absorbed pollute Air,

(*b*) Removed visible particle by using pad filter and pleated filter,

(*c*) Carbon filtering it removes odors and capture gases and organic compound from smoke,

(*d*) Capture microscope particles,

(*e*) Supply fresh air,

(*f*) Changing filter.

The organic and inorganic components are filtered naturally. To increase the air quality at indoor normally used air purifier, there are few natural way have noticed some of them increase Vegetation, Salt lamp, Active charcoal, House plant, Essential oil like oregano, cinnamon , rosemary. Bacteria and other harmful micro-pollutants are removed by the using … plates. Sometimes, it is very difficult to choose a geographical area for the installation of air purifier treatment plant because of its hierarchical nature. Till now, many researchers around the globe presented their models for the selection of land for the installation of air purifier treatment plant. Based on a review of the accumulated scientific evidence some guidelines were well documented in WHO manual (WHO 2005) on four most common air pollutants- particulate matter , ozone , nitrogen dioxide and sulfur dioxide for maintaining the quality of natural air . So, the location of installation of air purifier treatment plant has also decide the type of air treatment process, steps of treatment procedures to be implemented in the pollute air treatment plant for supply of fresh air in desired level of purity. That is why the selection of location for installation or relocation of air purifier treatment plant can be treated as a MADM problem. The selection of attribute is a high factor for identifying the location for the installation of the air purifier treatment plant. For the selection of location for the installation of

air purifier treatment plant, the decision maker need to choose some attributes so that they can easily select the suitable location.

In [4], Choudhury et al. have selected four attributes namely (*i*) Technological Constraints, (*ii*) Environmental Constraints, (*iii*) Economical Constraints, (*iv*) Political Constraints, for the selection of geographical land for the installation of surface water treatment plant.

Here, we have also selected the above mentioned four attributes to select the suitable location for the installation of Air Purifier.

1. Technological Constrains (Θ_1):

Among many parameter technological constraints is one of the most important factor for the air purifier treatment like machinery selection, Engineering process, use of electrify, air quality index level (briefly AQI), different components level in air etc. Technology can help to maintain the all factors including the animal health and environment. This is why we considering technical constrain is the first criterion in this study.

2. Environmental Constrains (Θ_2):

Environmental constrains is another essential parameter for air purifier treatment plant. In case of place selection in highly pollutant and population area is a difficult task. Outside of city are not suitable since the consumption air purifier needed to install in extremely pollute places. But some time hospital, stadium, school area has to decide for installing air purifier for minimizes the other issues. The work of conventional smoke tower plants clean polluted air and minimizes air pollution.

3. Economical Constrains (Θ_3):

For the air treatment plant economical constrains is the one of the most important factor to execute the installation plane. The cost of distribution of air from the treatment to the environment and other expenditure like, machinery, labor, plates and other technology used to minimize. The use of the technological and geographical selection can help the economical constrains.

4. Political Constrains (Θ_4):

The political constrains is another basic factor for the proper installation of air purifier. The balancing of clean air management economic rationale of demand management is the work of political constrains.

By using the decision makers' evaluation information about the alternatives against each attributes we get the following decision matrix:

Table-1

	$\acute{A}_1$	$\acute{A}_2$	$\acute{A}_3$	$\acute{A}_4$
Θ_1	(0.8,0.6,0.2,0.3,0.7)	(0.6, 0.4,0.1,0.5,0.6)	(0.5,0.5,0.7,0.4,0.3)	(0.8, 0.5,0.4,0.2,0.4)
Θ_2	(0.8, 0.6,0.4,0.2,0.5)	(0.7, 0.2,0.4,0.6,0.7)	(0.9, 0.3,0.4,0.3,0.8)	(0.7, 0.2,0.3,0.6,0.4)
Θ_3	(0.9, 0.5,0.7,0.2,0.4)	(0.6, 0.2,0.4,0.7,0.4)	(0.5, 0.4,0.4,0.3,0.7)	(0.8, 0.2,0.4,0.6,0.5)

By using eq. (6) and eq. (7), we get the weight vector $\hat{w}$= ($\hat{w}_1$, $\hat{w}_2$, $\hat{w}_3$, $\hat{w}_4$) = (0.2900302, 0.2205438, 0.2356495, 0.2537764) for the previously mentioned four attributes corresponding to the four alternatives.

Now, the decision maker can find the aggregate value of all the attributes with respect to the corresponding alternative by using eq. (4) and eq. (5). The aggregate value is given in the following table.

Table-2

	Aggregate Value
SVPNWHM (Θ_1: $\acute{A}_1$, $\acute{A}_2$, $\acute{A}_3$, $\acute{A}_4$)	(0.99771196, 0.70732183, 0.02562288, 0.15405526, 0.19856120)
SVPNWHM (Θ_2: $\acute{A}_1$, $\acute{A}_2$, $\acute{A}_3$, $\acute{A}_4$)	(0.99820335, 0.75082793, 0.02466152, 0.15405526, 0.39608476)
SVPNWHM (Θ_3: $\acute{A}_1$, $\acute{A}_2$, $\acute{A}_3$, $\acute{A}_4$)	(0.99686998, 0.68982898, 0.05410631, 0.17649619, 0.19856120)

Now, we have, $S(\Theta_1)$=0.8653589, $S(\Theta_2)$=0.8348459, $S(\Theta_3)$=0.8515071, by using the eq. (1). This implies, $S(\Theta_2) < S(\Theta_3) < S(\Theta_1)$. Hence, Θ_1 is the most suitable location for the installation of the outdoor Air purifier.

6. Conclusions:

In this study, we have grounded the notion of SVPNHM operator, SVPNWHM operator, and provide some appropriate examples on them. Then, we have proposed a SVPNWHM operator based MADM technique under the SVPN-set environment. Finally, we have given a real world numerical example namely "Location Selection for the Installation of Outdoor Air Purifier" demonstrate the practical significance of our recommended MADM methodology.

The recommended MADM methodology can also be utilized for dealing with the other decision-making problems such as brick selection [20], surface water treatment plant location selection [4], logistic center location selection [26, 27], tender selection [9], etc.

Furthermore, the proposed MADM technique is expected to open up new avenues for investigation in the SVPN-Set setting.

Conflict of Interest: The authors affirm that they have no conflicts of interest.

Authors Contribution: All authors contributed equally to the preparation of this article.

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Received: March 1, 2024. Accepted: July 25, 2024

NSS
Neutrosophic Sets and Systems, Vol. 70, 2024
University of New Mexico

New Neutrosophic Sentiment Analysis Method Based on NeutroAlgebra for the Evaluation of Ethics in Companies

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Abstract. Ethics is a discipline that has broken the limits of its field of study, which is related to having the right behavior towards other people, but also to respect nature and life. This has been extended to the accounting and organizational management of companies. This paper proposes a sentiment analysis method based on the NeutroGroup obtained from offuninorms defined by the combining functions in Prospector. The advantage of the proposed method is that in addition to "like" and "dislike", we obtain a result of indeterminacy that can be due either to neutrality or to a contradiction in the short informal writing. The method is illustrated with an example. The application of neutrosophy in sentiment analysis is well-known since neutrality is included in the analysis. However, for the first time, to the authors' knowledge, the indeterminacy of two different types is modeled.

Keywords: Ethics, Business Management, NeutroAlgebra, Prospector, Sentiment Analysis.

1 Introduction

In the contemporary business context, ethics has expanded beyond its traditional scope, encompassing new areas of application, including accounting and organizational management. Ethics, in its broad sense, is defined as the discipline of having the right behavior. It has also application in the accounting field involving the evaluation of integrity, transparency, and responsibility within business organizations. These ethical principles are essential for building a solid and trustworthy corporate culture that not only complies with legal regulations but also promotes morally responsible business conduct.

Integrity in organizations refers to adherence to sound moral and ethical principles, including honesty and consistency between actions and the company's stated values. Transparency, on the other hand, relates to clarity and openness in the communication of information, allowing all stakeholders to have access to accurate and understandable data about the company's operations and decisions. Finally, responsibility implies the capacity and willingness of the organization to be accountable for its actions and decisions, assuming the consequences thereof.

To address the evaluation of these ethical aspects in business organizations, neutrosophic sentiment analysis methods offer an innovative and effective approach. Neutrosophy, a theory developed by Florentin Smarandache, focuses on the principle that every idea or concept has a degree of truthfulness, falsehood, and indeterminacy [1, 2]. Applied to sentiment analysis, this theory allows for a nuanced and precise evaluation of the perceptions and opinions that exist about an organization, considering not only positive and negative feelings but also neutral and ambiguous ones [3-13].

On the other hand, sentiment analysis also known as Opinion Mining defines the processes of extraction, analysis, and identification of opinions or emotions [14]. This is becoming increasingly important due to the possibility that new technologies offer to capture customer opinions. Users must constantly and globally express their opinions or likings about a service, product, or video, among others provided online. This helps to improve the quality of these features and at the same time personalize them. In this research, we propose a framework to measure integrity, transparency, and accountability in business organizations using neutrosophic sentiment analysis methods.

The use of neutrosophic methods in this research will allow us to capture a complete and detailed vision of how these ethical aspects are perceived in business organizations. This approach will consider not only the positive and negative extremes of opinions but also the neutral and ambiguous perceptions that are often ignored in traditional analyses. This will achieve a rich and accurate assessment of integrity, transparency, and accountability, providing a solid foundation for continuous improvement in business practices.

The incorporation of ethics in accounting and business management, through the use of neutrosophic methods

of sentiment analysis, represents a significant advance in the promotion of ethical and responsible business practices. This research will contribute to the development of tools and methodologies that facilitate the evaluation and improvement of these ethical principles in organizations, promoting a fair and sustainable business environment.

This article proposes a new method for sentiment analysis using a NeutroGroup based on combining function in Prospector [15-18]. A NeutroAlgebra is an algebraic structure where the definitions, operations between elements, axioms, etc., at least one of them is satisfied by only a part of the elements. The NeutroGroup on which we base this method has been applied to solve several real-life problems [15, 16]. The idea on which we are based is that this NeutroGroup offers an operation between pairs of elements where the indeterminacy that may be due to neutrality or contradiction is included.

To achieve this objective, the article is divided into several sections. Below we will find a Preliminaries section where some basic notions of Sentiment Analysis and NeutroAlgebras are explained. The following section contains the details of the proposed method applied to ethics with an illustrative example. The last section is dedicated to giving the conclusions.

2 Preliminaries

2.1 Sentiment Analysis

Sentiment Analysis refers to the use of natural language processing, text analysis, and computational linguistics to identify and extract subjective information from resources [14]. Regarding text mining, Sentiment Analysis can be dedicated to the task of classifying the polarity of information massively.

For sentiment analysis, there are key categories that are: lexical affinity, statistical methods, and concept-level techniques. However, we cannot escape the complexity involved in the task of measuring the feelings of a person or a group of people, due to the subjectivity that is an intrinsic part of the problem. This is because there are affective states that are temporary and that can be reflected in one way in a survey, but that moments later can change to a different state.

Specialists agree that one of the elements that cannot be missing from the measurement scale is neutral opinion, because the person is not able to discern their opinion between a negative or positive state, or because their opinion or feeling is in a neutral state that is neither positive nor negative. This is why the use of Neutrosophy is pertinent, since neutrality is part of this theory, in addition to the positive and negative. This can also be useful in studying the connotation of words within the text, which makes the task even more complex.

2.2 Neutral Algebra generated by the combining function in Prospector

For a certain natural number $n > 0$, NeutroGroup is defined from the combining function of Prospector. Prospector is the well-known Expert System used to model mining problems [17, 18]. The NeutroGroup set consists of all integers between $-n$ and n in addition to the symbolic element I to represent indeterminacy. This is $NG_5 = \{-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, I\}$ and $\oplus_5$ is used. This is defined according to the following Cayley table:

$\oplus_5$	-5	-4	-3	-2	-1	0	I	1	2	3	4	5
-5	-5	-5	-5	-5	-5	-5	-5	-5	-5	-5	-5	I
-4	-5	-5	-5	-5	-4	-4	-4	-4	-3	-2	0	5
-3	-5	-5	-4	-4	-4	-3	-3	-2	-1	0	2	5
-2	-5	-5	-4	-3	-3	-2	-2	-1	0	1	3	5
-1	-5	-4	-4	-3	-2	-1	-1	0	1	2	4	5
0	-5	-4	-3	-2	-1	0	I	1	2	3	4	5
I	-5	-4	-3	-2	-1	I	I	I	I	I	I	I
1	-5	-4	-2	-1	0	1	I	2	3	4	4	5
2	-5	-3	-1	0	1	2	I	3	3	4	5	5
3	-5	-2	0	1	2	3	I	4	4	4	5	5
4	-5	0	2	3	4	4	I	4	5	5	5	5
5	I	5	5	5	5	5	I	5	5	5	5	5

Table 1. Cayley table corresponding to $\oplus_5$. Source: [18].

$\oplus_5$ satisfies the properties of commutativity, and associativity and has 0 as the null element. Additionally, it satisfies each of the following properties:

- If x, y < 0 then $x \oplus_5 y \leq min(x, y)$,
- If x, y > 0 then $x \oplus_5 y \geq max(x, y)$,
- If x < 0 and y > 0 or if x > 0 and y < 0, then we have $min(x, y) \leq x \oplus_5 y \leq max(x, y)$.
- $\forall x \in G, x \oplus_5 0 = x$.
- $(-5) \oplus_5 5 = 5 \oplus_5 (-5) = I$.

3 New Neutrosophic Sentiment Analysis Model

The sentiment analysis method in this research uses neutrosophy to evaluate integrity, transparency, and accountability in business organizations. Through this theory, opinions and perceptions are analyzed considering degrees of positivity, negativity, and indeterminacy. This allows for capturing not only positive and negative but also neutral and ambiguous feelings, providing an accurate and nuanced assessment of business ethics, and improving the understanding of how these ethical aspects are perceived within organizations.

The aspects to measure in the sentiment analysis for ethics within the company are the following:

1. Integrity: The company and its members present themselves to others in a dignified manner. There is self-control over the behavior of each member personally and of the company as an entity.
2. Honesty: Evaluation of the perception of truthfulness and sincerity in the company's communications and actions.
3. Congruence: Analysis of the alignment between the values declared by the company and its actual practices.
4. Ethical Compliance: Measurement of adherence to recognized ethical standards and legal regulations.
5. Transparency: The members of the organization respond and express themselves openly to any question or demand from other members of the company. The company also maintains the data of its financial and economic management in public within the established limits.
6. Information Clarity: Evaluation of the accessibility and understanding of the information provided by the company.
7. Access to Information: Measurement of the availability of relevant information for interested parties.
8. Open Communication: Analysis of the openness and frequency of communication between the company and its interested parties.
9. Responsibility: The members of the organization and the company respond to their duties and obligations and in case of error they assume the consequences of them.
10. Accountability: Assessment of the company's willingness to take responsibility for its actions.
11. Incident Response: Measurement of effectiveness and transparency in response to problems or incidents.
12. Social and Environmental Impact: Analysis of the company's actions about their impact on society and the environment.

The other step to perform appears in the article [19]. The method we propose is applied to short informal texts, as in the case of the technique that appears above. A set of words must be determined that are classified as positive, negative, or neutral with a strength value that is evaluated in a range of -5 to 5 or indeterminate.

When the indeterminate form is evaluated in this way, it is because we cannot decipher what the person thinks about the topic. This may happen because the text is unintelligible or because the semantics of what the person means are not clear.

Another possibility is that in the same text, each of the previous variables is manifested with an evaluation of maximum positive degree (+5) and maximum negative degree (-5), which is a contradiction and therefore is evaluated as indeterminate I.

Let us remember that the function on which the PROSPECT expert system is based, means the degree of evidence that the expert has on a specific aspect. When this evidence is aggregated on two aspects for which there is maximum evidence of opposite degrees, then there is a contradiction. This is why we talk about the advantage of this method using this NeutroOperation, where the indeterminacy can be due to two origins.

Some aspects that we have used are taken from SentiStrength Sentiment Strength Detection Algorithm [19].

- Terms referring to each of the previous variables are classified as Positive, Negative, or Neutral in a list that is made up of linguistic values. Each of these terms is also associated with a value between -5 and 5, and even I, according to its degree of strength of how positive or negative it is. For example, the term “I like it” is modified with a greater positive value when “I like it a lot” is written, while “I don't like it” is similarly intensified with a more negative value when it is written “I don't like it very much”.

What applies is that for the word “very” or “a lot” which modifies one of the positive or negative classifying words, is used $x \oplus_5 x$, and for “too much” is used $x \oplus_5 x \oplus_5 x$, where x is the value that is associated with the word. For example, $x > 0$ results in “very” with an even more positive value. On the other hand, when $x < 0$, the result is more negative.

Additionally, the modification of “quite” is converted to $\left[sig(x)\sqrt{|x|}\right]$.

- Words that invert the meaning of what is said are also taken into account. In this case, the sign is changed. For example, “I like it” with a value of $x = 3$, when it comes to “I don't like it” it is calculated as $x = -3$, both have the same strength, but with an opposite sense.
- In this algorithm, very complex cases, where there are exclamation or question marks, are ignored, since we want to evaluate what members of the organization or clients write, in case it makes sense, about each of the twelve aspects of the ethics indicated in the previous points.
- Another aspect that is taken into account in the proposed algorithm taken from the precedent is the evaluation

of emoticons.

- Spelling correction is also applied here.

The next step is the evaluation of a short informal text written by a person. To do this, natural language processing is done where words that express feelings or opinions about each of the twelve aforementioned aspects are searched. Let us denote these aspects as: $V = \{v_1, v_2, \cdots, v_{12}\}$.

Then, within the text processing, the words referring to each of these variables are identified. These words are identified with a value of -5 to 5 or I. Let us denote this as follows, for the i[th] variable, the set X_i of evaluations of words that appear in the text:

$v_i \rightarrow X_i = \{x_{i1}, x_{i2}, \ldots, x_{im_i}\}$, where x_{ij} is the set of elements in -5 to 5 or I, used to qualify the words that refer to the i[th] variable.

Keep in mind that even the individual evaluation of each word can be complicated. For example, when modifiers like "very" appear, then the value of the modified word changes. Also when there are spelling errors that make some evaluation illegible, it is necessary to use the value I. The final value associated with each v_i is:

$$x_{total,i} = x_{i1} \oplus_5 x_{i2} \oplus_5 \ldots \oplus_5 x_{im_i} \quad (1)$$

Let us note that we consider it not convenient to obtain an aggregated ethics value for all variables since the separate value is more useful to have an idea of individual opinion or sentiment.

If we have a set of people whose opinion on ethics in the company is studied. Let us denote this set of people by $P = \{p_1, p_2, \cdots, p_l\}$, such that the values are taken into account, $x_{total,i,j}$ is the total value of the i[th] variable of ethics in the organization, according to the j[th] person.

It is calculated:

$$\bar{x}_{total,i} = \frac{\sum_{j=1}^{l} x_{total,i,j}}{l} \quad (2)$$

That is, the arithmetic mean of each of the variables is calculated.

Figure 1 represents graphically a scheme of the proposed method:

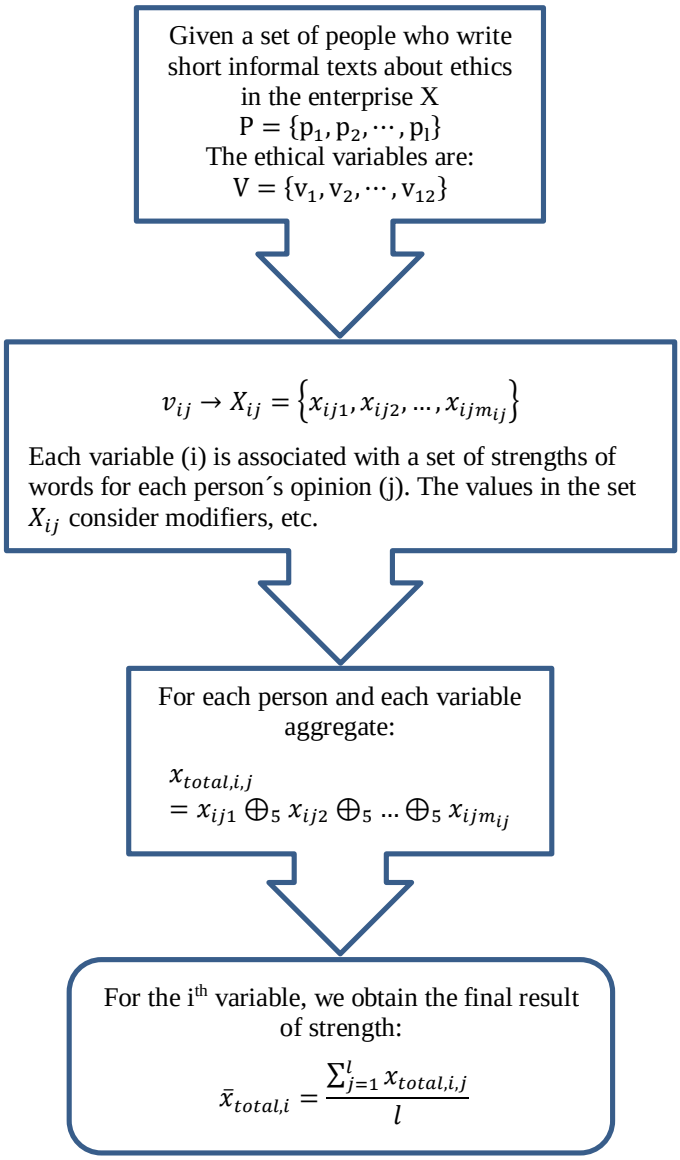
Given a set of people who write short informal texts about ethics in the enterprise X
$P = \{p_1, p_2, \cdots, p_l\}$
The ethical variables are:
$V = \{v_1, v_2, \cdots, v_{12}\}$
$v_{ij} \rightarrow X_{ij} = \{x_{ij1}, x_{ij2}, \ldots, x_{ijm_{ij}}\}$
Each variable (i) is associated with a set of strengths of words for each person´s opinion (j). The values in the set X_{ij} consider modifiers, etc.
For each person and each variable aggregate:
$x_{total,i,j} = x_{ij1} \oplus_5 x_{ij2} \oplus_5 \ldots \oplus_5 x_{ijm_{ij}}$
For the i[th] variable, we obtain the final result of strength:
$\bar{x}_{total,i} = \frac{\sum_{j=1}^{l} x_{total,i,j}}{l}$

Figure 1: Schematic representation of the proposed method.

Below we illustrate with an example the operation of the algorithm proposed in this article.

Example 1. Suppose that in company X a website is enabled where workers can freely express their opinion on ethics within the company as part of internal control. Comments are expressed anonymously to ensure sincerity. The text written by the workers is processed using the method proposed in this article.

Suppose that regarding honesty, the first worker expressed in the same text that "the company's honesty is enough good" and in another part that "it is not enough good". In this case, the following is assigned to the variable v_2:

$v_{21} \rightarrow X_{i1} = \left\{\left[sig(3)\sqrt{|3|}\right], \left[sig(-3)\sqrt{|-3|}\right]\right\}$, that is, the first person feels that there is enough honesty (degree $\left[sig(3)\sqrt{|3|}\right] = 2$ because 3 is the assignment to good honesty according to the assignment of a number to each word and this is the formula to obtain a fair value), and the opposite for it is assigned $\left[sig(-3)\sqrt{|-3|}\right] = -2$ (and therefore the sign is changed to change the meaning), and then $2 \oplus_5 (-2) = 0$. That is, this result is completely canceled and any other result is taken into account. For example, if somewhere else he/she wrote that there is a lot of honesty in the company, then it is calculated $2 \oplus_5 (-2) \oplus_5 (3 \oplus_5 3) = 0 \oplus_5 4 = 4$.

Here "a lot of honesty" is understood as $3 \oplus_5 3 = 4$, according to the proposal we made.

If a second worker considers that dishonesty is maximum and honesty is also maximum; for example: "there is too much honesty" and "there is too much dishonesty", then, in this case, it is calculated:

$3 \oplus_5 3 \oplus_5 3 = 5$ for the modifier in "too much honesty" and $(-3) \oplus_5 (-3) \oplus_5 (-3) = -5$ for "too much dishonesty." So in this case they are not canceled, but rather indeterminate $5 \oplus_5 (-5) = \mathrm{I}$.

From Table 1 it is seen that this indeterminacy disappears when the operator with a third negative value is applied, for example, $5 \oplus_5 (-5) \oplus_5 (-3) = \mathrm{I} \oplus_5 (-3) = -3$. However, when the third value is positive, indeterminacy is maintained with the idea of always taking a "pessimistic" opinion.

Note also that extreme values exercise veto power over any other opinion. That is, if the opinion is 5 or -5, then any aggregation results in 5 or -5, respectively.

On the other hand, in writing where an unintelligible text about honesty appears, either due to syntactic or semantic problems, then the value I is directly assigned.

Conclusion

This paper describes a new neutrosophic model of sentiment analysis. The model is based on the NeutroGroup generated by an offuninorm [20, 21, 22] for values on a scale from -5 to 5 and also the symbolic value I to denote indeterminacy. Twelve ethics variables were set in the company and an algorithm was designed to calculate the value of each of these variables within the previous scale in the company. To do this, workers are asked to express their opinion on each of these variables according to a short informal written text. The model allows us to deal with indeterminacy both due to contradiction and because the written text does not make sense. The effectiveness of the created model was illustrated with an example.

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Received: March 1, 2024. Accepted: July 25, 2024

NSS

Neutrosophic Sets and Systems, Vol. 70, 2024

University of New Mexico

A New Development of Entropy and Similarity Measures in Temporal Complex Neutrosophic Environments for Tourist Destination Selection

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Abstract. In human existence, making decisions is a common event. Various techniques have been devised to tackle decision-making troubles in practical situations. Over the past ten years, a great deal of study has concentrated on quantifying the degree of ambiguity and unpredictability in knowledge using the concept of neutrosophic sets or extensions thereof. An efficient framework for handling information in decision-making problems involving uncertain, indeterminate, and time-related aspects is the Temporal Complex Neurosophic Set (TCNS). Measures of entropy and similarity can be helpful for evaluating data to solve multicriteria decision-making (MCDM) challenges in practical situations. However, TCNS information measurements were of no concern or relevance to the current technique. In the context of the TCNS, this work suggests multiple novel similarity and entropy measurements. The proposed metrics have been validated and shown to comply with the explicit definition of the entropy measure and similarity for the TCNS. The novel similarity and entropy measures on the TCNS environment are proposed in this research. The four similarity measures on the TCNS contain Dice, Jaccard, Cosine, and Cotangent. Also, a numerical example concerning selecting a Vietnam tourist destination is provided to validate the usefulness of the suggested measures. The practical application shows that proposed TCNS similarity and entropy metrics can produce accurate and significant outcomes for real-world decision-making problems.

Keywords: Similarity measure; Entropy measure; Neutrosophic set; Temporal Complex Neutrosophic Set; MCDM;

1. Introduction

In human activities, decision-making (DM) is a highly typical routine. A multi-attribute decision-making (MADM) problem is determining the most suitable option from a group of options that all need to be evaluated simultaneously. Numerous DM problems have been solved using MADM models, which have attracted the attention of multiple researchers in diverse domains [1, 2] However, real-world issues are excessively complex because they frequently need clarification, clarity, or insufficient data. As a result, decision-makers tackle those issues through the fuzzification process, which is a vital method to address humanistic systems that exist in real-world problems.

The concept of classical fuzzy sets (FS) was introduced by Zadeh [3] in 1965 as a way to handle uncertain data. By enabling elements to be a part of sets with varying levels of membership, FS successfully captures the fuzziness and ambiguity that are inherent in many common problems. After that, it was used by numerous scientists and scholars to look into various real-world events in different fields and recommend the best courses of action. However, the nature of covert human assessments of discontent was not explained by the FS theory, Atanassov [4] was prompted by this to suggest an intuitionistic fuzzy set (IFS), wherein the levels of each element's membership and non-membership are determined separately. Then, an extension of IFS, namely neutrosophic set (NS), is introduced by Smarandache [5] by introducing an uncertain membership function with degrees of falsity, indeterminacy, and truth into IFS. It's also an essential and effective technique for handling conflicting, ambiguous, and incomplete information in certain real-life situations [6, 7, 8].

Although FS or its extension versions have shown remarkable effectiveness in addressing issues emerging from vagueness and uncertainty in many real-world scenarios, they cannot solve the periodicity of some ambiguous information. It is challenging to express periodic forms using crisp sets, FS, IFS, or NS. In order to address this challenge, Ramot et al. [9] introduced the notion of the complex fuzzy set (CFS), which adds information to a phase component in order to provide details about a specific higher dimensional periodic problem. Since its introduction, many researchers have researched and developed CFS because it is considered a valuable tool for representing information with ambiguous, uncertain, and periodic components [10, 11].

Furthermore, the complex neutrosophic set (CNS) was described by Ali and Smarandache [12]. With the intention of integrating the "complex" component to make NS more flexible and sensitive to occasionally ambiguous information, CNS is an integrated version of the CFS and NS. Uncertainty, inconsistency, and indeterminacy are the three membership degree functions that define a CNS. These functions have a complex-valued range in the complex space unit circle. The CNS is limited if the total of the negative, abstinence, and positive grades is three

or less. Following its introduction, CNS demonstrated its value in characterizing data that included unpredictable, inconsistent, periodic, and indeterminate factors.

In real-world decision-making problems, we encounter complex natural phenomena that require adding a second dimension to represent the degree of membership. By establishing the second dimension, information is fully gathered, and the problem of information loss is minimized. Therefore, since the beginning of the CNS, many hybrid versions have been developed and used in a range of life domains, such as Interval CNS [13], Bipolar CNS [14], Complex neutrosophic soft set (CNSS) [15], Interval-valued complex neutrosophic soft set (IVCNSS) [16]... In light of the theories mentioned above, it is evident that many MCDM researchers have found the information's uncertainty and time-periodic element interesting. Many of the decision-making scenarios we encounter in real life involve periodic events, which call for the representation of various time intervals (e.g., seasonal variations in weather information). However, the present experiments have yet to take into account the effect of temporal factors by CNS in an MCDM model concurrently. Recently, Lan et al. [17] introduced an extension of CNS, namely Temporal Complex neutrosophic set (TCNS), to represent data with dynamic and temporal factors of several real-life decision-making contexts. It covers decision-making issues with uncertain, temporal, and periodic aspects. It is a novel opening with several practical applications that concern the time cycle and time element.

Making decisions involves systematically solving issues in the real world and selecting the best option after determining which feasible alternatives are available. All possible options are sorted, allowing users to choose the suitable solution. When faced with multi-criteria decision-making difficulties, decision-makers can select the best option primarily based on information measures, including cross-entropy, entropy, similarity, and distance measures. Because of this, entropy and similarity measurements are consistently two crucial subjects that have been investigated widely by many scholars from various backgrounds. Nevertheless, prior research has yet to focus on refining and strengthening the theory of the TCNS environments using similarity metrics, mathematical operations, and entropy measures. That is the primary motivation for carrying out this research.

(1) Define novel measures of entropy and similarity for the Temporal Complex Neutronosophic Set, which addresses the unpredictable, periodic, and temporal aspects of DM problems.

(2) Build the decision-making model using the proposed entropy and similarity measurements in the TCNS environment.

(3) Using a real-world case study of selecting a tourism location in Vietnam, illustrate the viability and logic of applying the suggested model's decision-making process.

(4) Prove our proposed model's effectiveness and potential for real-life case study through a comparative analysis with related MCDM techniques.

The remaining portions of the paper are organized in this manner. Section 2 outlines the research gap and a few connected works. Next, in Section 3, we go over a few of the significant concepts of the TCNS. Novel metrics of entropy and similarity depending on the TCNS framework are defined in Section 4. The multi-criteria technique for making decisions using proposed indicators is provided in Section 5. In Section 6, a comparative experiment is utilized to demonstrate how our theory might be used in practice in a particular tourist location. The conclusions are finally presented in Section 7.

2. **Literature review**

2.1. *Related works*

Similarity is essential to finding solutions in many intricate and unpredictable situations in human life. Both the entropy and similarity measurements were able to show how similar two objects were in an unexpected and uncertain environment. As a result, many scholars have concentrated on using NS or extensions to quantify fuzziness and ambiguity in data. Many different entropy and similarity metrics established and have effectively dealt with practical applications such as clustering analysis [18, 19], decision-making [20, 21], health diagnostics [22, 23] and pattern recognition [24, 25].

According to Hausdorff distance, Broumi and Smarandache [26] presented some similarities for NS in the neutrosophic environment. Several metrics of entropy and similarity of a single-valued NS were proposed by Majumdar et al. [27]. The approach based on score function and divergence measure is suggested to determine the weights of decision experts using a single-valued NS [28]. Bin Ji et al. [29] published the multi-parameter similarity measure for interval-valued neutrosophic sets based on the tangent function. The Q-Neutrosophic Soft set was enhanced with similarity and entropy tools by Abu Qamar and Hassan [30], who also evaluated how well they worked for issues involving medical diagnosis. Additionally, entropy and similarity are frequently employed as crucial metrics for resolving real-world multi-criteria decision-making issues [31, 32, 33]

Since entropy and similarity measurements are two significant topics in the CFS and CNS, they have been comprehensively investigated from a variety of perspectives in multiple studies. Utilizing mixed models to handle the ambiguities of periodical data—where time is a crucial component in its representation—researchers made essential contributions to the literature on similarity measures domains. Axiomatic formulations of the similarity measure and entropy for Complex Multi Fuzzy Soft Sets were presented by Al-Qudah and Hassan [34]. Using a complex fuzzy soft set, Ganeshsree Selvachandran et al. [35] introduced metrics of distance and similarity to address pattern recognition challenges involving digital photographs. The Cosine similarity measures were used by the authors in [36] to calculate the coefficients of

similarity and dissimilarity between two Complex linear Diophantine fuzzy sets. The Complex Multi-Fuzzy Hypersoft Set's Entropy and Similarity Measure concept was introduced in [37]. To demonstrate the validity and significance of the developed measures, some associated theorems and an application for a person's selection decision issue who intends to buy a car are established.

Keeping up with this development and improving the modeling of some real-world issues using our approach (CNS theory and its hybrid structures), novel dice similarity measurements utilizing CNS were proposed by Ali et al. [38]. The authors then used the pattern recognition model and the criteria they provided to test the validity and superiority of the pre-existing methods. Jaccard, Dice, and cosine are three examples of complex neutrosophic similarity metrics proposed by Mondal et al. [39]. A numerical case study demonstrates the suggested methods for handling multi-attribute decision-making scenarios in which students select an appropriate course for post-secondary education after passing a secondary exam. Xu et al. [40] introduced some distance measures based on Hamming, Hausdorff, and Euclidean metrics to deal with the interval CNS information. In order to help with decision-making in medical diagnosis, Faisal et al. [41] developed several similarity metrics of interval complex neutrosophic soft sets (ICNSS) based on the distance measurements: Euclidean and Hamming. An axiomatic definition of single-valued CNS entropy and normalized distance formulas are provided by [42]. Also, to demonstrate the usefulness and feasibility of our suggested entropy metric, we offer a real-world example that involves the choice of green suppliers.

Because physical problems are inherently complicated and unpredictable, the difficulty of researching MCDM has concentrated on the issues with criteria values in the form of intervals, drawing attention to this attractive study area. The research indicates they are effective at encapsulating and illustrating vague and ambiguous information, enabling decision-makers to make intelligent decisions.

2.2. *Research gap analysis*

The evaluations above suggest that these several measures of similarity and entropy and their applications play a significant role in efficiently managing uncertainty and inconsistency in decision-making contexts. However, there are several extant research gaps, as indicated below:

- All entropy and similarity measures in the literature are established based on the theory of CFS, CNS, and these hybrid structures.

- Entropy and similarity measures in the literature only deal with uncertain, imprecise, and periodic information. None of the above theories have the capability of managing information involving dynamic, temporal, and time-cycle factors in real-life problems.

- Previous theories focused on the time-varying component but neglected the temporal, time-cycle factor, and the impact of temporal factors.

- Many entropy measures exist for CFS, CNS, or extension of CNS in the literature that fail to deal with the problem of providing reasonable or consistent results to the decision-makers in the case of a TCNS environment.

- There hasn't been any study on entropy and similarity metrics especially created for TCNS information theory, even though they are essential to fuzzy set theory.

When taking into account the time cycle, periodic, and time components, TCNS is a novel opening with several practical applications. Unfortunately, previous studies have not concentrated on improving and solidifying the theory of the TCNS settings through entropy and similarity metrics. The above facts and the research gap in CNS neutrosophic information theory prompted us to pursue this work. Thus, the paper aims to develop novel entropy and similarity metrics for the TCNS.

3. Preliminaries

This section summarizes the core concepts of TCNS and the three operations (intersection, union, and complement) that will be applied in our study.

Definition 3.1 ([17]). Let U be a universal set, and let $\tilde{\tau} = \{\tau_1, \tau_2, ..., \tau_{n_\tau}\}$ be consecutive time periods. On U, a temporal complex neutrosophic set (TCNS) A is represented by form below:

$$\mathrm{A}(\theta, \tilde{\tau}) = \{\theta, \langle p(\theta, \tau_l).e^{j\mu(\theta,\tau_l)}, q(\theta, \tau_l).e^{j\nu(\theta,\tau_l)}, r(\theta, \tau_l).e^{j\eta(\theta,\tau_l)}\rangle \mid \theta \in U\} \tag{1}$$

Where $p(\theta, \tau_l)$, $q(\theta, \tau_l)$, $r(\theta, \tau_l) \in [0, 1]$ and $\mu(\theta, \tau_l)$, $\nu(\theta, \tau_l)$, $\eta(\theta, \tau_l) \in [0, 2\pi]$

Definition 3.2 ([17]). Suppose there are two TCNSs $\Omega_1(\theta, \tilde{\tau})$ and $\Omega_2(\theta, \tilde{\tau})$ respectively.

$$\Omega_1(\theta, \tilde{\tau}) = \left\{\theta, \left\langle p_{\Omega_1}(\theta, \tilde{\tau}) e^{j\mu_{\Omega_1}(\theta,\tilde{\tau})}, q_{\Omega_1}(\theta, \tilde{\tau}) e^{j\nu_{\Omega_1}(\theta,\tilde{\tau})}, r_{\Omega_1}(\theta, \tilde{\tau}) e^{j\eta_{\Omega_1}(\theta,\tilde{\tau})} \right\rangle\right\}$$
$$\Omega_2(\theta, \tilde{\tau}) = \left\{\theta, \left\langle p_{\Omega_2}(\theta, \tilde{\tau}) e^{j\mu_{\Omega_2}(\theta,\tilde{\tau})}, q_{\Omega_2}(\theta, \tilde{\tau}) e^{j\nu_{\Omega_2}(\theta,\tilde{\tau})}, r_{\Omega_2}(\theta, \tilde{\tau}) e^{j\eta_{\Omega_2}(\theta,\tilde{\tau})} \right\rangle\right\}$$

The following describes the fundamental functions of TCNS:

Complement:

$$C(\Omega_1(\theta, \tilde{\tau})) = \left\{\theta, \left\langle \begin{array}{l} r_{\Omega_1}(\theta, \tilde{\tau}) e^{j\left(2\pi - \eta_{\Omega_1}(\theta,\tilde{\tau})\right)}, \\ (1 - q_{\Omega_1}(\theta, \tilde{\tau})) e^{j\left(2\pi - v_{\Omega_1}(\theta,\tilde{\tau})\right)}, \\ p_{\Omega_1}(\theta, \tilde{\tau}) e^{j\left(2\pi - \mu_{\Omega_1}(\theta,\tilde{\tau})\right)} \end{array} \right\rangle \mid \theta \in U\right\} \tag{2}$$

Union:

$$\Omega_1(\theta, \tilde{\tau}) \cup \Omega_2(\theta, \tilde{\tau}) = \left\{\theta, \left\langle \begin{array}{l} (p_{\Omega_1}(\theta, \tilde{\tau}) \vee p_{\Omega_2}(\theta, \tilde{\tau})) e^{j\mu_{\Omega_1 \cup \Omega_2}(\theta,\tilde{\tau})}, \\ (q_{\Omega_1}(\theta, \tilde{\tau}) \wedge q_{\Omega_2}(\theta, \tilde{\tau})) e^{j\nu_{\Omega_1 \cup \Omega_2}(\theta,\tilde{\tau})}, \\ (r_{\Omega_1}(\theta, \tilde{\tau}) \wedge r_{\Omega_2}(\theta, \tilde{\tau})) e^{j\eta_{\Omega_1 \cup \Omega_2}(\theta,\tilde{\tau})} \end{array} \right\rangle \mid \theta \in U\right\} \tag{3}$$

Intersect:

$$\Omega_1(\theta,\tilde{\tau})\cap\Omega_2(\theta,\tilde{\tau})=\left\{\theta,\left\langle\begin{array}{l}(p_{\Omega_1}(\theta,\tilde{\tau})\wedge p_{\Omega_2}(\theta,\tilde{\tau}))\,e^{j\mu_{\Omega_1\cap\Omega_2}(\theta,\tilde{\tau})},\\(q_{\Omega_1}(\theta,\tilde{\tau})\vee q_{\Omega_2}(\theta,\tilde{\tau}))\,e^{j\nu_{\Omega_1\cap\Omega_2}(\theta,\tilde{\tau})},\\(r_{\Omega_1}(\theta,\tilde{\tau})\vee r_{\Omega_2}(\theta,\tilde{\tau}))\,e^{j\eta_{\Omega_1\cap\Omega_2}(\theta,\tilde{\tau})}\end{array}\right\rangle\mid\theta\in U\right\}\tag{4}$$

4. **Propose new measures of Temporal complex neutrosophic set**

The study presents a few TCNS similarity metrics in this part, including the Dice, Jaccard, Cosine, and Cotangent similarity measures. Additionally, it is suggested that decision-making models in temporally complex neutrosophic contexts start with an entropy assessment of TCNS.

4.1. *TCNS - Dice similarity measure*

Definition 4.1. Assume that $\Omega_1=\left\langle p_{\Omega_1}(\theta_i,\tau_l)\,e^{j\mu_{\Omega_1}(\theta_i,\tau_l)},q_{\Omega_1}(\theta_i,\tau_l)\,e^{j\nu_{\Omega_1}(\theta_i,\tau_l)},r_{\Omega_1}(\theta_i,\tau_l)\,e^{j\eta_{\Omega_1}(\theta_i,\tau_l)}\right\rangle$ and $\Omega_2=\left\langle p_{\Omega_2}(\theta_i,\tau_l)\,e^{j\mu_{\Omega_2}(\theta_i,\tau_l)},q_{\Omega_2}(\theta_i,\tau_l)\,e^{j\nu_{\Omega_2}(\theta_i,\tau_l)},r_{\Omega_2}(\theta_i,\tau_l)\,e^{j\eta_{\Omega_2}(\theta_i,\tau_l)}\right\rangle$ be are two TCNSs on universe of discourse U for $\theta_i\,(1,2,...,n_U)$ and $\tau_l\,(1,2,...,n_\tau)$.

Suppose $Sim_D(\Omega_1,\Omega_2)$ is a TCNS - Dice similarity measure between Ω_1 and Ω_2, $Sim_D(\Omega_1,\Omega_2)$ is defined as follows:

$$\begin{aligned}&Sim_D(\Omega_1,\Omega_2)\\&=\frac{1}{n_U*n_\tau}\sum_{i=1}^{n_U}\sum_{l=1}^{n_\tau}\frac{2*\left[\begin{array}{l}(p_{\Omega_1}(\theta_i,\tau_l)+\mu_{\Omega_1}(\theta_i,\tau_l))\,(p_{\Omega_2}(\theta_i,\tau_l)+\mu_{\Omega_2}(\theta_i,\tau_l))\\+(q_{\Omega_1}(\theta_i,\tau_l)+\nu_{\Omega_1}(\theta_i,\tau_l))\,(q_{\Omega_2}(\tau_l)+\nu_{\Omega_2}(\theta_i,\tau_l))\\+(r_{\Omega_1}(\theta_i,\tau_l)+\eta_{\Omega_1}(\theta_i,\tau_l))\,(r_{\Omega_2}(\theta_i,\tau_l)+\eta_{\Omega_2}(\theta_i,\tau_l))\end{array}\right]}{\left(\begin{array}{l}(p_{\Omega_1}(\theta_i,\tau_l)+\mu_{\Omega_1}(\theta_i,\tau_l))^2+(q_{\Omega_1}(\theta_i,\tau_l)+\nu_{\Omega_1}(\theta_i,\tau_l))^2\\+(r_{\Omega_1}(\theta_i,\tau_l)+\eta_{\Omega_1}(\theta_i,\tau_l))^2+(p_{\Omega_2}(\theta_i,\tau_l)+\mu_{\Omega_2}(\theta_i,\tau_l))^2\\+(q_{\Omega_2}(\theta_i,\tau_l)+\nu_{\Omega_2}(\theta_i,\tau_l))^2+(r_{\Omega_2}(\theta_i,\tau_l)+\eta_{\Omega_2}(\theta_i,\tau_l))^2\end{array}\right)}\end{aligned}\tag{5}$$

Example 4.2. Let $U=\{\theta_1,\theta_2\}$ be a universe of discourse; $\tilde{\tau}=\{\tau_1,\tau_2,\tau_3\}$ be time period; Ω_1 and Ω_2 be two TCNS in U

$$\Omega_1(\theta_1)=\left\{\begin{array}{l}\left\langle 0.1e^{j0.2},0.6e^{j0.45},0.3e^{j0.56}\right\rangle,\left\langle 0.8e^{j0.77},0.2e^{j0.86},0.4e^{j0.65}\right\rangle,\\\left\langle 0.6e^{j0.52},0.2e^{j0.2},0.36e^{j0.2}\right\rangle\end{array}\right\}$$

$$\Omega_1(\theta_2)=\left\{\begin{array}{l}\left\langle 0.5e^{j0.12},0.2e^{j0.65},0.5e^{j0.85}\right\rangle,\left\langle 0.3e^{j0.8},0.7e^{j0.23},0.1e^{j0.32}\right\rangle,\\\left\langle 0.74e^{j0.6},0.5e^{j0.15},0.66e^{j0.41}\right\rangle\end{array}\right\}$$

$$\Omega_2(\theta_1)=\left\{\begin{array}{l}\left\langle 0.6e^{j0.8},0.9e^{j0.25},0.4e^{j0.52}\right\rangle,\left\langle 0.6e^{j0.62},0.5e^{j0.24},0.3e^{j0.18}\right\rangle,\\\left\langle 0.3e^{j0.82},0.3e^{j0.6},0.4e^{j0.71}\right\rangle\end{array}\right\}$$

$$\Omega_2(\theta_2) = \left\{ \begin{array}{l} \left\langle 0.3e^{j0.22}, 0.4e^{j0.56}, 0.6e^{j0.26} \right\rangle, \left\langle 0.1e^{j0.38}, 0.5e^{j0.43}, 0.5e^{j0.24} \right\rangle, \\ \left\langle 0.2e^{j0.29}, 0.4e^{j0.72}, 0.5e^{j0.65} \right\rangle \end{array} \right\}$$

Then, the TCNS - Dice similarity measure between Ω_1 and Ω_2 can be: $Sim_D(\Omega_1, \Omega_2) = 0.87942$

Theorem 4.3. *Let Ω_1 and Ω_2 be two TCNSs then,*

(1) $0 \leq Sim_D(\Omega_1, \Omega_2) \leq 1$

(2) $Sim_D(\Omega_1, \Omega_2) = Sim_D(\Omega_2, \Omega_1)$

(3) $Sim_D(\Omega_1, \Omega_2) = 1$ *, if and only if* $\Omega_1 = \Omega_2$

(4) if Ω_3 is a TCNS in U and $\Omega_1 \subseteq \Omega_2 \subseteq \Omega_3$ then $Sim_D(\Omega_1, \Omega_3) \leq Sim_D(\Omega_1, \Omega_2)$ and $Sim_D(\Omega_1, \Omega_3) \leq Sim_D(\Omega_1, \Omega_2)$

Proof

(1) We can have,

$$2 * \left[\begin{array}{l} \left(p_{\Omega_1}(\theta_i, \tau_l) + \mu_{\Omega_1}(\theta_i, \tau_l)\right)\left(p_{\Omega_2}(\theta_i, \tau_l) + \mu_{\Omega_2}(\theta_i, \tau_l)\right) \\ + \left(q_{\Omega_1}(\theta_i, \tau_l) + \nu_{\Omega_1}(\theta_i, \tau_l)\right)\left(q_{\Omega_2}(\theta_i, \tau_l) + \nu_{\Omega_2}(\theta_i, \tau_l)\right) \\ + \left(r_{\Omega_1}(\theta_i, \tau_l) + \eta_{\Omega_1}(\theta_i, \tau_l)\right)\left(r_{\Omega_2}(\theta_i, \tau_l) + \eta_{\Omega_2}(\theta_i, \tau_l)\right) \end{array} \right]$$

$$\leq \left(\begin{array}{l} \left(p_{\Omega_1}(\theta_i, \tau_l) + \mu_{\Omega_1}(\theta_i, \tau_l)\right)^2 + \left(q_{\Omega_1}(\theta_i, \tau_l) + \nu_{\Omega_1}(\theta_i, \tau_l)\right)^2 \\ + \left(r_{\Omega_1}(\theta_i, \tau_l) + \eta_{\Omega_1}(\theta_i, \tau_l)\right)^2 + \left(p_{\Omega_2}(\theta_i, \tau_l) + \mu_{\Omega_2}(\theta_i, \tau_l)\right)^2 \\ + \left(q_{\Omega_2}(\theta_i, \tau_l) + \nu_{\Omega_2}(\theta_i, \tau_l)\right)^2 + \left(r_{\Omega_2}(\theta_i, \tau_l) + \eta_{\Omega_2}(\theta_i, \tau_l)\right)^2 \end{array} \right)$$

Hence, $0 \leq Sim_D(\Omega_1, \Omega_2) \leq 1$. Thus, the first inequality is shown to be true.

(2) It is easy to observe that this is true

(3) When $\Omega_1 = \Omega_2$, then it implies that $Sim_D(\Omega_1, \Omega_2) = 1$. On the other hand, if $Sim_D(\Omega_1, \Omega_2) = 1$ then,

$p_{\Omega_1}(\theta_i, \tau_l) = p_{\Omega_2}(\theta_i, \tau_l)$; $\mu_{\Omega_1}(\theta_i, \tau_l) = \mu_{\Omega_2}(\theta_i, \tau_l)$; $q_{\Omega_1}(\theta_i, \tau_l) = q_{\Omega_2}(\theta_i, \tau_l)$; $\nu_{\Omega_1}(\theta_i, \tau_l) = \nu_{\Omega_2}(\theta_i, \tau_l)$; $r_{\Omega_1}(\theta_i, \tau_l) = r_{\Omega_2}(\theta_i, \tau_l)$; $\eta_{\Omega_1}(\theta_i, \tau_l) = \eta_{\Omega_2}(\theta_i, \tau_l)$;

This implies that $\Omega_1 = \Omega_2$. Thus, the third inequality is proved.

(4) When $\Omega_1 \subseteq \Omega_2 \subseteq \Omega_3$, we can have

$p_{\Omega_1}(\theta_i, \tau_l) + \mu_{\Omega_1}(\theta_i, \tau_l) \leq p_{\Omega_2}(\theta_i, \tau_l) + \mu_{\Omega_2}(\theta_i, \tau_l) \leq p_{\Omega_3}(\theta_i, \tau_l) + \mu_{\Omega_3}(\theta_i, \tau_l)$;

$q_{\Omega_1}(\theta_i, \tau_l) + \nu_{\Omega_1}(\theta_i, \tau_l) \geq q_{\Omega_2}(\theta_i, \tau_l) + \nu_{\Omega_2}(\theta_i, \tau_l) \geq q_{\Omega_3}(\theta_i, \tau_l) + \nu_{\Omega_3}(\theta_i, \tau_l)$;

$r_{\Omega_1}(\theta_i, \tau_l) + \eta_{\Omega_1}(\theta_i, \tau_l) \geq r_{\Omega_2}(\theta_i, \tau_l) + \eta_{\Omega_2}(\theta_i, \tau_l) \geq r_{\Omega_3}(\theta_i, \tau_l) + \eta_{\Omega_3}(\theta_i, \tau_l)$;

So, $Sim_D(\Omega_1, \Omega_3) \leq Sim_D(\Omega_1, \Omega_2)$ and $Sim_D(\Omega_1, \Omega_3) \leq Sim_D(\Omega_2, \Omega_3)$. So, the fourth inequality is proved

4.2. *TCNS - Jaccard similarity measure*

Definition 4.4. Let two TCNSs

$\Omega_1 = \left\langle p_{\Omega_1}(\theta_i,\tau_l)\, e^{j\mu_{\Omega_1}(\theta_i,\tau_l)}, q_{\Omega_1}(\theta_i,\tau_l)\, e^{j\nu_{\Omega_1}(\theta_i,\tau_l)}, r_{\Omega_1}(\theta_i,\tau_l)\, e^{j\eta_{\Omega_1}(\theta_i,\tau_l)} \right\rangle$ and $\Omega_2 = \left\langle p_{\Omega_2}(\theta_i,\tau_l)\, e^{j\mu_{\Omega_2}(\theta_i,\tau_l)}, q_{\Omega_2}(\theta_i,\tau_l)\, e^{j\nu_{\Omega_2}(\theta_i,\tau_l)}, r_{\Omega_2}(\theta_i,\tau_l)\, e^{j\eta_{\Omega_2}(\theta_i,\tau_l)} \right\rangle$ and $\theta_i; i = 1,2,...,n_U$ and $\tau_l; l = 1,2,...,n_\tau$.

$Sim_J(\Omega_1,\Omega_2))$ is a TCNS - Jaccard similarity measure between Ω_1 and Ω_2 and $Sim_J(\Omega_1,\Omega_2))$ is defined as follows:

$$Sim_J(\Omega_1,\Omega_2) = \frac{1}{n_U * n_\tau}\sum_{i=1}^{n_U}\sum_{l=1}^{n_\tau}\frac{\begin{array}{l}(p_{\Omega_1}(\theta_i,\tau_l)+\mu_{\Omega_1}(\theta_i,\tau_l))(p_{\Omega_2}(\theta_i,\tau_l)+\mu_{\Omega_2}(\theta_i,\tau_l)) \\ +(q_{\Omega_1}(\theta_i,\tau_l)+\nu_{\Omega_1}(\theta_i,\tau_l))(q_{\Omega_2}(\theta_i,\tau_l)+q_{\Omega_2}(\theta_i,\tau_l)) \\ +(r_{\Omega_1}(\theta_i,\tau_l)+r_{\Omega_1}(\theta_i,\tau_l))(r_{\Omega_2}(\theta_i,\tau_l)+r_{\Omega_2}(\theta_i,\tau_l))\end{array}}{\begin{array}{l}(p_{\Omega_1}(\theta_i,\tau_l)+\mu_{\Omega_1}(\theta_i,\tau_l))^2+(q_{\Omega_1}(\theta_i,\tau_l)+\nu_{\Omega_1}(\theta_i,\tau_l))^2 \\ +(r_{\Omega_1}(\theta_i,\tau_l)+r_{\Omega_1}(\theta_i,\tau_l))^2+(p_{\Omega_2}(\theta_i,\tau_l)+\mu_{\Omega_2}(\theta_i,\tau_l))^2 \\ +(q_{\Omega_2}(\theta_i,\tau_l)+q_{\Omega_2}(\theta_i,\tau_l))^2+(r_{\Omega_2}(\theta_i,\tau_l)+r_{\Omega_2}(\theta_i,\tau_l))^2 \\ -\left(\begin{array}{l}(p_{\Omega_1}(\theta_i,\tau_l)+\mu_{\Omega_1}(\theta_i,\tau_l))(p_{\Omega_2}(\theta_i,\tau_l)+\mu_{\Omega_2}(\theta_i,\tau_l)) \\ +(q_{\Omega_1}(\theta_i,\tau_l)+\nu_{\Omega_1}(\theta_i,\tau_l))(q_{\Omega_2}(\theta_i,\tau_l)+q_{\Omega_2}(\theta_i,\tau_l)) \\ +(r_{\Omega_1}(\theta_i,\tau_l)+r_{\Omega_1}(\theta_i,\tau_l))(r_{\Omega_2}(\theta_i,\tau_l)+r_{\Omega_2}(\theta_i,\tau_l))\end{array}\right)\end{array}} \quad (6)$$

Example 4.5. Let $U = \{\theta_1,\theta_2\}$;$\tilde{\tau} = \{\tau_1,\tau_2,\tau_3\}$; and two TCNS Ω_1, Ω_2 in U.

$$\Omega_1(\theta_1) = \left\{\begin{array}{l}\left\langle 0.1e^{j0.2}, 0.6e^{j0.45}, 0.3e^{j0.56}\right\rangle, \left\langle 0.8e^{j0.77}, 0.2e^{j0.86}, 0.4e^{j0.65}\right\rangle, \\ \left\langle 0.6e^{j0.52}, 0.2e^{j0.2}, 0.36e^{j0.2}\right\rangle\end{array}\right\}$$

$$\Omega_1(\theta_2) = \left\{\begin{array}{l}\left\langle 0.5e^{j0.12}, 0.2e^{j0.65}, 0.5e^{j0.85}\right\rangle, \left\langle 0.3e^{j0.8}, 0.7e^{j0.23}, 0.1e^{j0.32}\right\rangle, \\ \left\langle 0.74e^{j0.6}, 0.5e^{j0.15}, 0.66e^{j0.41}\right\rangle\end{array}\right\}$$

$$\Omega_2(\theta_1) = \left\{\begin{array}{l}\left\langle 0.6e^{j0.8}, 0.9e^{j0.25}, 0.4e^{j0.52}\right\rangle, \left\langle 0.6e^{j0.62}, 0.5e^{j0.24}, 0.3e^{j0.18}\right\rangle, \\ \left\langle 0.3e^{j0.82}, 0.3e^{j0.6}, 0.4e^{j0.71}\right\rangle\end{array}\right\}$$

$$\Omega_2(\theta_2) = \left\{\begin{array}{l}\left\langle 0.3e^{j0.22}, 0.4e^{j0.56}, 0.6e^{j0.26}\right\rangle, \left\langle 0.1e^{j0.38}, 0.5e^{j0.43}, 0.5e^{j0.24}\right\rangle, \\ \left\langle 0.2e^{j0.29}, 0.4e^{j0.72}, 0.5e^{j0.65}\right\rangle\end{array}\right\}$$

$Sim_J(\Omega_1,\Omega_2)$ is The TTCNS - Jaccard similarity measure between Ω_1, Ω_2 can be: $Sim_J(\Omega_1,\Omega_2) = 0.78807$

Theorem 4.6. *Let Ω_1 and Ω_2 be two TCNSs then,*

(1) $0 \le Sim_J(\Omega_1,\Omega_2) \le 1$

(2) $Sim_J(\Omega_1,\Omega_2) = Sim_J(\Omega_2,\Omega_1)$

(3) $Sim_J(\Omega_1,\Omega_2) = 1$*, if and only if* $\Omega_1 = \Omega_2$

(4) if Ω_3 is a TCNS in U and $\Omega_1 \subseteq \Omega_2 \subseteq \Omega_3$ then $Sim_J(\Omega_1,\Omega_3) \le Sim_J(\Omega_1,\Omega_2)$ and $Sim_J(\Omega_1,\Omega_3) \le Sim_J(\Omega_2,\Omega_3)$

Proof

(1) Since:

$$\left(\begin{array}{l}\left(p_{\Omega_1}(\theta_i,\tau_l)+\mu_{\Omega_1}(\theta_i,\tau_l)\right)\left(p_{\Omega_2}(\theta_i,\tau_l)+\mu_{\Omega_2}(\theta_i,\tau_l)\right)\\+\left(q_{\Omega_1}(\theta_i,\tau_l)+\nu_{\Omega_1}(\theta_i,\tau_l)\right)\left(q_{\Omega_2}(\theta_i,\tau_l)+\nu_{\Omega_2}(\theta_i,\tau_l)\right)\\+\left(r_{\Omega_1}(\theta_i,\tau_l)+\eta_{\Omega_1}(\theta_i,\tau_l)\right)\left(r_{\Omega_2}(\theta_i,\tau_l)+\eta_{\Omega_2}(\theta_i,\tau_l)\right)\end{array}\right)$$

$$\leq\left(\begin{array}{l}\left(p_{\Omega_1}(\theta_i,\tau_l)+\mu_{\Omega_1}(\theta_i,\tau_l)\right)^2+\left(q_{\Omega_1}(\theta_i,\tau_l)+\nu_{\Omega_1}(\theta_i,\tau_l)\right)^2\\+\left(r_{\Omega_1}(\theta_i,\tau_l)+\eta_{\Omega_1}(\theta_i,\tau_l)\right)^2+\left(p_{\Omega_2}(\theta_i,\tau_l)+\mu_{\Omega_2}(\theta_i,\tau_l)\right)^2\\+\left(q_{\Omega_2}(\theta_i,\tau_l)+\nu_{\Omega_2}(\theta_i,\tau_l)\right)^2+\left(r_{\Omega_2}(\theta_i,\tau_l)+\eta_{\Omega_2}(\theta_i,\tau_l)\right)^2\\-\left[\begin{array}{l}\left(p_{\Omega_1}(\theta_i,\tau_l)+\mu_{\Omega_1}(\theta_i,\tau_l)\right)\left(p_{\Omega_2}(\theta_i,\tau_l)+\mu_{\Omega_2}(\theta_i,\tau_l)\right)\\+\left(q_{\Omega_1}(\theta_i,\tau_l)+\nu_{\Omega_1}(\theta_i,\tau_l)\right)\left(q_{\Omega_2}(\theta_i,\tau_l)+\nu_{\Omega_2}(\theta_i,\tau_l)\right)\\+\left(r_{\Omega_1}(\theta_i,\tau_l)+\eta_{\Omega_1}(\theta_i,\tau_l)\right)\left(r_{\Omega_2}(\theta_i,\tau_l)+\eta_{\Omega_2}(\theta_i,\tau_l)\right)\end{array}\right]\end{array}\right)$$

Hence, we have, $0 \leq Sim_J(\Omega_1,\Omega_2) \leq 1$

(2) It is easily observed.

(3) When $\Omega_1=\Omega_2$, then, $Sim_J(\Omega_1,\Omega_2)=1$. On the other hand, if $Sim_J(\Omega_1,\Omega_2)=1$ then, $p_{\Omega_1}(\theta_i,\tau_l)=p_{\Omega_2}(\theta_i,\tau_l)$; $\mu_{\Omega_1}(\theta_i,\tau_l)=\mu_{\Omega_2}(\theta_i,\tau_l)$; $q_{\Omega_1}(\theta_i,\tau_l)=q_{\Omega_2}(\theta_i,\tau_l)$; $\nu_{\Omega_1}(\theta_i,\tau_l)=\nu_{\Omega_2}(\theta_i,\tau_l)$; $r_{\Omega_1}(\theta_i,\tau_l)=r_{\Omega_2}(\theta_i,\tau_l)$; $\eta_{\Omega_1}(\theta_i,\tau_l)=\eta_{\Omega_2}(\theta_i,\tau_l)$

This implies that $\Omega_1=\Omega_2$.

(4) When $\Omega_1 \subseteq \Omega_2 \subseteq \Omega_3$, we can write $p_{\Omega_1}(\theta_i,\tau_l)+\mu_{\Omega_1}(\theta_i,\tau_l) \leq p_{\Omega_2}(\theta_i,\tau_l)+\mu_{\Omega_2}(\theta_i,\tau_l) \leq p_{\Omega_3}(\theta_i,\tau_l)+\mu_{\Omega_3}(\theta_i,\tau_l)$;

$q_{\Omega_1}(\theta_i,\tau_l)+\nu_{\Omega_1}(\theta_i,\tau_l) \geq q_{\Omega_2}(\theta_i,\tau_l)+\nu_{\Omega_2}(\theta_i,\tau_l) \geq q_{\Omega_3}(\theta_i,\tau_l)+\nu_{\Omega_3}(\theta_i,\tau_l)$;

$r_{\Omega_1}(\theta_i,\tau_l)+\eta_{\Omega_1}(\theta_i,\tau_l) \geq r_{\Omega_2}(\theta_i,\tau_l)+\eta_{\Omega_2}(\theta_i,\tau_l) \geq r_{\Omega_3}(\theta_i,\tau_l)+\eta_{\Omega_3}(\theta_i,\tau_l)$;

So, $Sim_J(\Omega_1,\Omega_3) \leq S_J(\Omega_1,\Omega_2)$ and $Sim_J(\Omega_1,\Omega_3) \leq Sim_J(\Omega_2,\Omega_3)$.

4.3. *TCNS - Cosine Similarity*

The cosine similarity measure is one of the practical measures in the decision-making process. The similarity between two TCNS can be computed with the use of the temporal complex neutrosophic cosine measure. Time factors and time cycles are of interest to this metric, which employs complex neutrosophic numbers.

The cosine similarity measure for TCNSs is defined as follows:

Definition 4.7. Let $\Omega_1 = \left\langle p_{\Omega_1}(\theta_i,\tau_l)\,e^{j\mu_{\Omega_1}(\theta_i,\tau_l)}, q_{\Omega_1}(\theta_i,\tau_l)\,e^{j\nu_{\Omega_1}(\theta_i,\tau_l)}, r_{\Omega_1}(\theta_i,\tau_l)\,e^{j\eta_{\Omega_1}(\theta_i,\tau_l)}\right\rangle$ and $\Omega_2 = \left\langle p_{\Omega_2}(\theta_i,\tau_l)\,e^{j\mu_{\Omega_2}(\theta_i,\tau_l)}, q_{\Omega_2}(\theta_i,\tau_l)\,e^{j\nu_{\Omega_2}(\theta_i,\tau_l)}, r_{\Omega_2}(\theta_i,\tau_l)\,e^{j\eta_{\Omega_2}(\theta_i,\tau_l)}\right\rangle$ be two TCNSs and $\theta_i\,(1,2,3,...,n_U)$ belong to U and $\tau_l\,(1,2,3,...,n_\tau)$.

$Sim_{\text{Cos}}(\Omega_1, \Omega_2)$ is A TCNS - Cosine similarity measure between TCNSs Ω_1 and Ω_2 and $Sim_{\text{Cos}}(\Omega_1, \Omega_2)$ is proposed as follows:

$$
\begin{aligned}
&Sim_{\text{Cos}}(\Omega_1, \Omega_2) \\
&= \frac{1}{n_U * n_\tau} \sum_{i=1}^{n_U} \sum_{l=1}^{n_\tau} \frac{\begin{array}{l}(p_{\Omega_1}(\theta_i, \tau_l) + \mu_{\Omega_1}(\theta_i, \tau_l))(p_{\Omega_2}(\theta_i, \tau_l) + \mu_{\Omega_2}(\theta_i, \tau_l)) \\ + (q_{\Omega_1}(\theta_i, \tau_l) + \nu_{\Omega_1}(\theta_i, \tau_l))(q_{\Omega_2}(\theta_i, \tau_l) + q_{\Omega_2}(\theta_i, \tau_l)) \\ + (r_{\Omega_1}(\theta_i, \tau_l) + r_{\Omega_1}(\theta_i, \tau_l))(r_{\Omega_2}(\theta_i, \tau_l) + r_{\Omega_2}(\theta_i, \tau_l))\end{array}}{\begin{array}{l}\sqrt{\begin{array}{l}(p_{\Omega_1}(\theta_i, \tau_l) + \mu_{\Omega_1}(\theta_i, \tau_l))^2 + (q_{\Omega_1}(\theta_i, \tau_l) + \nu_{\Omega_1}(\theta_i, \tau_l))^2 \\ + (r_{\Omega_1}(\theta_i, \tau_l) + r_{\Omega_1}(\theta_i, \tau_l))^2\end{array}} \\ \times \sqrt{\begin{array}{l}(p_{\Omega_2}(\theta_i, \tau_l) + \mu_{\Omega_2}(\theta_i, \tau_l))^2 + (q_{\Omega_2}(\theta_i, \tau_l) + q_{\Omega_2}(\theta_i, \tau_l))^2 \\ + (r_{\Omega_2}(\theta_i, \tau_l) + r_{\Omega_2}(\theta_i, \tau_l))^2\end{array}}\end{array}}
\end{aligned} \tag{7}
$$

Example 4.8. Let $U = \{\theta_1, \theta_2\}; \tilde{\tau} = \{\tau_1, \tau_2, \tau_3\}$ and Ω_1 and Ω_2 are two TCNS in U

$$
\Omega_1(\theta_1) = \left\{ \begin{array}{l} \left\langle 0.1e^{j0.2}, 0.6e^{j0.45}, 0.3e^{j0.56} \right\rangle, \left\langle 0.8e^{j0.77}, 0.2e^{j0.86}, 0.4e^{j0.65} \right\rangle, \\ \left\langle 0.6e^{j0.52}, 0.2e^{j0.2}, 0.36e^{j0.2} \right\rangle \end{array} \right\}
$$

$$
\Omega_1(\theta_2) = \left\{ \begin{array}{l} \left\langle 0.5e^{j0.12}, 0.2e^{j0.65}, 0.5e^{j0.85} \right\rangle, \left\langle 0.3e^{j0.8}, 0.7e^{j0.23}, 0.1e^{j0.32} \right\rangle, \\ \left\langle 0.74e^{j0.6}, 0.5e^{j0.15}, 0.66e^{j0.41} \right\rangle \end{array} \right\}
$$

$$
\Omega_2(\theta_1) = \left\{ \begin{array}{l} \left\langle 0.6e^{j0.8}, 0.9e^{j0.25}, 0.4e^{j0.52} \right\rangle, \left\langle 0.6e^{j0.62}, 0.5e^{j0.24}, 0.3e^{j0.18} \right\rangle, \\ \left\langle 0.3e^{j0.82}, 0.3e^{j0.6}, 0.4e^{j0.71} \right\rangle \end{array} \right\}
$$

$$
\Omega_2(\theta_2) = \left\{ \begin{array}{l} \left\langle 0.3e^{j0.22}, 0.4e^{j0.56}, 0.6e^{j0.26} \right\rangle, \left\langle 0.1e^{j0.38}, 0.5e^{j0.43}, 0.5e^{j0.24} \right\rangle, \\ \left\langle 0.2e^{j0.29}, 0.4e^{j0.72}, 0.5e^{j0.65} \right\rangle \end{array} \right\}
$$

We have the following results:

$Sim_{\text{Cos}}(\Omega_1, \Omega_2) = 0.73203$

Theorem 4.9. *Let Ω_1 and Ω_2 be two TCNSs then,*

(1) $0 \leq Sim_{\text{Cos}}(\Omega_1, \Omega_2) \leq 1$

(2) $Sim_{\text{Cos}}(\Omega_1, \Omega_2) = Sim_{\text{Cos}}(\Omega_2, \Omega_1)$

(3) $Sim_{\text{Cos}}(\Omega_1, \Omega_2) = 1$*, if and only if* $\Omega_1 = \Omega_2$

(4) if Ω_3 is a TCNS in U and $\Omega_1 \subseteq \Omega_2 \subseteq \Omega_3$ then $Sim_{\text{Cos}}(\Omega_1, \Omega_3) \leq Sim_{\text{Cos}}(\Omega_1, \Omega_2)$ *and* $Sim_{\text{Cos}}(\Omega_1, \Omega_3) \leq Sim_{\text{Cos}}(\Omega_2, \Omega_3)$

Proof

(1) We can have,

$$\left(\begin{array}{l}\left(p_{\Omega_1}\left(\theta_i,\tau_l\right)+\mu_{\Omega_1}\left(\theta_i,\tau_l\right)\right)\left(p_{\Omega_2}\left(\theta_i,\tau_l\right)+\mu_{\Omega_2}\left(\theta_i,\tau_l\right)\right)\\+\left(q_{\Omega_1}\left(\theta_i,\tau_l\right)+\nu_{\Omega_1}\left(\theta_i,\tau_l\right)\right)\left(q_{\Omega_2}\left(\theta_i,\tau_l\right)+\nu_{\Omega_2}\left(\theta_i,\tau_l\right)\right)\\+\left(r_{\Omega_1}\left(\theta_i,\tau_l\right)+\eta_{\Omega_1}\left(\theta_i,\tau_l\right)\right)\left(r_{\Omega_2}\left(\theta_i,\tau_l\right)+\eta_{\Omega_2}\left(\theta_i,\tau_l\right)\right)\end{array}\right)^2$$
$$\leq\left(\begin{array}{l}\left(p_{\Omega_1}\left(\theta_i,\tau_l\right)+\mu_{\Omega_1}\left(\theta_i,\tau_l\right)\right)^2+\left(q_{\Omega_1}\left(\theta_i,\tau_l\right)+\nu_{\Omega_1}\left(\theta_i,\tau_l\right)\right)^2\\+\left(r_{\Omega_1}\left(\theta_i,\tau_l\right)+\eta_{\Omega_1}\left(\theta_i,\tau_l\right)\right)^2+\left(p_{\Omega_2}\left(\theta_i,\tau_l\right)+\mu_{\Omega_2}\left(\theta_i,\tau_l\right)\right)^2\\+\left(q_{\Omega_2}\left(\theta_i,\tau_l\right)+\nu_{\Omega_2}\left(\theta_i,\tau_l\right)\right)^2+\left(r_{\Omega_2}\left(\theta_i,\tau_l\right)+\eta_{\Omega_2}\left(\theta_i,\tau_l\right)\right)^2\end{array}\right)$$

Hence, $0 \leq S_{\mathrm{Cos}}\left(\Omega_1,\Omega_2\right) \leq 1$

(2) It is easily observed.

(3) Since $\Omega_1=\Omega_2$, then $Sim_{\mathrm{Cos}}\left(\Omega_1,\Omega_2\right)=1$. On the other hand, if $Sim_{\mathrm{Cos}}\left(\Omega_1,\Omega_2\right)=1$, we have,

$p_{\Omega_1}\left(\theta_i,\tau_l\right)=p_{\Omega_2}\left(\theta_i,\tau_l\right)$;$\mu_{\Omega_1}\left(\theta_i,\tau_l\right)=\mu_{\Omega_2}\left(\theta_i,\tau_l\right)$; $q_{\Omega_1}\left(\theta_i,\tau_l\right)=q_{\Omega_2}\left(\theta_i,\tau_l\right)$.; $\nu_{\Omega_1}\left(\theta_i,\tau_l\right)=\nu_{\Omega_2}\left(\theta_i,\tau_l\right)$; $r_{\Omega_1}\left(\theta_i,\tau_l\right)=r_{\Omega_2}\left(\theta_i,\tau_l\right)$; $\eta_{\Omega_1}\left(\theta_i,\tau_l\right)=\eta_{\Omega_2}\left(\theta_i,\tau_l\right)$;

This implies that $\Omega_1=\Omega_2$

(4) When $\Omega_1 \subseteq \Omega_2 \subseteq \Omega_3$, we can have

$p_{\Omega_1}\left(\theta_i,\tau_l\right)+\mu_{\Omega_1}\left(\theta_i,\tau_l\right) \leq p_{\Omega_2}\left(\theta_i,\tau_l\right)+\mu_{\Omega_2}\left(\theta_i,\tau_l\right) \leq p_{\Omega_3}\left(\theta_i,\tau_l\right)+\mu_{\Omega_3}\left(\theta_i,\tau_l\right)$;

$q_{\Omega_1}\left(\theta_i,\tau_l\right)+\nu_{\Omega_1}\left(\theta_i,\tau_l\right) \geq q_{\Omega_2}\left(\theta_i,\tau_l\right)+\nu_{\Omega_2}\left(\theta_i,\tau_l\right) \geq q_{\Omega_3}\left(\theta_i,\tau_l\right)+\nu_{\Omega_3}\left(\theta_i,\tau_l\right)$;

$r_{\Omega_1}\left(\theta_i,\tau_l\right)+\eta_{\Omega_1}\left(\theta_i,\tau_l\right) \geq r_{\Omega_2}\left(\theta_i,\tau_l\right)+\eta_{\Omega_2}\left(\theta_i,\tau_l\right) \geq r_{\Omega_3}\left(\theta_i,\tau_l\right)+\eta_{\Omega_3}\left(\theta_i,\tau_l\right)$;

So, $Sim_J\left(\Omega_1,\Omega_3\right) \leq Sim_J\left(\Omega_1,\Omega_2\right)$ and $Sim_J\left(\Omega_1,\Omega_3\right) \leq Sim_J\left(\Omega_2,\Omega_3\right)$

Hence, Theorem 4.9 is proved.

4.4. *TCNS - Cotangent Similarity*

In this part, we utilize the special characteristics of the Cotangent function to present a novel method of similarity measurement in the context of TCNS.

Definition 4.10. Let two TCNSs

$\Omega_1 = \left\langle p_{\Omega_1}\left(\theta_i,\tau_l\right)e^{j\mu_{\Omega_1}\left(\theta_i,\tau_l\right)}, q_{\Omega_1}\left(\theta_i,\tau_l\right)e^{j\nu_{\Omega_1}\left(\theta_i,\tau_l\right)}, r_{\Omega_1}\left(\theta_i,\tau_l\right)e^{j\eta_{\Omega_1}\left(\theta_i,\tau_l\right)}\right\rangle$ and $\Omega_2 = \left\langle p_{\Omega_2}\left(\theta_i,\tau_l\right)e^{j\mu_{\Omega_2}\left(\theta_i,\tau_l\right)}, q_{\Omega_2}\left(\theta_i,\tau_l\right)e^{j\nu_{\Omega_2}\left(\theta_i,\tau_l\right)}, r_{\Omega_2}\left(\theta_i,\tau_l\right)e^{j\eta_{\Omega_2}\left(\theta_i,\tau_l\right)}\right\rangle$ for all $\theta_i\left(1,2,3,...,n_U\right)$ belong to U and $\tau_l\left(1,2,3,...,n_\tau\right)$.

$Sim_{\text{Cot\_1}}(\Omega_1, \Omega_2)$ is a TCNS - Cotangent similarity measure between TCNSs Ω_1 and Ω_2 and $Sim_{\text{Cot\_1}}(\Omega_1, \Omega_2)$ is defined as follows:

$$Sim_{\text{Cot\_1}}(\Omega_1, \Omega_2) = \cot\left(\frac{\pi}{4} + \frac{\pi}{8} \times \frac{1}{n_U * n_\tau} \sum_{i=1}^{n_U} \sum_{l=1}^{n_\tau} \left[\begin{array}{l} max\left(\begin{array}{l} \mid p_{\Omega_1}(\theta_i, \tau_l) - p_{\Omega_2}(\theta_i, \tau_l) \mid, \\ \mid q_{\Omega_1}(\theta_i, \tau_l) - q_{\Omega_2}(\theta_i, \tau_l) \mid, \\ \mid r_{\Omega_1}(\theta_i, \tau_l) - r_{\Omega_2}(\theta_i, \tau_l) \mid \end{array} \right) \\ + \frac{1}{2\pi} max\left(\begin{array}{l} \mid \mu_{\Omega_1}(\theta_i, \tau_l) - \mu_{\Omega_2}(\theta_i, \tau_l) \mid, \\ \mid \nu_{\Omega_1}(\theta_i, \tau_l) - \nu_{\Omega_2}(\theta_i, \tau_l) \mid, \\ \mid \eta_{\Omega_1}(\theta_i, \tau_l) - \eta_{\Omega_2}(\theta_i, \tau_l) \mid \end{array} \right) \end{array} \right] \right) \tag{8}$$

$$Sim_{\text{Cot\_2}}(\Omega_1, \Omega_2) = \cot\left(\frac{\pi}{4} + \frac{\pi}{24} \times \frac{1}{n_U * n_\tau} \sum_{i=1}^{n_U} \sum_{l=1}^{n_\tau} \left[\begin{array}{l} \left(\begin{array}{l} \mid p_{\Omega_1}(\theta_i, \tau_l) - p_{\Omega_2}(\theta_i, \tau_l) \mid \\ + \mid q_{\Omega_1}(\theta_i, \tau_l) - q_{\Omega_2}(\theta_i, \tau_l) \mid \\ + \mid r_{\Omega_1}(\theta_i, \tau_l) - r_{\Omega_2}(\theta_i, \tau_l) \mid \end{array} \right) \\ + \frac{1}{2\pi} max\left(\begin{array}{l} \mid \mu_{\Omega_1}(\theta_i, \tau_l) - \mu_{\Omega_2}(\theta_i, \tau_l) \mid \\ + \mid \nu_{\Omega_1}(\theta_i, \tau_l) - \nu_{\Omega_2}(\theta_i, \tau_l) \mid \\ + \mid \eta_{\Omega_1}(\theta_i, \tau_l) - \eta_{\Omega_2}(\theta_i, \tau_l) \mid \end{array} \right) \end{array} \right] \right) \tag{9}$$

Example 4.11. Let $U = \{\theta_1, \theta_2\}$; $\tilde{\tau} = \{\tau_1, \tau_2, \tau_3\}$; Ω_1 and Ω_2 are two TCNS in U

$$\Omega_1(\theta_1) = \left\{ \begin{array}{l} \left\langle 0.1e^{j0.2}, 0.6e^{j0.45}, 0.3e^{j0.56} \right\rangle, \left\langle 0.8e^{j0.77}, 0.2e^{j0.86}, 0.4e^{j0.65} \right\rangle, \\ \left\langle 0.6e^{j0.52}, 0.2e^{j0.2}, 0.36e^{j0.2} \right\rangle \end{array} \right\}$$

$$\Omega_1(\theta_2) = \left\{ \begin{array}{l} \left\langle 0.5e^{j0.12}, 0.2e^{j0.65}, 0.5e^{j0.85} \right\rangle, \left\langle 0.3e^{j0.8}, 0.7e^{j0.23}, 0.1e^{j0.32} \right\rangle, \\ \left\langle 0.74e^{j0.6}, 0.5e^{j0.15}, 0.66e^{j0.41} \right\rangle \end{array} \right\}$$

$$\Omega_2(\theta_1) = \left\{ \begin{array}{l} \left\langle 0.6e^{j0.8}, 0.9e^{j0.25}, 0.4e^{j0.52} \right\rangle, \left\langle 0.6e^{j0.62}, 0.5e^{j0.24}, 0.3e^{j0.18} \right\rangle, \\ \left\langle 0.3e^{j0.82}, 0.3e^{j0.6}, 0.4e^{j0.71} \right\rangle \end{array} \right\}$$

$$\Omega_2(\theta_2) = \left\{ \begin{array}{l} \left\langle 0.3e^{j0.22}, 0.4e^{j0.56}, 0.6e^{j0.26} \right\rangle, \left\langle 0.1e^{j0.38}, 0.5e^{j0.43}, 0.5e^{j0.24} \right\rangle, \\ \left\langle 0.2e^{j0.29}, 0.4e^{j0.72}, 0.5e^{j0.65} \right\rangle \end{array} \right\}$$

Since Definitions 4.10, we have: $Sim_{Cot\_1}(\Omega_1, \Omega_2) = 0.690485$; $Sim_{Cot\_2}(\Omega_1, \Omega_2) = 0.82243$

Theorem 4.12. *Let Ω_1 and Ω_2 be two TCNSs then,*

(1) $0 \leq Sim_{Cot}(\Omega_1, \Omega_2) \leq 1$

(2) $Sim_{Cot}(\Omega_1, \Omega_2) = Sim_{Cot}(\Omega_2, \Omega_1)$

(3) $Sim_{Cot}(\Omega_1, \Omega_2) = 1$, *if and only if* $\Omega_1 = \Omega_2$

(4) if Ω_3 *is a TCNS in* U *and* $\Omega_1 \subseteq \Omega_2 \subseteq \Omega_3$ *then* $Sim_{Cot}(\Omega_1, \Omega_3) \leq Sim_{Cot}(\Omega_1, \Omega_2)$ *and* $Sim_{Cot}(\Omega_1, \Omega_3) \leq Sim_{Cot}(\Omega_2, \Omega_3)$

Proof

The proof follows the same general approach as that used for Equations (5) and (6), with specific adaptations for this case.

(1) We have,

Hence, $0 \leq Sim_{Cot}(\Omega_1, \Omega_2) \leq 1$

(2) It is easily observed.

(3) When $\Omega_1 = \Omega_2$, then obviously$S_{Cot_2}(\Omega_1, \Omega_2) = 1$. On the other hand, if $S_{Cot\_2}(\Omega_1, \Omega_2) = 1$ then,

$p_{\Omega_1}(\theta_i, \tau_l) = p_{\Omega_2}(\theta_i, \tau_l)$;$\mu_{\Omega_1}(\theta_i, \tau_l) = \mu_{\Omega_2}(\theta_i, \tau_l)$;$q_{\Omega_1}(\theta_i, \tau_l) = q_{\Omega_2}(\theta_i, \tau_l)$; $\nu_{\Omega_1}(\theta_i, \tau_l) = \nu_{\Omega_2}(\theta_i, \tau_l)$; $r_{\Omega_1}(\theta_i, \tau_l) = r_{\Omega_2}(\theta_i, \tau_l)$; $\eta_{\Omega_1}(\theta_i, \tau_l) = \eta_{\Omega_2}(\theta_i, \tau_l)$;

This implies that $\Omega_1 = \Omega_2$

(4) When $\Omega_1 \subseteq \Omega_2 \subseteq \Omega_3$, wen can have

$p_{\Omega_1}(\theta_i, \tau_l) + \mu_{\Omega_1}(\theta_i, \tau_l) \leq p_{\Omega_2}(\theta_i, \tau_l) + \mu_{\Omega_2}(\theta_i, \tau_l) \leq p_{\Omega_3}(\theta_i, \tau_l) + \mu_{\Omega_3}(\theta_i, \tau_l)$;

$q_{\Omega_1}(\theta_i, \tau_l) + \nu_{\Omega_1}(\theta_i, \tau_l) \geq q_{\Omega_2}(\theta_i, \tau_l) + \nu_{\Omega_2}(\theta_i, \tau_l) \geq q_{\Omega_3}(\theta_i, \tau_l) + \nu_{\Omega_3}(\theta_i, \tau_l)$;

$r_{\Omega_1}(\theta_i, \tau_l) + \eta_{\Omega_1}(\theta_i, \tau_l) \geq r_{\Omega_2}(\theta_i, \tau_l) + \eta_{\Omega_2}(\theta_i, \tau_l) \geq r_{\Omega_3}(\theta_i, \tau_l) + \eta_{\Omega_3}(\theta_i, \tau_l)$;

So, $Sim_{Cot\_2}(\Omega_1, \Omega_3) \leq Sim_{Cot\_2}(\Omega_1, \Omega_2)$ and $Sim_{Cot\_2}(\Omega_1, \Omega_3) \leq Sim_{Cot\_2}(\Omega_2, \Omega_3)$. The Theorem 4.12 is proved.

4.5. *Entropy measures of Temporal complex neutrosophic set*

When Zadeh [3] first proposed the entropy metric, it was intended to quantify fuzziness in information. A measure of the uncertainty and fuzziness present in any set: fuzzy set, complex fuzzy set, neutrophilic, etc.—is the entropy measure. Since the TCNS can handle ambiguous and uncertain data in this work, finding a TCNS's entropy is particularly important.

Definition 4.13. A function E qualifies as a TCNS - Entropy measure on a finite set U if *E: TCNS(U)* $\rightarrow$ [0, 1] and it satisfies the following properties:

(a) $E(\Omega) = 0$ if Ω is a crisp set

(b) $E(\Omega) = 1$ if $p_\Omega(\theta, \tau) = 0.5$;$q_\Omega(\theta, \tau) = 0.5$;$r_\Omega(\theta, \tau) = 0.5$ and $\mu_\Omega(\theta, \tau) = \pi$;$\nu_\Omega(\theta, \tau) = \pi$;$\eta_\Omega(\theta, \tau) = \pi$ for $\forall \tau \in \tilde{\tau}$;$\forall \theta \in U$.

(c) $E(\Omega_1) \leq E(\Omega_2)$ if Ω_2 is more uncertain that Ω_1, that is $| p_{\Omega_1}(\theta, \tau) - 0.5 | \geq | p_{\Omega_2}(\theta, \tau) - 0.5 |$; $| q_{\Omega_1}(\theta, \tau) - 0.5 | \geq | q_{\Omega_2}(\theta, \tau) - 0.5 |$; $| r_{\Omega_1}(\theta, \tau) - 0.5 | \geq | r_{\Omega_2}(\theta, \tau) - 0.5 |$ and

$| \mu_{\Omega_1}(\theta, \tau) - \pi | \geq | \mu_{\Omega_2}(\theta, \tau) - \pi |$; $| \nu_{\Omega_1}(\theta, \tau) - \pi | \geq | \nu_{\Omega_2}(\theta, \tau) - \pi |$; $| \eta_{\Omega_1}(\theta, \tau) - \pi | \geq | \eta_{\Omega_2}(\theta, \tau) - \pi |$

(d) $E(\Omega) = E(\Omega^c)$, Ω^c is a TCNS complement of Ω

Theorem 4.14. *Let a Temporal Complex Neutrosophic Set Ω on U. A TCNS - The following equation suggests entropy metric of Ω:*

$$E(\Omega) = \tan\left(\frac{\pi}{6 * n_U * n_\tau} \sum_{i=1}^{n_U} \sum_{l=1}^{n_\tau} e_{il}(\Omega)\right) \tag{10}$$

Where

$$e_{il}(\Omega) = \begin{pmatrix} p_\Omega(\theta_i,\tau_l)(1 - p_\Omega(\theta_i,\tau_l)) + q_\Omega(\theta_i,\tau_l)(1 - q_\Omega(\theta_i,\tau_l)) \\ + r_\Omega(\theta_i,\tau_l)(1 - r_\Omega(\theta_i,\tau_l)) + \left(\frac{\mu_\Omega(\theta_i,\tau_l)}{2\pi}\right)\left(1 - \frac{\mu_\Omega(\theta_i,\tau_l)}{2\pi}\right) \\ + \left(\frac{\nu_\Omega(\theta_i,\tau_l)}{2\pi}\right)\left(1 - \frac{\nu_\Omega(\theta_i,\tau_l)}{2\pi}\right) + \left(\frac{\eta_\Omega(\theta_i,\tau_l)}{2\pi}\right)\left(1 - \frac{\eta_\Omega(\theta_i,\tau_l)}{2\pi}\right) \end{pmatrix}$$

Proof.

$E(\Omega)$ must strictly adhere to all conditions outlined in Definition 4.13.

(a) Assume that $E(\Omega) = 0$, it implies that $p(\theta_i,\tau_l) = 0$; $q(\theta_i,\tau_l) = 0$; $r(\theta_i,\tau_l) = 0$; $\mu(\theta_i,\tau_l) = 0$; $v(\theta_i,\tau_l) = 0$; $r(\theta_i,\tau_l) = 0$or $p(\theta_i,\tau_l) = 1$; $q(\theta_i,\tau_l) = 1$; $r(\theta_i,\tau_l) = 1$; $\mu(\theta_i,\tau_l) = 2\pi$; $v(\theta_i,\tau_l) = 2\pi$; $r(\theta_i,\tau_l) = 2\pi$;

(b) $E(\Omega) = 1$, if and only if $p(\theta_i,\tau_l) = 0.5$; $q(\theta_i,\tau_l) = 0.5$; $r(\theta_i,\tau_l) = 0.5$; $\mu(\theta_i,\tau_l) = \pi$; $v(\theta_i,\tau_l) = \pi$; $r(\theta_i,\tau_l) = \pi$

(c) Assum Ω_1, Ω_2 and Ω_2 more uncertain than that Ω_2. It implies that, if $p_{\Omega_1}(\theta,\tau) \leq p_{\Omega_2}(\theta,\tau) \leq 0.5$; $q_{\Omega_1}(\theta,\tau) \leq q_{\Omega_2}(\theta,\tau) \leq 0.5$; $r_{\Omega_1}(\theta,\tau) \leq r_{\Omega_2}(\theta,\tau) \leq 0.5$ or $p_{\Omega_1}(\theta,\tau) \geq p_{\Omega_2}(\theta,\tau) \geq 0.5$; $q_{\Omega_1}(\theta,\tau) \geq q_{\Omega_2}(\theta,\tau) \geq 0.5$; $r_{\Omega_1}(\theta,\tau) \geq r_{\Omega_2}(\theta,\tau) \geq 0.5$ for each $\theta_i \in X$.

Hence,

$p_{\Omega 1}(\theta_i,\tau_l)(1 - p_{\Omega_1}(\theta_i,\tau_l)) \leq p_{\Omega 2}(\theta_i,\tau_l)(1 - p_{\Omega_2}(\theta_i,\tau_l))$
$q_{\Omega 1}(\theta_i,\tau_l)(1 - q_{\Omega_1}(\theta_i,\tau_l)) \leq q_{\Omega 2}(\theta_i,\tau_l)(1 - q_{\Omega_2}(\theta_i,\tau_l))$;
$r_{\Omega 1}(\theta_i,\tau_l)(1 - r_{\Omega_1}(\theta_i,\tau_l)) \leq r_{\Omega 2}(\theta_i,\tau_l)(1 - r_{\Omega_2}(\theta_i,\tau_l))$;
and $\frac{\mu_{\Omega_1}(\theta_i,\tau_l)}{2\pi}\left(1 - \frac{\mu_{\Omega_1}(\theta_i,\tau_l)}{2\pi}\right) \leq \frac{\mu_{\Omega_2}(\theta_i,\tau_l)}{2\pi}\left(1 - \frac{\mu_{\Omega_2}(\theta_i,\tau_l)}{2\pi}\right)$;
$\frac{\nu_{\Omega 1}(\theta_i,\tau_l)}{2\pi}\left(1 - \frac{\nu_{\Omega_1}(\theta_i,\tau_l)}{2\pi}\right) \leq \frac{\nu_{\Omega 2}(\theta_i,\tau_l)}{2\pi}\left(1 - \frac{\nu_{\Omega_2}(\theta_i,\tau_l)}{2\pi}\right)$;
$\frac{\eta_{\Omega 1}(\theta_i,\tau_l)}{2\pi}\left(1 - \frac{\eta_{\Omega 1}(\theta_i,\tau_l)}{2\pi}\right) \leq \frac{\eta_{\Omega_2}(\theta_i,\tau_l)}{2\pi}\left(1 - \frac{\eta_{\Omega 2}(\theta_i,\tau_l)}{2\pi}\right)$;
which implies $E(\Omega_1) \leq E(\Omega_2)$

(d) $E(\Omega) = E(\Omega^c)$ is trivial

Example 4.15. Let $U = \{\theta_1, \theta_2\}$;$\tilde{\tau} = \{\tau_1, \tau_2, \tau_3\}$ and Ω_1 and Ω_2 are two TCNS in U

$$\Omega_1(\theta_1) = \left\{ \begin{array}{l} \left\langle 0.1e^{j0.2}, 0.6e^{j0.45}, 0.3e^{j0.56} \right\rangle, \left\langle 0.8e^{j0.77}, 0.2e^{j0.86}, 0.4e^{j0.65} \right\rangle, \\ \left\langle 0.6e^{j0.52}, 0.2e^{j0.2}, 0.36e^{j0.2} \right\rangle \end{array} \right\}$$

$$\Omega_1(\theta_2) = \left\{ \begin{array}{l} \left\langle 0.5e^{j0.12}, 0.2e^{j0.65}, 0.5e^{j0.85} \right\rangle, \left\langle 0.3e^{j0.8}, 0.7e^{j0.23}, 0.1e^{j0.32} \right\rangle, \\ \left\langle 0.74e^{j0.6}, 0.5e^{j0.15}, 0.66e^{j0.41} \right\rangle \end{array} \right\}$$

$$\Omega_2(\theta_1) = \left\{ \begin{array}{l} \left\langle 0.6e^{j0.8}, 0.9e^{j0.25}, 0.4e^{j0.52} \right\rangle, \left\langle 0.6e^{j0.62}, 0.5e^{j0.24}, 0.3e^{j0.18} \right\rangle, \\ \left\langle 0.3e^{j0.82}, 0.3e^{j0.6}, 0.4e^{j0.71} \right\rangle \end{array} \right\}$$

$$\Omega_2(\theta_2) = \left\{ \begin{array}{l} \left\langle 0.3e^{j0.22}, 0.4e^{j0.56}, 0.6e^{j0.26} \right\rangle, \left\langle 0.1e^{j0.38}, 0.5e^{j0.43}, 0.5e^{j0.24} \right\rangle, \\ \left\langle 0.2e^{j0.29}, 0.4e^{j0.72}, 0.5e^{j0.65} \right\rangle \end{array} \right\}$$

Entropy values can be estimated by applying Definition 7 as follows: $E_{\Omega_1} = 0.44476$; $E_{\Omega_2} = 0.47580$;

5. **Multi-Criteria Decision Making**

Consider that $\tilde{A}$, $\tilde{C}$, $\tilde{D}$ are sets of alternatives, requirements and decision makers; where $\tilde{A} = \{A_1, A_2, ..., A_{n_A}\}$ and $\tilde{C} = \{C_1, C_2, ..., C_{n_C}\}$ and $\tilde{D} = \{D_1, D_2, ..., D_{n_D}\}$. With regard to the decision maker D_{i_d}; $i_d = 1, 2, ..., n_D$, the evaluation description of an alternatives A_{i_a}; $i_a = 1, 2, ..., n_A$ on an attribute C_{i_c}; $i_c = 1, 2, ..., n_C$ in given time period τ_l; $l = 1, 2, ..., n_\tau$ is represented by matrix $U^{i_d}(\tau_l) = \left(\theta^{i_d}_{i_a i_c}(\tau_l)\right)_{m \times n}$. where $\theta^{i_d}_{i_a i_c}(\tau_l)$ chosen as the language identifier of CNS by given period τ_l.

Let

$$\theta_{i_a i_c i_d}(\tau_l) = \left\langle p_{i_a i_c i_d}(\tau_l) e^{j\mu_{i_a i_c i_d}(\tau_l)}, q_{i_a i_c i_d}(\tau_l) e^{j\nu_{i_a i_c i_d}(\tau_l)}, r_{i_a i_c i_d}(\tau_l) e^{j\eta_{i_a i_c i_d}(\tau_l)} \right\rangle$$

Step 1. In the absence of any pre-existing information regarding criterion weights, we proceed to calculate the weight w_{i_c} of criterion C_{i_c} as follows:

Equation (11) can be used to evaluate the decision makers' averaged rating.

$$\theta_{i_a i_c} = \frac{1}{n_D * n_\tau} \overset{n_\tau}{\underset{l=1}{\oplus}} \overset{n_D}{\underset{i_a=1}{\oplus}} x_{i_a i_c i_d}(\tau_l) = \left\langle \tilde{T}_{i_a i_c}, \tilde{I}_{i_a i_c}, \tilde{F}_{i_a i_c} \right\rangle \tag{11}$$

The weight w_{i_c} of criterion C_{i_c} is calculated by Equation (12)

$$w_{i_c} = \frac{\left(1 - E_{C_{i_c}}\right)}{n_c - \sum_{i=1}^{n_c} E_{C_{i_c}}} \tag{12}$$

Where $E_{C_{i_c}} = \frac{1}{n_A} \sum_{i_a=1}^{n_A} E(\theta_{i_a i_c})$; each $E(\theta_{i_a i_c})$ is calculated using Equation (10)

Step 2. The temporal complex neutrosophic positive ideal solution and temporal complex neutrosophic negative ideal solution are represented by two values, $TCN-PIS$ and $TCN-NIS$, respectively, and are determined in the following way:

$$TCN-PIS=\left\{\theta,\left\langle \max\left(p_{i_ai_c}\left(\theta,\tilde{\tau}\right)\right)e^{j\max(\mu_{i_ai_c}(\theta,\tilde{\tau}))},0,0\right\rangle\right\} \tag{13}$$

$$TCN-NIS= \left\{\theta,\left\langle 0,\min\left(q_{i_ai_c}\left(\theta,\tilde{\tau}\right)\right)e^{j\min(\nu_{i_ai_c}(\theta,\tilde{\tau}))},\min\left(r_{i_ai_c}\left(\theta,\tilde{\tau}\right)\right)e^{j\min(\eta_{i_ai_c}(\theta,\tilde{\tau}))}\right\rangle\right\} \tag{14}$$

Step 3 For alternatives to $TCN-PIS$ and $TCN-NIS$, we can analyze the weighted similarity index using Theorems 4.3-4.12. And equation (15) is used to calculate the weighted similarity measure $S_{i_a}^{+}$ and $S_{i_a}^{-}$ of A_{i_a} $(i_a=1,2,...,n_A)$-(16)

$$S_{i_a}^{+}=\overset{n_A}{\underset{i_a=1}{\oplus}} w_{i_c}Sim\left(\theta_{i_ai_c},TCN-PIS\right) \tag{15}$$

$$S_{i_a}^{-}=\overset{n_A}{\underset{i_a=1}{\oplus}} w_{i_c}S\left(\theta_{i_ai_c},TCN-NIS\right) \tag{16}$$

Step 4. We determine the relative closeness of alternative A_{i_a} to the ideal solution using the following calculation

$$RR_{i_a}=\frac{S_{i_a}^{-}}{S_{i_a}^{+}+S_{i_a}^{-}} \tag{17}$$

Step 5. Through a comparison based on relative closeness values, we have identified the most promising alternatives, listed in descending order of their suitability.

6. **A case study for choosing a tourist destination**

A practical use of recommended metrics to a decision problem analyzing a real-world scenario is shown in this section. The case study is a particular illustration of the challenge of selecting a travel destination in Vietnam. This model can be applied to travel companies supporting and advising tourists.

6.1. *Problem description*

The viability and efficacy of the suggested decision technique are illustrated by an example provided by Lan et al. [17] regarding the selection of a tourism location in Vietnam. Figure 1 presents an MCDM model for choosing a tourism destination in Vietnam. Assume for the moment that a business is looking for a tourism attraction in Vietnam. For example, L_i; $i=1,2,3,4,5$ is one of the five tourist locations that the company recommended. There are twenty sub-criteria that are specific to each place and three group criterion that determine which site is the best tourist destination. This enables a more precise evaluation of the different locations.

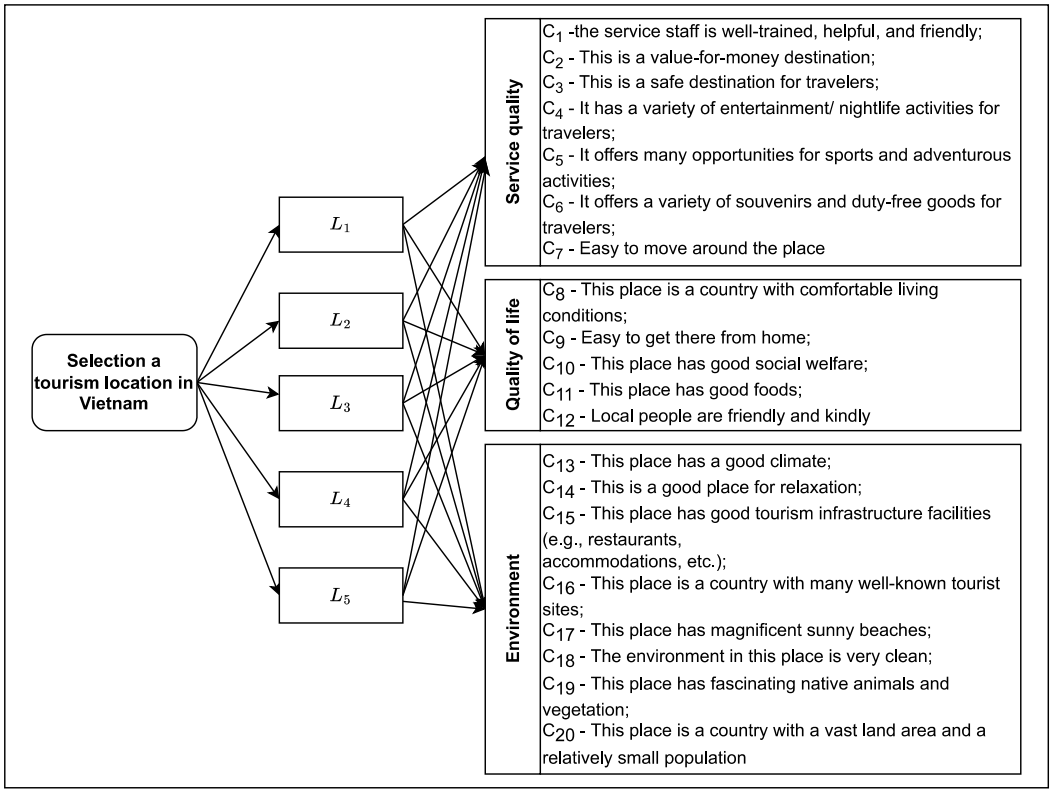
Selection a tourism location in Vietnam
L_1
L_2
L_3
L_4
L_5
Service quality
C_1 -the service staff is well-trained, helpful, and friendly;
C_2 - This is a value-for-money destination;
C_3 - This is a safe destination for travelers;
C_4 - It has a variety of entertainment/ nightlife activities for travelers;
C_5 - It offers many opportunities for sports and adventurous activities;
C_6 - It offers a variety of souvenirs and duty-free goods for travelers;
C_7 - Easy to move around the place
Quality of life
C_8 - This place is a country with comfortable living conditions;
C_9 - Easy to get there from home;
C_{10} - This place has good social welfare;
C_{11} - This place has good foods;
C_{12} - Local people are friendly and kindly
Environment
C_{13} - This place has a good climate;
C_{14} - This is a good place for relaxation;
C_{15} - This place has good tourism infrastructure facilities (e.g., restaurants, accommodations, etc.);
C_{16} - This place is a country with many well-known tourist sites;
C_{17} - This place has magnificent sunny beaches;
C_{18} - The environment in this place is very clean;
C_{19} - This place has fascinating native animals and vegetation;
C_{20} - This place is a country with a vast land area and a relatively small population

FIGURE 1. Selection a tourism location in Vietnam

TABLE 1. Language variables

Label	Number
Very low (VL)	$\langle 0.15e^{j0.55}, 0.65e^{j0.45}, 0.65e^{j0.35}\rangle$
Low (L)	$\langle 0.25e^{j0.65}, 0.55e^{j0.55}, 0.65e^{j0.45}\rangle$
Medium (M)	$\langle 0.40e^{j0.75}, 0.50e^{j0.65}, 0.45e^{j0.55}\rangle$
High (H)	$\langle 0.55e^{j0.85}, 0.45e^{j0.75}, 0.35e^{j0.65}\rangle$
Very high (VH)	$\langle 0.65e^{j0.95}, 0.25e^{j0.85}, 0.25e^{j0.75}\rangle$

Three experts with varying backgrounds and specialties have been invited to conduct a thorough study of the four locations. The estimates made by the expert are represented as TCN numbers.

6.2. *Solution approach using the proposed measures*

TCNS is a technology that helps characterize periodic, ambiguous information that we encounter in our daily lives. In order to show that the suggested similarity measures are useful in resolving real-world issues, such as selecting tourism sites in multi-criteria decision-making problems, we provide a real-world instance in the TCNS environment in this section.

Table 2. Similarity measure Values at step 3

Measures	Ideal solution	Alternatives				
		L_1	L_2	L_3	L_4	L_5
Sim_D	S^+	0.63893	0.63931	0.63727	0.63702	0.63459
	S^-	0.65671	0.65388	0.65677	0.66025	0.66090
Sim_j	S^+	0.46947	0.46989	0.46768	0.46740	0.46479
	S^-	0.48892	0.48580	0.48899	0.49283	0.49354
Sim_{Cos}	S^+	0.58630	0.58890	0.58585	0.58270	0.58158
	S^-	0.55933	0.55967	0.56067	0.56004	0.56200
Sim_{Cot_1}	S^+	0.64133	0.62623	0.63643	0.66239	0.65205
	S^-	0.59517	0.57743	0.59477	0.62114	0.61690
Sim_{Cot_2}	S^+	0.718926	0.71437	0.71740	0.72466	0.72094
	S^-	0.78670	0.76771	0.78476	0.81370	0.80613

Step 1. Equations 11 and 12 can be used for calculating the weight of each of the twenty criteria.

$w_1 = 0.04920; w_2 = 0.05787; w_3 = 0.05274; w_4 = 0.04981; w_5 = 0.03842;$

$w_6 = 0.04216; w_7 = 0.04805; w_8 = 0.05547; w_9 = 0.05323; w_{10} = 0.04190;$

$w_{11} = 0.04967; w_{12} = 0.0439; w_{13} = 0.04418; w_{14} = 0.04968; w_{15} = 0.05861;$

$w_{16} = 0.05000; w_{17} = 0.053227; w_{18} = 0.05968; w_{19} = 0.05397; w_{20} = 0.05745;$

Step 2. The positive ideal solution (TCN-PIS) and negative ideal solution (TCN-NIS) of the temporal complex neutrosophic should be estimated. Here's how the values are derived:

$$TCN-PIS = \left\{ \begin{array}{l} \left\langle 0.696238.\ \mathrm{e}^{\mathrm{j}0.916667}, 0, 0 \right\rangle, \left\langle 0.696238.\mathrm{e}^{\mathrm{j}0.916667}, 0, 0 \right\rangle, \\ \left\langle 0.696238.\mathrm{e}^{\mathrm{j}0.916667}, 0, 0 \right\rangle, \left\langle 0.696238.\mathrm{e}^{\mathrm{j}0.916667}, 0, 0 \right\rangle \end{array} \right\}$$

$$TCN-NIS = \left\{ \begin{array}{l} \left\langle 0, 0.30411.\mathrm{e}^{\mathrm{j}0.583333}, 0.279672.\mathrm{e}^{0.483333} \right\rangle, \\ \left\langle 0, 0.30411.\mathrm{e}^{\mathrm{j}0.583333}, 0.279672.\mathrm{e}^{0.483333} \right\rangle, \\ \left\langle 0, 0.30411.\mathrm{e}^{\mathrm{j}0.583333}, 0.279672.\mathrm{e}^{0.483333} \right\rangle, \\ \left\langle 0, 0.30411.\mathrm{e}^{\mathrm{j}0.583333}, 0.279672.\mathrm{e}^{0.483333} \right\rangle \end{array} \right\}$$

Step 3. Determine TCN-PIS and TCN-NIS's similarity metrics for five different places as shown in Table 2.

Step 4. Calculate the proximity of the optimal solution to five different places. They are described in Table 3.

Step 5. Table 3 displays the five places that are ordered according to their relative proximity.

Table 3. The relative proximity of places determined via similarity metrics

Mesuares		Alternatives				
		L_1	L_2	L_3	L_4	L_5
Sim_D	RR	0.50686	0.50563	0.50753	0.50895	**0.51015**
	In Order	4	5	3	2	1
Sim_J	RR	0.51015	0.50832	0.51114	0.51324	**0.51500**
	In Order	4	5	3	2	1
Sim_{Cos}	RR	0.48823	0.48728	0.48902	0.49009	**0.49144**
	In Order	4	5	3	2	1
Sim_{Cot_1}	RR	0.48134	0.47973	0.48308	0.48393	**0.48615**
	In Order	4	5	3	2	1
Sim_{Cot_2}	RR	0.52251	0.51800	0.52242	**0.52894**	0.52789
	In Order	3	5	4	1	2

6.3. *Result and Discussion*

In this section, We validate the proposed model's efficacy through comparison with Ali et al.'s models (2020) [38] in temporal complex neutrosophic environments. In Table 3, the ranking results of 5 locations at four different periods, $\tau_1, \tau_4, \tau_3, \tau_4$, are suitably explained for the problem of selecting a tourist destination. According to Table 4, the rankings of five locations at four times each of τ_1, τ_4, τ_3, and τ_4 are $L_5 \succ L_4 \succ L_1 \succ L_3 \succ L_2$; $L_5 \succ L_4 \succ L_3 \succ L_1 \succ L_2$; $L_5 \succ L_4 \succ L_3 \succ L_1 \succ L_2$ and $L_5 \succ L_4 \succ L_1 \succ L_3 \succ L_2$ and in all four time periods $L_5$is assigned as the optimal location. In the meantime, the suggested method's overall outcome at four different periods based on the metrics: Dice, Jaccard, Cosine, and Cotangent similarity, respectively, are $L_5 \succ L_4 \succ L_3 \succ L_1 \succ L_2$; $L_5 \succ L_4 \succ L_3 \succ L_1 \succ L_2$; $L_5 \succ L_4 \succ L_3 \succ L_1 \succ L_2$; $L_4 \succ L_5 \succ L_1 \succ L_3 \succ L_2$. The best location according to the combined results of the Cosine, Jaccard, and Dice similarity measures is L_5; the best location according to the Cotangent similarity measure is L_4.

This finding demonstrates how the benefits and practical application of the suggested measures handle time-related decision-making challenges in a TCNS. Furthermore, in TCN contexts, it is more flexible and generalized than the technique of Ali et al. [38]. The ability to handle periodic and temporal factors in the data effectively is one of TCNS's advantages over standard CNS. The proposed approach has the considerable advantage of taking into account the impact of temporal aspects in CNS disorders while also reducing information loss. In other words, we argue that new similarity and distance measures will provide decision-makers with various options based on their optimistic and pessimistic behavior during the decision-making process.

Table 4. Results of Ali et al. method for locations at four periods

Measures		Alternitives				
		L_1	L_2	L_3	L_4	L_5
τ_1	RR	0.81427	0.80866	0.81220	0.81627	**0.81948**
	In Order	3	5	4	2	1
τ_2	RR	0.81330	0.81059	0.81335	0.81762	**0.82221**
	In Order	4	5	3	2	1
τ_3	RR	0.81413	0.80887	0.81348	0.81707	**0.82197**
	In Order	4	5	3	2	1
τ_4	RR	0.81316	0.80860	0.81309	0.81665	**0.81934**
	In Order	3	5	4	2	1

7. Conclusion, Limitations, and Future Works

TCNS theory is a valuable tool for solving problems concerning the uncertain, temporal, and periodical factors of decision-making. In this paper, we have introduced four similarity measures: Dice, Jaccard, Cosine, and cotangent in the context of the TCNS. Furthermore, we have established an entropy measure of TCNS to ascertain the weights of unknown attributes in MCDM. Next, based on suggested entropy and similarity metrics in the TCNS environment, a new MCDM technique has also been devised. Lastly, the TCN environment provides a numerical example of decision-making issues when selecting a tourist location in Vietnam. It is provided to highlight the benefits and real-world suitability of the suggested actions. It is demonstrated through the case study that the suggested TCNS entropy and similarity measurements can yield credible outcomes for decision-making issues. However, the proposed measures only use the discrete temporal variables and apply to only a practical problem of tourist destinations chosen in Vietnam. Hence, in future follow-up research, we can use TCNS to account for continuous time variables and apply our model to other real-life problems, including selecting workers, medical treatment, logistics center choosing, etc.

Funding: This work was funded by the Research Project: THTETN.05/23-24, Vietnam Academy of Science and Technology.

Acknowledgments: The authors would like to thank Professor Florentin Smarandache for valuable comments and suggestions.

Conflicts of Interest: The authors declare no conflict of interest.

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Received: March 1, 2024. Accepted: July 25, 2024

NSS

Neutrosophic Sets and Systems, Vol. 70, 2024

University of New Mexico

Statistical Convergence Sequences in Neutrosophic Metric Spaces

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Abstract. In this study, motivated by the notion of statistical convergence in intuitionistic fuzzy metric space, we present the ideas of statistical convergence and cauchy sequences with regard to the neutrosophic metric spaces. This study also discusses the statistical completeness on neutrosophic metric space with an example and results.

Keywords: Neutrosophic metric; Statistically convergent sequence; Statistically Cauchy sequence

1. Introduction

In 2004, utilising the intuitionistic fuzzy set [1], Fuzzy Metric Spaces ($\mathfrak{FMS}$) has been widened to Intuitionistic Fuzzy Metric Space ($\mathfrak{IFMS}$) by Park [13]. Park used continuous triangular norms as well as continuous triangular conorms to describe this idea. For $\mathfrak{FMS}$, Changqing et al. [4] developed Statistical Convergence ($\mathfrak{SC}$) sequences in 2020. On $\mathfrak{IFMS}$, similar ideas were used by several auhtors and their work, who achieved considerable outcomes. Fast [5] established the concept of $\mathfrak{SC}$ in 1951, garnering interest from researchers across both purely mathematical and practical disciplines. $\mathfrak{FMS}$ and $\mathfrak{IFS}$ were used for investigating several new breakthroughs, including fixed point theories and convergence by Granados et. al [7]. In $\mathfrak{NMS}$, similar outcomes can be derived by Jeyaraman et al [9]. In [10], Kramosil and Michalek presented the idea of $\mathfrak{FMS}$. Fuzzy sets were first presented by Zadeh [21], and numerous writers have since addressed their notions under numerous contexts, including $\mathfrak{FMS}$.

The concept of generalization of convergence, denoted by $\mathfrak{SC}$, is defined as follows: Let $\mathfrak{M} \subseteq \mathfrak{I}N$, where $\mathfrak{I}N$ represents the set of all positive integers. For every $\mathfrak{n} \in \mathfrak{I}N$, $\mathfrak{M}(\mathfrak{n})$ is defined as $\{\lambda \leq \mathfrak{n} : \lambda \in \mathfrak{M}\}$. The natural (or asymptotic) density of $\mathfrak{M}$, denoted by $\Delta(\mathfrak{M})$, is defined by $\Delta(\mathfrak{M}) = \lim_{n\to\infty} \frac{|\mathfrak{M}(\mathfrak{n})|}{\mathfrak{n}}$ if the limit exists, where $|\mathfrak{M}(\mathfrak{n})|$ represents the cardinality of the set $\mathfrak{M}(\mathfrak{n})$. The value of $\Delta(\mathfrak{M})$ lies in the interval $[0, 1]$ and satisfies $\Delta(\mathfrak{I}N\backslash\mathfrak{M}) = 1 - \Delta(\mathfrak{M})$ if $\Delta(\mathfrak{M})$ exists.

For instance, $\Delta(\mathfrak{I}\varrho) = 1$, $\Delta(\mathfrak{P}) = \frac{1}{2}$ where $\mathfrak{P}$ denotes the set of even positive integers, and $\Delta(\mathfrak{B}) = 0$ where $\mathfrak{B}$ is a finite subset of $\mathfrak{I}N$. A set $\mathfrak{M}$ is considered statistically dense if $\Delta(\mathfrak{M}) = 1$. A sequence $(\zeta_{\mathfrak{n}}) \subset \mathfrak{I}R$ is said to be $\mathfrak{SC}$ to $\zeta_0 \in \mathfrak{I}R$ if for every $\mathfrak{e} > 0$, the condition $\Delta(\{\mathfrak{n} \in \mathfrak{I}N : |\zeta_{\mathfrak{n}} - \zeta_0| < \mathfrak{e}\}) = 1$ is satisfied, where $\mathfrak{I}R$ denotes the set of all real numbers. Numerous significant results on $\mathfrak{SC}$ have been presented by various authors ([?]- [19]).

in this work, convergent and $\mathfrak{SC}$ relations in $\mathfrak{NMS}$ are then investigated. On $\mathfrak{NMS}$, we also investigate $\mathfrak{SCa}$ sequences and statistical completeness.

2. **Preliminaries**

Definition 2.1. [7] Let κ, ϱ, and φ be fuzzy sets on $\Omega^2 \times (0, \infty)$, where $\odot$ is a continuous triangular norm and $\oplus$ is a continuous triangular conorm. We say that $(\kappa, \varrho, \varphi)$ forms a Neutrosophic Metric ($\mathfrak{NM}$) on Ω if κ and ϱ satisfy the following conditions:

(1) $\kappa(\zeta, \xi, \varpi) + \varrho(\zeta, \xi, \varpi) + \varphi(\zeta, \xi, \varpi) \leq 3$;
(2) $0 < \kappa(\zeta, \xi, \varpi) < 1$, $0 < \varrho(\zeta, \xi, \varpi) < 1$ and $0 < \varphi(\zeta, \xi, \varpi) < 1$;
(3) $\kappa(\zeta, \xi, \varpi) > 0$;
(4) $\kappa(\zeta, \xi, \varpi) = 1 \iff \zeta = \xi$;
(5) $\kappa(\zeta, \xi, \varpi) = \kappa(\xi, \zeta, \varpi)$;
(6) $\kappa(\zeta, \xi, \varpi) \odot \kappa(\xi, \tilde{v}, \mathfrak{u}) \leq \kappa(\zeta, \tilde{v}, \varpi + \mathfrak{u})$;
(7) $\kappa(\zeta, \xi, .) : (0, \infty) \to (0, 1]$ is continuous;
(8) $\varrho(\zeta, \xi, \varpi) > 0$;
(9) $\varrho(\zeta, \xi, \varpi) = 0 \iff \zeta = \xi$;
(10) $\varrho(\zeta, \xi, \varpi) = \varrho(\xi, \zeta, \varpi)$;
(11) $\varrho(\zeta, \xi, \varpi) \oplus \varrho(\xi, \tilde{v}, \mathfrak{u}) \geq \varrho(\zeta, \tilde{v}, \varpi + \mathfrak{u})$;
(12) $\varrho(\zeta, \xi, .) : (0, \infty) \to (0, 1]$ is continuous.
(13) $\varphi(\zeta, \xi, \varpi) > 0$;
(14) $\varphi(\zeta, \xi, \varpi) = 0 \iff \zeta = \xi$;
(15) $\varphi(\zeta, \xi, \varpi) = \varphi(\xi, \zeta, \varpi)$;
(16) $\varphi(\zeta, \xi, \varpi) \oplus \varphi(\xi, \tilde{v}, \mathfrak{u}) \geq \varphi(\zeta, \tilde{v}, \varpi + \mathfrak{u})$;
(17) $\varphi(\zeta, \xi, .) : (0, \infty) \to (0, 1]$ is continuous.

A 6-tuple $(\Omega, \kappa, \varrho, \varphi, \odot, \oplus)$ is called $\mathfrak{NMS}$.
The functions $\kappa(\zeta, \xi, \varpi)$, $\varrho(\zeta, \xi, \varpi)$ and $\varphi(\zeta, \xi, \varpi)$ denote nearness degree, non-nearness degree and indeterminacy degree of ζ to ξ at ϖ, respectively.

Example 2.2. Let $\Omega = \mathfrak{I}R, \vartheta \odot \omega = \vartheta\omega$ and $\vartheta \oplus \omega = \min\{\vartheta + \omega, 1\}$ for all $\vartheta, \omega \in [0, 1]$. Define κ, ϱ and φ by $\kappa(\zeta, \xi, \varpi) = \frac{\varpi}{\varpi + |\zeta - \xi|}$, $\varrho(\zeta, \xi, \varpi) = \frac{|\zeta - \xi|}{\varpi + |\zeta - \xi|}$, $\varphi(\zeta, \xi, \varpi) = \frac{|\zeta - \xi|}{\varpi}$ and for all $\zeta, \xi \in \Omega$ and $\varpi > 0$. Then $(\mathfrak{I}R, \kappa, \varrho, \odot, \oplus)$ is a $\mathfrak{NMS}$.

Definition 2.3. Let $(\Omega, \kappa, \varrho, \varphi, \odot, \oplus)$ be a $\mathfrak{NMS}$ and $\varpi > 0, \mathfrak{v} \in (0, 1)$ and $\zeta \in \Omega$. The set $\mathfrak{B}_\zeta(\mathfrak{v}, \varpi) = \{\xi \in \Omega : \kappa(\zeta, \xi, \varpi) > 1 - \mathfrak{v}, \varrho(\zeta, \xi, \varpi) < \mathfrak{v}, \varphi(\zeta, \xi, \varpi) < \mathfrak{v}\}$ is described as an open sphere with center ζ and radius $\mathfrak{v}$ with regard to ϖ.

Example 2.4. Let $\Omega = \mathfrak{I}R, \vartheta \odot \omega = \vartheta\omega$ and $\vartheta \oplus \omega = \min\{\vartheta + \omega, 1\}$ for all $\vartheta, \omega \in [0, 1]$. Define κ, ϱ and φ by $\kappa(\zeta, \xi, \varpi) = \frac{\varpi}{\varpi + |\zeta - \xi|}$, $\varrho(\zeta, \xi, \varpi) = \frac{|\zeta - \xi|}{\varpi + |\zeta - \xi|}$, $\varphi(\zeta, \xi, \varpi) = \frac{|\zeta - \xi|}{\varpi}$ and for all $\zeta, \xi \in \Omega$ and $\varpi > 0$. Then $(\mathfrak{I}R, \kappa, \varrho, \odot, \oplus)$ is an $\mathfrak{NMS}$.
Now define a sequence $(\zeta_\mathfrak{n})$ by $\zeta_\mathfrak{n} = \begin{cases} \frac{1}{\sqrt{\mathfrak{n}}}, & \mathfrak{n} = \lambda^3, \lambda \in \mathfrak{I}N; \\ 0, & \text{otherwise} \end{cases}$
Then, for every $\mathfrak{v} \in (0, 1)$ and for any $\varpi > 0$. we have $(\zeta_\mathfrak{n})$. Here $(\zeta_\mathfrak{n}) \in \mathfrak{B}_\zeta(0, \varpi)$.

Definition 2.5. Let $(\Omega, \kappa, \varrho, \varphi, \odot, \oplus)$ be a $\mathfrak{NMS}$.

(i) A sequence $(\zeta_\mathfrak{n})$ is known to be convergent to ζ if for all $\varpi > 0$ and $\mathfrak{v} \in (0, 1)$ there exists $\mathfrak{n}_0 \in \mathfrak{I}N$ so that $\kappa(\zeta_\mathfrak{n}, \zeta, \varpi) > 1 - \mathfrak{v}$, $\varrho(\zeta_\mathfrak{n}, \zeta, \varpi) < \mathfrak{v}$ and $\varphi(\zeta_\mathfrak{n}, \zeta, \varpi) < \mathfrak{v}$ for every $\mathfrak{n} \geq \mathfrak{n}_0$. It is symbolised as $\zeta_\mathfrak{n} \to \zeta$ as $\mathfrak{n} \to \infty$.
(ii) $\odot\kappa(\zeta_\mathfrak{n}, \zeta, \varpi) \to 1$, $\varrho(\zeta_\mathfrak{n}, \zeta, \varpi) \to 0$ and $\varphi(\zeta_\mathfrak{n}, \zeta, \varpi) \to 0$ as $\mathfrak{n} \to \infty$ for each $\varpi > 0$.
(iii) $(\zeta_\mathfrak{n})$ is known to be a Cauchy sequence if, for $\varpi > 0$ and $\mathfrak{v} \in (0, 1)$, there is $\mathfrak{n}_0 \in \mathfrak{I}N$ such that $\kappa(\zeta_\mathfrak{n}, \zeta_\mathfrak{m}, \varpi) > 1 - \mathfrak{v}$, $\varrho(\zeta_\mathfrak{n}, \zeta_\mathfrak{m}, \varpi) < \mathfrak{v}$ and $\varphi(\zeta_\mathfrak{n}, \zeta_\mathfrak{m}, \varpi) < \mathfrak{v}$ for all $\mathfrak{n}, \mathfrak{m} \geq \mathfrak{n}_0$.
(iv) $(\Omega, \kappa, \varrho, \varphi, \odot, \oplus)$ is called $(\kappa, \varrho, \varphi)$-complete if every Cauchy sequence is convergent.

Example 2.6. Let $\Omega = \mathfrak{I}R, \vartheta \odot \omega = \vartheta\omega$ and $\vartheta \oplus \omega = \min\{\vartheta + \omega, 1\}$ for all $\vartheta, \omega \in [0, 1]$. Define κ, ϱ and φ by $\kappa(\zeta, \xi, \varpi) = \frac{\varpi}{\varpi + |\zeta - \xi|}$, $\varrho(\zeta, \xi, \varpi) = \frac{|\zeta - \xi|}{\varpi + |\zeta - \xi|}$, $\varphi(\zeta, \xi, \varpi) = \frac{|\zeta - \xi|}{\varpi}$ and for all $\zeta, \xi \in \Omega$ and $\varpi > 0$. Then $(\mathfrak{I}R, \kappa, \varrho, \odot, \oplus)$ is an $\mathfrak{IFMS}$.
Now define a sequence $(\zeta_\mathfrak{n})$ by $\zeta_\mathfrak{n} = \begin{cases} \frac{1}{\sqrt[3]{\mathfrak{n}}}, & \mathfrak{n} = \lambda^2, \lambda \in \mathfrak{I}N; \\ 0, & \text{otherwise} \end{cases}$
Then, for every $\mathfrak{v} \in (0, 1)$ and for any $\varpi > 0$,
$\kappa(\zeta_\mathfrak{n}, 0, \varpi) \leq 1 - \mathfrak{v}, \varrho(\zeta_\mathfrak{n}, 0, \varpi) \geq \mathfrak{v}, \varphi(\zeta_\mathfrak{n}, 0, \varpi) \geq \mathfrak{v}$.
$\zeta_\mathfrak{n}$ is both convergent and Cauchy sequence in Ω.

Definition 2.7. Let $(\Omega, \kappa, \odot)$ be a $\mathfrak{FMS}$.

(1) A sequence $(\zeta_{\mathfrak{n}}) \subset \Omega$ is termed $\mathfrak{SC}$ to $\zeta_0 \in \Omega$ if, for every $\mathfrak{v} \in (0,1)$ and $\varpi > 0$, the condition $\Delta(\{\mathfrak{n} \in \mathfrak{I}N : \kappa(\zeta_{\mathfrak{n}}, \zeta_0, \varpi) > 1 - \mathfrak{v}, \varrho(\zeta_{\mathfrak{n}}, 0, \varpi) < \mathfrak{v}\}) = 1$ holds.

(2) A sequence $(\zeta_{\mathfrak{n}}) \subset \Omega$ is referred to as $\mathfrak{SCa}$ if, for every $\mathfrak{v} \in (0,1)$ and $\varpi > 0$, there exists an $\mathfrak{m} \in \mathfrak{I}N$ such that $\Delta(\{\mathfrak{n} \in \mathfrak{I}N : \kappa(\zeta_{\mathfrak{n}}, \zeta_{\mathfrak{m}}, \varpi) > 1 - \mathfrak{v}, \varrho(\zeta_{\mathfrak{n}}, \zeta_{\mathfrak{m}}, \varpi) < \mathfrak{v}\}) = 0$.

Example 2.8. Let $\Omega = \mathfrak{I}R, \vartheta \odot \omega = \vartheta\omega$ and $\vartheta \oplus \omega = \min\{\vartheta + \omega, 1\}$ for all $\vartheta, \omega \in [0,1]$. Define κ, ϱ and φ by $\kappa(\zeta, \xi, \varpi) = \frac{\varpi}{\varpi + |\zeta - \xi|}$, $\varrho(\zeta, \xi, \varpi) = \frac{|\zeta - \xi|}{\varpi + |\zeta - \xi|}$, $\varphi(\zeta, \xi, \varpi) = \frac{|\zeta - \xi|}{\varpi}$ and for all $\zeta, \xi \in \Omega$ and $\varpi > 0$. Then $(\mathfrak{I}R, \kappa, \varrho, \odot, \oplus)$ is an $\mathfrak{IFMS}$.

Now define a sequence $(\zeta_{\mathfrak{n}})$ by $\zeta_{\mathfrak{n}} = \begin{cases} \frac{\sqrt{n}}{2}, & \mathfrak{n} = \lambda^3, \lambda \in \mathfrak{I}N; \\ 0, & \text{otherwise} \end{cases}$

Then, for every $\mathfrak{v} \in (0,1)$ and for any $\varpi > 0$, let $\mathfrak{M} = \{\mathfrak{n} \leq \mathfrak{m} : \kappa(\zeta_{\mathfrak{n}}, 0, \varpi) \leq 1 - \mathfrak{v}, \varrho(\zeta_{\mathfrak{n}}, 0, \varpi) \geq \mathfrak{v}, \varphi(\zeta_{\mathfrak{n}}, 0, \varpi) \geq \mathfrak{v}\} = \left\{\mathfrak{n} \leq \mathfrak{m} : \frac{\varpi}{\varpi + |\zeta_{\mathfrak{n}}|} \leq 1 - \mathfrak{v}, \frac{|\zeta_{\mathfrak{n}}|}{\varpi + |\zeta_{\mathfrak{n}}|} \geq \mathfrak{v}, \frac{|\zeta_{\mathfrak{n}}|}{\varpi} \geq \mathfrak{v}\right\} = \{\mathfrak{n} \leq \mathfrak{m} : |\zeta_{\mathfrak{n}}| \geq \frac{\mathfrak{v}\varpi}{1 - \mathfrak{v}} > 0\} = \{\mathfrak{n} \leq \mathfrak{m} : \zeta_{\mathfrak{n}} = \frac{\sqrt{n}}{2}\} = \{\mathfrak{n} \leq \mathfrak{m} : \mathfrak{n} = \lambda^2, \lambda \in \mathfrak{I}N\}$, and we obtain $\frac{1}{\mathfrak{m}}|\mathfrak{M}| \leq \frac{1}{\mathfrak{m}}|\{\mathfrak{n} \leq \mathfrak{m} : \mathfrak{n} = \lambda^3, \mathfrak{n} \in \mathfrak{I}N\}| \leq \frac{\sqrt[3]{\mathfrak{m}}}{\mathfrak{m}} \to 0, \mathfrak{m} \to \infty$. As a result, we have $(\zeta_{\mathfrak{n}})$ is $\mathfrak{SC}$ to 0 with regard to the $\mathfrak{NMS}$ $(\Omega, \kappa, \varrho, \varphi, \odot, \oplus)$.

3. Statical Convergence in Neutrosophic Metric Spaces

Definition 3.1. Let $(\Omega, \kappa, \varrho, \varphi, \odot, \oplus)$ be a $\mathfrak{NMS}$. A sequence $(\zeta_{\mathfrak{n}}) \subset \Omega$ is called $\mathfrak{SC}$ to $\zeta_0 \in \Omega$ with respect to $\mathfrak{NM}$ obtained that, for every $\mathfrak{v} \in (0,1)$ and $\varpi > 0$, $\Delta(\{\mathfrak{n} \in \mathfrak{I}N : \kappa(\zeta_{\mathfrak{n}}, \zeta_0, \varpi) > 1 - \mathfrak{v}, \varrho(\zeta_{\mathfrak{n}}, \zeta_0, \varpi) < \mathfrak{v}, \varphi(\zeta_{\mathfrak{n}}, \zeta_0, \varpi) < \mathfrak{v}\}) = 1$.

We assert that $(\zeta_{\mathfrak{n}})$ is $\mathfrak{SC}$ to ζ_0. We can observe as $\Delta(\{\mathfrak{n} \in \mathfrak{I}N : \kappa(\zeta_{\mathfrak{n}}, \zeta_0, \varpi) > 1 - \mathfrak{v}, \varrho(\zeta_{\mathfrak{n}}, \zeta, \varpi) < \mathfrak{v}, \varrho(\zeta_{\mathfrak{n}}, \zeta, \varpi) < \mathfrak{v}\}) = 1$

$\Leftrightarrow \lim_{n \to \infty} \frac{|\{k \leq n : \kappa(\zeta_\lambda, \zeta_0, \varpi) > 1 - \mathfrak{v}, \varrho(\zeta_\lambda, \zeta_0, \varpi) < \mathfrak{v}, \varphi(\zeta_\lambda, \zeta_0, \varpi) < \mathfrak{v}\}|}{\mathfrak{n}} = 1$

Example 3.2. Let $\Omega = \mathfrak{I}R, \vartheta \odot \omega = \vartheta\omega$ and $\vartheta \oplus \omega = \min\{\vartheta + \omega, 1\}$ for all $\vartheta, \omega \in [0,1]$. Define κ, ϱ and φ by $\kappa(\zeta, \xi, \varpi) = \frac{\varpi}{\varpi + |\zeta - \xi|}$, $\varrho(\zeta, \xi, \varpi) = \frac{|\zeta - \xi|}{\varpi + |\zeta - \xi|}$, $\varphi(\zeta, \xi, \varpi) = \frac{|\zeta - \xi|}{\varpi}$ and for all $\zeta, \xi \in \Omega$ and $\varpi > 0$. Then $(\mathfrak{I}R, \kappa, \varrho, \odot, \oplus)$ is an $\mathfrak{IFMS}$.

Now define a sequence $(\zeta_{\mathfrak{n}})$ by $\zeta_{\mathfrak{n}} = \begin{cases} 1, & \mathfrak{n} = \lambda^2, \lambda \in \mathfrak{I}N; \\ 0, & \text{otherwise} \end{cases}$

Then, for every $\mathfrak{v} \in (0,1)$ and for any $\varpi > 0$, let $\mathfrak{M} = \{\mathfrak{n} \leq \mathfrak{m} : \kappa(\zeta_{\mathfrak{n}}, 0, \varpi) \leq 1 - \mathfrak{v}, \varrho(\zeta_{\mathfrak{n}}, 0, \varpi) \geq \mathfrak{v}, \varphi(\zeta_{\mathfrak{n}}, 0, \varpi) \geq \mathfrak{v}\} = \left\{\mathfrak{n} \leq \mathfrak{m} : \frac{\varpi}{\varpi + |\zeta_{\mathfrak{n}}|} \leq 1 - \mathfrak{v}, \frac{|\zeta_{\mathfrak{n}}|}{\varpi + |\zeta_{\mathfrak{n}}|} \geq \mathfrak{v}, \frac{|\zeta_{\mathfrak{n}}|}{\varpi} \geq \mathfrak{v}\right\} = \{\mathfrak{n} \leq \mathfrak{m} : |\zeta_{\mathfrak{n}}| \geq \frac{\mathfrak{v}\varpi}{1 - \mathfrak{v}} > 0\} = \{\mathfrak{n} \leq \mathfrak{m} : \zeta_{\mathfrak{n}} = 1\} = \{\mathfrak{n} \leq \mathfrak{m} : \mathfrak{n} = \lambda^2, \lambda \in \mathfrak{I}N\}$, and we obtain $\frac{1}{\mathfrak{m}}|\mathfrak{M}| \leq \frac{1}{\mathfrak{m}}|\{\mathfrak{n} \leq \mathfrak{m} : \mathfrak{n} = \lambda^2, \mathfrak{n} \in \mathfrak{I}N\}| \leq \frac{\sqrt{\mathfrak{m}}}{\mathfrak{m}} \to 0, \mathfrak{m} \to \infty$. As a result, we have $(\zeta_{\mathfrak{n}})$ is $\mathfrak{SC}$ to 0 with regard to the $\mathfrak{NMS}$ $(\Omega, \kappa, \varrho, \varphi, \odot, \oplus)$.

Lemma 3.3. *Let $(\Omega, \kappa, \varrho, \varphi, \odot, \oplus)$ be an $\mathfrak{NMS}$. Then, for every $\mathfrak{v} \in (0,1)$ and $\varpi > 0$. These are comparable to each other:*

(i) $(\zeta_{\mathfrak{n}})$ is $\mathfrak{SC}$ to ζ_0;

(ii) $\Delta(\{\mathfrak{n} \in \mathfrak{I}N : \kappa(\zeta_{\mathfrak{n}}, \zeta_0, \varpi) \leq 1 - \mathfrak{v}\}) = \Delta(\{\varrho(\zeta_{\mathfrak{n}}, \zeta_0, \varpi) \geq \mathfrak{v}, \Delta(\{\varphi(\zeta_{\mathfrak{n}}, \zeta_0, \varpi) \geq \mathfrak{v}\}) = 0$;

(iii) $\Delta(\{\mathfrak{n} \in \mathfrak{I}N : \kappa(\zeta_\mathfrak{n}, \zeta_0, \varpi) > 1 - \mathfrak{v}\}) = \Delta(\{\varrho(\zeta_\mathfrak{n}, \zeta_0, \varpi) < \mathfrak{v}, \Delta(\{\varphi(\zeta_\mathfrak{n}, \zeta_0, \varpi) < \mathfrak{v}\}) = 1.$

Proof: Utilizing Definition (3.1) and we possess a lemma regarding density characteristics.

Theorem 3.4. *Let* $(\Omega, \kappa, \varrho, \varphi, \odot, \oplus)$ *be* $\mathfrak{NMS}$. *When a sequence* $(\zeta_\mathfrak{n})$ *is* $\mathfrak{SC}$ *with regards to the* $\mathfrak{NM}$, *then the* $\mathfrak{SC}$ *limit is unique.*

Proof: Let us consider $(\zeta_\mathfrak{n})$ be $\mathfrak{SC}$ to ζ_1 and ζ_2. Given $\mathfrak{v} \in (0, 1)$, take $\varpi > 0$ such that $(1\varpi) \odot (1\varpi) > 1\mathfrak{v}$ and $\varpi \oplus \varpi < \mathfrak{v}$.

Then classify the following sets, for any $\epsilon > 0$:

$\mathfrak{M}_{\kappa 1}(\varpi, \epsilon) := \{\mathfrak{n} \in \mathfrak{I}N : \kappa(\zeta_\mathfrak{n}, \zeta_1, \epsilon) > 1 - \varpi\}$

$\mathfrak{M}_{\kappa 2}(\varpi, \epsilon) := \{\mathfrak{n} \in \mathfrak{I}N : \kappa(\zeta_\mathfrak{n}, \zeta_2, \epsilon) > 1 - \varpi\}$

$\mathfrak{M}_{\varrho 1}(\varpi, \epsilon) := \{\mathfrak{n} \in \mathfrak{I}N : \varrho(\zeta_\mathfrak{n}, \zeta_1, \epsilon) < \varpi\}$

$\mathfrak{M}_{\varrho 2}(\varpi, \epsilon) := \{\mathfrak{n} \in \mathfrak{I}N : \varrho(\zeta_\mathfrak{n}, \zeta_2, \epsilon) < \varpi\}$

$\mathfrak{M}_{\varphi 1}(\varpi, \epsilon) := \{\mathfrak{n} \in \mathfrak{I}N : \varphi(\zeta_\mathfrak{n}, \zeta_1, \epsilon) < \varpi\}$

$\mathfrak{M}_{\varphi 2}(\varpi, \epsilon) := \{\mathfrak{n} \in \mathfrak{I}N : \varphi(\zeta_\mathfrak{n}, \zeta_2, \epsilon) < \varpi\}$

Since $(\zeta_\mathfrak{n})$ is $\mathfrak{SC}$ with respect to ζ_1 and ζ_2, we get

$\Delta\{\mathfrak{M}_{\kappa 1}(\varpi, \epsilon)\} = \Delta\{\mathfrak{M}_{\varrho 1}(\varpi, \epsilon)\} = \Delta\{\mathfrak{M}_{\varphi 1}(\varpi, \epsilon)\} = 1$ and $\Delta\{\mathfrak{M}_{\kappa 2}(\varpi, \epsilon)\} = \Delta\{\mathfrak{M}_{\varrho 2}(\varpi, \epsilon)\} = 1$, for every $\epsilon > 0$.

Let $K_{\kappa\varrho\varphi}(\varpi, \epsilon) := \{\mathfrak{M}_{\kappa 1}(\varpi, \epsilon) \cup \mathfrak{M}_{\kappa 2}(\varpi, \epsilon)\} \cap \{\mathfrak{M}_{\varrho 1}(\varpi, \epsilon) \cup \mathfrak{M}_{\varrho 2}(\varpi, \epsilon)\} \cap \{\mathfrak{M}_{\varphi 1}(\varpi, \epsilon) \cup \mathfrak{M}_{\varphi 2}(\varpi, \epsilon)\}$.

Therefore, $\Delta\{\mathfrak{M}_{\kappa\varrho\varphi}(\varpi, \epsilon)\} = 1$ which implies that $\Delta\{\mathfrak{I}N \backslash \mathfrak{M}_{\kappa\varrho\varphi}(\varpi, \epsilon)\} = 0$.

When $\mathfrak{n} \in \mathfrak{I}N \backslash \mathfrak{M}_{\kappa\varrho\varphi}(t, \epsilon)$, then there are two potential outcomes:

$\mathfrak{n} \in \mathfrak{I}N \backslash \{\mathfrak{M}_{\kappa 1}(\varpi, \epsilon) \cup \mathfrak{M}_{\kappa 2}(\varpi, \epsilon)\}$ or $\mathfrak{n} \in \mathfrak{I}N \backslash \{\mathfrak{M}_{\varrho 1}(\varpi, \epsilon) \cup \mathfrak{M}_{\varrho 2}(\varpi, \epsilon)\}$ or $\mathfrak{n} \in \mathfrak{I}N \backslash \{\mathfrak{M}_{\varphi 1}(\varpi, \epsilon) \cup \mathfrak{M}_{\varphi 2}(\varpi, \epsilon)\}$.

Let us consider $\mathfrak{n} \in \mathfrak{I}N \backslash \{\mathfrak{M}_{\kappa 1}(\varpi, \epsilon) \cup \mathfrak{M}_{\kappa 2}(\varpi, \epsilon)\}$.

Then we achieve $\kappa(\zeta_1, \zeta_2, \epsilon) \geq \kappa\left(\zeta_1, \zeta_\mathfrak{n}, \frac{\epsilon}{2}\right) \odot \kappa\left(\zeta_\mathfrak{n}, \zeta_2, \frac{\epsilon}{2}\right) > (1 - \varpi) \odot (1 - \varpi) > 1 - \mathfrak{v}$.

Therefore, $\kappa(\zeta_1, \zeta_2, \epsilon) > 1\mathfrak{v}$ and since $\mathfrak{v} > 0$ is arbitrary, we achieve $\kappa(\zeta_1, \zeta_2, \epsilon) = 1$ for all $\epsilon > 0$, which infers $\zeta_1 = \zeta_2$.

Let us consider $\mathfrak{n} \in \mathfrak{I}N \backslash \{\mathfrak{M}_{\varrho 1}(\varpi, \epsilon) \cup \mathfrak{M}_{\varrho 2}(\varpi, \epsilon)\}$.

Then, $\varrho(\zeta_1, \zeta_2, \epsilon) \leq \varrho(\zeta_1, \zeta_\mathfrak{n}, \epsilon) \oplus \varrho(\zeta_\mathfrak{n}, \zeta_2, \epsilon) < \varpi \oplus \varpi < \mathfrak{v}$.

Since $\mathfrak{v} > 0$ is arbitrary, we obtain $\varrho(\zeta_1, \zeta_2, \epsilon) = 0$ for all $\epsilon > 0$, which suggests $\zeta_1 = \zeta_2$.

Now let us consider $\mathfrak{n} \in \mathfrak{I}N \backslash \{\mathfrak{M}_{\varphi 1}(\varpi, \epsilon) \cup \mathfrak{M}_{\varphi 2}(\varpi, \epsilon)\}$.

Then, $\varphi(\zeta_1, \zeta_2, \epsilon) \leq \varphi(\zeta_1, \zeta_\mathfrak{n}, \epsilon) \oplus \varphi(\zeta_\mathfrak{n}, \zeta_2, \epsilon) < \varpi \oplus \varpi < \mathfrak{v}$.

Since $\mathfrak{v} > 0$ is arbitrary, we obtain $\varphi(\zeta_1, \zeta_2, \epsilon) = 0$ for all $\epsilon > 0$, which refers $\zeta_1 = \zeta_2$.

Theorem 3.5. *Consider the sequence* $(\zeta_\mathfrak{n})$ *in the Neutrosophic Metric Space* $\mathfrak{NMS}$ $(\Omega, \kappa, \varrho, \varphi, \odot, \oplus)$. *If* $(\zeta_\mathfrak{n})$ *converges to* ζ_0 *with respect to the Neutrosophic Metric* $\mathfrak{NM}$, *then* $(\zeta_\mathfrak{n})$ *is* $\mathfrak{SC}$ *to* ζ_0 *in the context of the* $\mathfrak{NM}$.

Proof: Let $(\zeta_{\mathfrak{n}})$ be a sequence convergent to ζ_0. Then, for every $\mathfrak{v} \in (0,1)$ and $\varpi > 0$, there exists an $\mathfrak{n}_0 \in \mathfrak{I}N$ such that $\kappa(\zeta_{\mathfrak{n}}, \zeta_0, \varpi) > 1 - \mathfrak{v}$, $\varrho(\zeta_{\mathfrak{n}}, \zeta_0, \varpi) < \mathfrak{v}$, and $\varphi(\zeta_{\mathfrak{n}}, \zeta_0, \varpi) < \mathfrak{v}$. We obtain $|\{\lambda \leq \mathfrak{n} : \kappa(\zeta_{\mathfrak{n}}, \zeta_0, \varpi) > 1 - \mathfrak{v},\ \varrho(\zeta_{\mathfrak{n}}, \zeta_0, \varpi) < \mathfrak{v} \text{ and } \varphi(\zeta_{\mathfrak{n}}, \zeta_0, \varpi) < \mathfrak{v}\}| \geq \mathfrak{n}\mathfrak{n}_0$.
Hence, the set $\{\lambda \leq \mathfrak{n} : \kappa(\zeta_{\mathfrak{n}}, \zeta_0, \varpi) > 1 - \mathfrak{v}$, $\varrho(\zeta_{\mathfrak{n}}, \zeta_0, \varpi) < \mathfrak{v}$ and $\varrho(\zeta_{\mathfrak{n}}, \zeta_0, \varpi) < \mathfrak{v}\}$ has a finite number of terms.
Then, $\lim\limits_{\mathfrak{n}\to\infty} \dfrac{|\{\lambda \leq \mathfrak{n} : \kappa(\zeta_{\mathfrak{n}}, \zeta_0, \varpi) > 1 - \mathfrak{v}, \varrho(\zeta_{\mathfrak{n}}, \zeta_0, \varpi) < \mathfrak{v}, \varphi(\zeta_{\mathfrak{n}}, \zeta_0, \varpi) < \mathfrak{v}\}|}{\mathfrak{n}} \geq \lim\limits_{\mathfrak{n}\to\infty} \dfrac{\mathfrak{n}\mathfrak{n}_0}{\mathfrak{n}} = 1.$
Consequently, $\Delta(\{\mathfrak{n} \in \mathfrak{I}N : \kappa(\zeta_{\mathfrak{n}}, \zeta_0, \varpi) > 1 - \mathfrak{v}, \varrho(\zeta_{\mathfrak{n}}, \zeta_0, \varpi) < \mathfrak{v}, \varphi(\zeta_{\mathfrak{n}}, \zeta_0, \varpi) < \mathfrak{v}\}) = 1$.
The converse of the theorem need not true.

Example 3.6. Let $\Omega = [1,3]$, and define the operations $\vartheta \odot \omega = \vartheta\omega$ and $\vartheta \oplus \omega = \min\{\vartheta + \omega, 1\}$ for all $\vartheta, \omega \in [0,1]$. Define κ, ϱ, and φ by the following formulas: $\kappa(\zeta, \xi, \varpi) = \frac{\varpi}{\varpi + |\zeta - \xi|}$, $\varrho(\zeta, \xi, \varpi) = \frac{|\zeta - \xi|}{\varpi + |\zeta - \xi|}$, and $\varphi(\zeta, \xi, \varpi) = \frac{|\zeta - \xi|}{\varpi}$ for all $\zeta, \xi \in \Omega$ and $\varpi > 0$. Then $(\mathfrak{I}R, \kappa, \varrho, \varphi, \odot, \oplus)$ forms a Neutrosophic Metric Space $(\mathfrak{NMS})$.
Now define a sequence $(\zeta_{\mathfrak{n}})$ by $\zeta_{\mathfrak{n}} = \begin{cases} 2, & \mathfrak{n} = \lambda^2, \lambda \in \mathfrak{I}N; \\ 1, & \text{otherwise} \end{cases}$.
We can see that $(\zeta_{\mathfrak{n}})$ is not convergent to 1.
We need to show that $(\zeta_{\mathfrak{n}})$ is $\mathfrak{SC}$ to 1.
Let $\mathfrak{v} \in (0,1)$ and $\varpi > 0$. $\mathfrak{M} = \{\mathfrak{n} \in \mathfrak{I}N : \kappa(\zeta_{\mathfrak{n}}, 1, \varpi) > 1\mathfrak{v}, \varrho(\zeta_{\mathfrak{n}}, 1, \varpi) < \mathfrak{v}, \varphi(\zeta_{\mathfrak{n}}, 1, \varpi) < \mathfrak{v}\}$.
Case 1. $\mathfrak{v} \in \left(0, \frac{1}{\varpi+1}\right]$. If $n \neq \lambda^2$ for all $\lambda \in \mathfrak{I}N$, then $\kappa(\zeta_{\mathfrak{n}}, 1, \varpi) = 1 > 1 - \mathfrak{v}$, $\varrho(\zeta_{\mathfrak{n}}, 1, \varpi) = 0 < \mathfrak{v}$ and $\varphi(\zeta_{\mathfrak{n}}, 1, \varpi) = 0 < \mathfrak{v}$. If $\mathfrak{n} = \lambda^2$ for some $\lambda \in \mathfrak{I}N$, then $\kappa(\zeta_{\mathfrak{n}}, 1, \varpi) = \frac{\varpi}{\varpi+1} = 1\frac{1}{\varpi+1} \leq 1\mathfrak{v}$, $\varrho(\zeta_{\mathfrak{n}}, 1, \varpi) = \frac{1}{\varpi+1} \geq \mathfrak{v}$ and $\varphi(\zeta_{\mathfrak{n}}, 1, \varpi) = \frac{1}{\varpi+1} \geq \mathfrak{v}$.
Now, let $\mathfrak{n} \in \mathfrak{I}N$. If $\mathfrak{n} = \lambda_0^2$ for an $\lambda_0 \in \mathfrak{I}N$, then $\lim\limits_{\mathfrak{n}\to\infty} \dfrac{|\mathfrak{M}(\mathfrak{n})|}{\mathfrak{n}} = \lim\limits_{\lambda_0\to\infty} \dfrac{\lambda_0^2\lambda_0}{\lambda_0^2} = 1$. If $\mathfrak{n} \neq \lambda^2$ for all $\lambda \in \mathfrak{I}N$, Therefore, it follows that $\lambda_1 \in \mathfrak{I}N$ such that $\mathfrak{n} = \lambda_1^2 l$ with $l \in \mathfrak{I}N$ and $1 \leq l \leq \lambda_1$. $\lim_{\mathfrak{n}\to\infty} |\mathfrak{M}(\mathfrak{n})|\mathfrak{n} = \lim_{\lambda_1\to\infty} \frac{\lambda_1^2 l(\lambda_1 l)}{\lambda_1^2 l} = \lim_{\lambda_1\to\infty} \frac{\lambda_1^2\lambda_1 l + 1}{\lambda_1^2 l} = 1$.
Case 2. $\mathfrak{v} \in \left(\frac{1}{\varpi+1}, 1\right)$. If $\mathfrak{n} \neq \lambda^2$ for all $\lambda \in \mathfrak{I}N$, then $\kappa(\zeta_{\mathfrak{n}}, 1, \varpi) = 1 > 1 - \mathfrak{v}$, $\varrho(\zeta_{\mathfrak{n}}, 1, \varpi) = 0 < \mathfrak{v}$ and $\varphi(\zeta_{\mathfrak{n}}, 1, \varpi) = 0 < \mathfrak{v}$. If $\mathfrak{n} = \lambda^2$ for some $\lambda \in \mathfrak{I}N$, then $\kappa(\zeta_{\mathfrak{n}}, 1, \varpi) = \frac{\varpi}{\varpi+1} = 1\frac{1}{\varpi+1} > 1\mathfrak{v}$, $\varrho(\zeta_{\mathfrak{n}}, 1, \varpi) = \frac{1}{\varpi+1} < \mathfrak{v}$ and $\varphi(\zeta_{\mathfrak{n}}, 1, \varpi) = \frac{1}{\varpi+1} < \mathfrak{v}$.
Hence, $\kappa(\zeta_{\mathfrak{n}}, 1, \varpi) > 1 - \mathfrak{v}$, $\varrho(\zeta_{\mathfrak{n}}, 1, \varpi) < \mathfrak{v}$ and $\varphi(\zeta_{\mathfrak{n}}, 1, \varpi) < \mathfrak{v}$ for all $\mathfrak{n} \in \mathfrak{I}N$. Therefore, $\lim\limits_{\mathfrak{n}\to\infty} \dfrac{|\mathfrak{M}(\mathfrak{n})|}{\mathfrak{n}} = \lim\limits_{\mathfrak{n}\to\infty} \dfrac{\mathfrak{n}}{\mathfrak{n}} = 1$.
Therefore, $\Delta(\{\mathfrak{n} \in \mathfrak{I}N : \kappa(\zeta_{\mathfrak{n}}, 1, \varpi) > 1 - \mathfrak{v}, \varrho(\zeta_{\mathfrak{n}}, 1, \varpi) < \mathfrak{v}, \varphi(\zeta_{\mathfrak{n}}, 1, \varpi) < \mathfrak{v}\}) = 1$ for all $\mathfrak{v} \in (0,1)$ and $\varpi > 0$.

Theorem 3.7. *Consider the sequence $(\zeta_{\mathfrak{n}})$ in the Neutrosophic Metric Space $\mathfrak{NMS}$ $(\Omega, \kappa, \varrho, \varphi, \odot, \oplus)$. The sequence $(\zeta_{\mathfrak{n}})$ is $\mathfrak{SC}$ to ζ_0 if and only if there exists an increasing index sequence $\mathfrak{P} = \{\mathfrak{n}_i\}_{i \in \mathfrak{I}N}$ of natural numbers such that $(\zeta_{\mathfrak{n}_i})$ converges to ζ_0 and $\Delta(\mathfrak{P}) = 1$.*

Proof: Given that the sequence $(\zeta_{\mathfrak{n}})$ statistically converges to ζ_0, define $\mathfrak{M}_{\kappa\varrho\varphi}(j, \varpi) := \{\mathfrak{n} \in \mathfrak{I}N : \kappa(\zeta_{\mathfrak{n}}, \zeta_0, \varpi) > 1 - \frac{1}{j}, \varrho(\zeta_{\mathfrak{n}}, \zeta_0, \varpi) < \frac{1}{j}$, and $\varphi(\zeta_{\mathfrak{n}}, \zeta_0, \varpi) < \frac{1}{j}\}$ for any $\varpi > 0$ and $j \in \mathfrak{I}N$.

We establish that $\mathfrak{M}_{\kappa\varrho\varphi}(j+1,\varpi) \subset \mathfrak{M}_{\kappa\varrho\varphi}(j,\varpi)$ for all $\varpi > 0$ and $j \in \mathfrak{I}N$. Since $(\zeta_{\mathfrak{n}})$ is $\mathfrak{SC}$ to ζ_0,

$$\Delta(\mathfrak{M}_{\kappa\varrho\varphi}(j,\varpi)) = 1 \tag{1}$$

Take $\mathfrak{u}_1 \in \mathfrak{M}_{\kappa\varrho\varphi}(1,\varpi)$. Since $\Delta(\mathfrak{M}_{\kappa\varrho\varphi}(2,\varpi)) = 1$ (by Equation (1)) we have a number $\mathfrak{u}_2 \in (\mathfrak{M}_{\kappa\varrho\varphi}(2,\varpi)(\mathfrak{u}_2 > \mathfrak{u}_1)$ such that $\frac{|\{\lambda\leq\mathfrak{n}:\kappa(\zeta_\lambda,\zeta_0,\varpi)>1-\frac{1}{2},\varrho(\zeta_\lambda,\zeta_0,\varpi)<\frac{1}{2},\varphi(\zeta_\lambda,\zeta_0,\varpi)<\frac{1}{2}\}|}{\mathfrak{n}} > \frac{1}{2}$, for every $\mathfrak{n} \geq \mathfrak{u}_2$.

According to Equation (1), $\Delta(\mathfrak{M}_{\kappa\varrho\varphi}(3,\varpi)) = 1$. We can select $\mathfrak{u}_3 \in \mathfrak{M}_{\kappa\varrho\varphi}(3,\varpi)$ (where $\mathfrak{u}_3 > \mathfrak{u}_2$) such that for all $\mathfrak{n} \geq \mathfrak{u}_3$, the fraction $\frac{|\{\lambda\leq\mathfrak{n}:\kappa(\zeta_\lambda,\zeta_0,\varpi)>1-\frac{1}{3},\varrho(\zeta_\lambda,\zeta_0,\varpi)<\frac{1}{3},\varphi(\zeta_\lambda,\zeta_0,\varpi)<\frac{1}{3}\}|}{\mathfrak{n}} > \frac{2}{3}$. We proceed in this manner.

Subsequently, we can construct an increasing sequence of indices $\{\mathfrak{u}_j\}_{j\in\mathfrak{I}N}$ from natural numbers such that each $\mathfrak{u}_j$ belongs to $\mathfrak{M}_{\kappa\varrho\varphi}(j,\varpi)$.

$$\frac{|\{\lambda \leq \mathfrak{n} : \kappa(\zeta_\lambda,\zeta_0,\varpi) > 1-\frac{1}{j}, \varrho(\zeta_\lambda,\zeta_0,\varpi) < \frac{1}{j}, \varphi(\zeta_\lambda,\zeta_0,\varpi) < \frac{1}{j}\}|}{\mathfrak{n}} > \frac{j-1}{j}, \tag{2}$$

for all $\mathfrak{n} \geq \mathfrak{u}_j$, where j ranges over $\mathfrak{I}N$. We define the increasing index sequence $\mathfrak{P}$ as:

$$\mathfrak{P} := \{\mathfrak{n} \in \mathfrak{I}N : 1 < \mathfrak{n} < \mathfrak{u}_1\} \cup \left(\bigcup_{j\in\mathfrak{I}N} \{\mathfrak{n} \in \mathfrak{I}N : \mathfrak{u}_j \leq \mathfrak{n} < \mathfrak{u}_{j+1}\} \cap \mathfrak{M}_{\kappa\varrho\varphi}(j,\varpi) \right).$$

By Equation (2) and $\mathfrak{M}_{\kappa\varrho\varphi}(j+1,\varpi) \subset \mathfrak{M}_{\kappa\varrho\varphi}(j,\varpi)$, we write

$$\frac{|\{\lambda \leq \mathfrak{n} : \lambda \in \mathfrak{P}\}|}{\mathfrak{n}} \geq \frac{|\{\lambda \leq \mathfrak{n} : \kappa(\zeta_\lambda,\zeta_0,\varpi) > 1-\frac{1}{j}, \varrho(\zeta_\lambda,\zeta_0,\varpi) < \frac{1}{j}, \varphi(\zeta_\lambda,\zeta_0,\varpi) < \frac{1}{j}\}|}{\mathfrak{n}} > \frac{j-1}{j}$$

for all $\mathfrak{n}, (\mathfrak{u}_j \leq \mathfrak{n} < \mathfrak{u}_{j+1})$.

Given that $j \to \infty$, at which $\mathfrak{n} \to \infty$, we have $\lim_{\mathfrak{n}\to\infty} \frac{|\{\lambda\leq\mathfrak{n}:\lambda\in\mathfrak{P}\}|}{\mathfrak{n}} = 1$, i.e., $\Delta(\mathfrak{P}) = 1$.

Now, we demonstrate the convergence of $(\zeta_{\mathfrak{n}_i})$ to ζ_0. Let $\mathfrak{v} \in (0,1)$ and $\varpi > 0$. Choose $N_0 > \mathfrak{u}_2$ sufficiently large such that there exists an index $s_0 \in \mathfrak{I}N$ satisfying $\mathfrak{u}_{s_0} \leq N_0 < \mathfrak{u}_{s_0+1}$ and $\frac{1}{s_0} < \mathfrak{v}$. Consider $\mathfrak{n}_{\mathfrak{m}} \geq N_0$ where $\mathfrak{n}_{\mathfrak{m}} \in \mathfrak{P}$. By the definition of $\mathfrak{P}$, there exists $s \in \mathfrak{I}N$ such that $\mathfrak{u}_s \leq \mathfrak{n}_{\mathfrak{m}} < \mathfrak{u}_{s+1}$ and $\mathfrak{n}_{\mathfrak{m}} \in \mathfrak{M}_{\kappa\varrho\varphi}(s,\varpi)$ (with $s \geq s_0$). Thus,
$\kappa(\zeta_{\mathfrak{n}_{\mathfrak{m}}},\zeta_0,\varpi) \geq \kappa\left(\zeta_{\mathfrak{n}_{\mathfrak{m}}},\zeta_0,\frac{1}{s_0}\right) \geq \kappa\left(\zeta_{\mathfrak{n}_{\mathfrak{m}}},\zeta_0,\frac{1}{s}\right) > 1-\frac{1}{s} \geq 1-\frac{1}{s_0} > 1-\mathfrak{v}$, $\varrho(\zeta_{\mathfrak{n}_{\mathfrak{m}}},\zeta_0,\varpi) > \frac{1}{s_0} < \mathfrak{v}$ and $\varphi(\zeta_{\mathfrak{n}_{\mathfrak{m}}},\zeta_0,\varpi) > \frac{1}{s_0} < \mathfrak{v}$. Therefore, $(\zeta_{\mathfrak{n}i})$ converges to ζ_0.

On the otherhand, Let's suppose there exists a sequence of increasing indices $\mathfrak{P} = \{\mathfrak{n}_i\}_{i\in\mathfrak{I}N}$ of natural numbers such that $\Delta(\mathfrak{P}) = 1$ and $(\zeta_{\mathfrak{n}_i})$ converges to ζ_0. Take $\mathfrak{v} \in (0,1)$ and $\varpi > 0$. Then, there exists a number $\mathfrak{n}_0 \in \mathfrak{I}N$ such that for every $\mathfrak{n} \geq \mathfrak{n}_0$, the conditions $\kappa(\zeta_{\mathfrak{n}_i},\zeta_0,\varpi) > 1-\mathfrak{v}$, $\varrho(\zeta_{\mathfrak{n}_i},\zeta_0,\varpi) < \mathfrak{v}$, and $\varphi(\zeta_{\mathfrak{n}_i},\zeta_0,\varpi) < \mathfrak{v}$ are satisfied.

Let us define $\mathfrak{M}_{\kappa\varrho\varphi}(\mathfrak{v},\varpi) := \{\mathfrak{n} \in \mathfrak{I}N : \kappa(\zeta_{\mathfrak{n}i},\zeta_0,\varpi) \leq 1-\mathfrak{v} \text{ or } \varrho(\zeta_{\mathfrak{n}i},\zeta_0,\varpi) \geq \mathfrak{v} \text{ and } \varphi(\zeta_{\mathfrak{n}i},\zeta_0,\varpi) \geq \mathfrak{v}\}$. We have $\mathfrak{M}_{\kappa\varrho\varphi}(\mathfrak{v},\varpi) \subset \mathfrak{I}N\backslash\{\mathfrak{n}_{\mathfrak{n}_0},\mathfrak{n}_{\mathfrak{n}_0+1},\mathfrak{n}_{\mathfrak{n}_0+2},\dots\}$. Since $\Delta(\mathfrak{P}) = 1$, we have $\Delta(\mathfrak{I}N\backslash\{\mathfrak{n}_{\mathfrak{n}_0},\mathfrak{n}_{\mathfrak{n}_0+1},\mathfrak{n}_{\mathfrak{n}_0+2},\dots\}) = 0$, so we deduce $\Delta(\mathfrak{M}_{\kappa\varrho\varphi}(\mathfrak{v},\varpi)) = 0$. Hence, $\Delta(\{\mathfrak{n} \in \mathfrak{I}N : \kappa(\zeta_{\mathfrak{n}},\zeta_0,\varpi) < 1-\mathfrak{v}, \varrho(\zeta_{\mathfrak{n}},\zeta_0,\varpi) < \mathfrak{v} \text{ and } \varphi(\zeta_{\mathfrak{n}},\zeta_0,\varpi) < \mathfrak{v}\}) = 1$.

Therefore, $(\zeta_{\mathfrak{n}})$ $\mathfrak{SC}$ to ζ_0.

Corollary 3.8. *If* $(\zeta_{\mathfrak{n}})$ *is a sequence within an* $\mathfrak{NMS}$ $(\Omega, \kappa, \varrho, \varphi, \odot, \oplus)$ *that is both* $\mathfrak{SC}$ *to* ζ_0 *and convergent, then* $(\zeta_{\mathfrak{n}})$ *converges to* ζ_0.

Definition 3.9. Let $(\Omega_1, \kappa_1, \varrho_1, \varphi_1, \odot_1, \oplus_1)$ and $(\Omega_2, \kappa_2, \varrho_2, \varphi_2, \odot_2, \oplus_2)$ be two $\mathfrak{NM}$ spaces.

(i) A map $\chi : \Omega_1 \to \Omega_2$ is termed an isometry if for every $\zeta, \xi \in \Omega_1$ and $\varpi > 0, \kappa_1(\zeta, \xi, \varpi) = \kappa_2(\chi(\zeta), \chi(\xi), \varpi)$, $\varrho_1(\zeta, \xi, \varpi) = \varrho_2(\chi(\zeta), \chi(\xi), \varpi)$ and $\varphi_1(\zeta, \xi, \varpi) = \varphi_2(\chi(\zeta), \chi(\xi), \varpi)$.

(ii) $(\Omega_1, \kappa_1, \varrho_1, \varphi_1, \odot_1, \oplus_1)$ and $(\Omega_2, \kappa_2, \varrho_2, \varphi_2, \odot_2, \oplus_2)$ are termed isometric if there exists a bijective mapping (an isometry) from Ω_1 to Ω_2.

(iii) A neutrosophic completion of $(\Omega_1, \kappa_1, \varrho_1, \odot_1, \oplus_1)$ is defined as a complete $\mathfrak{NMS}$ $(\Omega_2, \kappa_2, \varrho_2, \odot_2, \oplus_2)$ such that $(\Omega_1, \kappa_1, \varrho_1, \odot_1, \oplus_1)$ is isometrically embedded as a dense subspace within Ω_2.

(iv) $(\Omega_1, \kappa_1, \varrho_1, \odot_1, \oplus_1)$ is termed completable if it can be extended to form a complete $\mathfrak{NMS}$.

Proposition 3.10. *Suppose* $(\zeta_{\mathfrak{n}})$ *is a sequence in a completable* $\mathfrak{NMS}$ $(\Omega, \kappa, \varrho, \varphi, \odot, \oplus)$. *If* $(\zeta_{\mathfrak{n}})$ *is a Cauchy sequence in* Ω *and it is statistically dense around* ζ_0, *then* $(\zeta_{\mathfrak{n}})$ *converges to* ζ_0.

Proof: Let $(\Omega_1, \kappa_1, \varrho_1, \varphi_1, \odot_1, \oplus_1)$ be the completion of $(\Omega, \kappa, \varrho, \varphi, \odot, \oplus)$. Consequently, there exists $\zeta_1 \in \Omega_1$ such that the sequence $(\zeta_{\mathfrak{n}})$ converges to ζ_1. We have $\kappa_1(\zeta_{\mathfrak{n}}, \zeta_0, \varpi) = \kappa(\zeta_{\mathfrak{n}}, \zeta_0, \varpi)$, $\varrho_1(\zeta_{\mathfrak{n}}, \zeta_0, \varpi) = \varrho(\zeta_{\mathfrak{n}}, \zeta_0, \varpi)$ and $\varphi_1(\zeta_{\mathfrak{n}}, \zeta_0, \varpi) = \varphi(\zeta_{\mathfrak{n}}, \zeta_0, \varpi)$ for all $\varpi > 0$ and $\mathfrak{n} \in \mathfrak{I}N$.

Let $\mathfrak{v} \in (0, 1)$ and $\varpi > 0$. Since $\Delta(\{\mathfrak{n} \in \mathfrak{I}N : \kappa(\zeta_{\mathfrak{n}}, \zeta_0, \varpi) > 1 - \mathfrak{v}, \varrho(\zeta_{\mathfrak{n}}, \zeta_0, \varpi) < \mathfrak{v} \text{ and } \varphi(\zeta_{\mathfrak{n}}, \zeta_0, \varpi) < \mathfrak{v}\}) = 1$, we obtain $\Delta(\{\mathfrak{n} \in \mathfrak{I}N : \kappa_1(\zeta_{\mathfrak{n}}, \zeta_0, \varpi) > 1 - \mathfrak{v}, \varrho_1(\zeta_{\mathfrak{n}}, \zeta_0, \varpi) < \mathfrak{v} \text{and } \varphi_1(\zeta_{\mathfrak{n}}, \zeta_0, \varpi) < \mathfrak{v}\}) = 1$. Hence, we see that $(\zeta_{\mathfrak{n}})$ statistically converges to $\zeta_0 \in \Omega_1$ with respect to $(\kappa_1, \varrho_1, \varphi_1)$. By Corollary (3.8), we have $\zeta_1 = \zeta_0$.

4. Statically Complete $\mathfrak{NMS}$

Definition 4.1. Consider a sequence $(\zeta_{\mathfrak{n}}) \subset \Omega$, where $(\Omega, \kappa, \varrho, \varphi, \odot, \oplus)$ is a $\mathfrak{NMS}$. The sequence is termed a $\mathfrak{SCa}$ if, for every $\mathfrak{v} \in (0, 1)$ and $\varpi > 0$, there exists $\mathfrak{m} \in \mathfrak{I}N$ such that $\Delta(\{\mathfrak{n} \in \mathfrak{I}N : \kappa(\zeta_{\mathfrak{n}}, \zeta_{\mathfrak{m}}, \varpi) > 1 - \mathfrak{v}, \varrho(\zeta_{\mathfrak{n}}, \zeta_{\mathfrak{m}}, \varpi) < \mathfrak{v}, \varphi(\zeta_{\mathfrak{n}}, \zeta_{\mathfrak{m}}, \varpi) < \mathfrak{v}\}) = 1$.

Example 4.2. In Example(2.4), $\mathfrak{M} = \{\mathfrak{n} \leq \mathfrak{m} : \kappa(\zeta_{\mathfrak{n}}, 0, \varpi) \leq 1 - \mathfrak{v}, \varrho(\zeta_{\mathfrak{n}}, 0, \varpi) \geq \mathfrak{v}, \varphi(\zeta_{\mathfrak{n}}, 0, \varpi) \geq \mathfrak{v}\} = \left\{\mathfrak{n} \leq \mathfrak{m} : \frac{\varpi}{\varpi + |\zeta_{\mathfrak{n}}|} \leq 1 - \mathfrak{v}, \frac{|\zeta_{\mathfrak{n}}|}{\varpi + |\zeta_{\mathfrak{n}}|} \geq \mathfrak{v}, \frac{|\zeta_{\mathfrak{n}}|}{\varpi} \geq \mathfrak{v}\right\} = \{\mathfrak{n} \leq \mathfrak{m} : |\zeta_{\mathfrak{n}}| \geq \frac{\mathfrak{v}\varpi}{1-\mathfrak{v}} > 0\} = \{\mathfrak{n} \leq \mathfrak{m} : \zeta_{\mathfrak{n}} = 1\} = \{\mathfrak{n} \leq \mathfrak{m} : \mathfrak{n} = \lambda^3, \lambda \in \mathfrak{I}N\}$, and we obtain $\frac{1}{\mathfrak{m}}|\mathfrak{M}| \leq \frac{1}{\mathfrak{m}}|\{\mathfrak{n} \leq \mathfrak{m} : \mathfrak{n} = \lambda^3, \mathfrak{n} \in \mathfrak{I}N\}| \leq \frac{\sqrt[3]{\mathfrak{m}}}{\mathfrak{m}} \to 0, \mathfrak{m} \to \infty$. As a result, we have $(\zeta_{\mathfrak{n}})$ is $\mathfrak{SC}$ to 0 with regard to the $\mathfrak{NMS}$ $(\Omega, \kappa, \varrho, \varphi, \odot, \oplus)$.

Theorem 4.3. *Let* $(\zeta_{\mathfrak{n}})$ *denote a sequence within the framework of* $\mathfrak{NMS}$ $(\Omega, \kappa, \varrho, \varphi, \odot, \oplus)$. *The following statements are equivalent:*

(i) $(\zeta_{\mathfrak{n}})$ *is classified as a* $\mathfrak{SCa}$ *sequence.*

(ii) There exists a progressively increasing index sequence $\mathfrak{M} = \{\mathfrak{n}_i\}_{i\in\mathfrak{I}N}$ *of natural numbers such that* $(\zeta_{\mathfrak{n}_i})$ *is characterized by being Cauchy and achieving* $\Delta(\mathfrak{M}) = 1$.

Proof: Direct

Theorem 4.4. *Consider a sequence* $(\zeta_{\mathfrak{n}})$ *within the framework of* $\mathfrak{NMS}$ $(\Omega, \kappa, \varrho, \varphi, \odot, \oplus)$. *If* $(\zeta_{\mathfrak{n}})$ *is* $\mathfrak{SC}$ *with respect to the*$\mathfrak{NM}$, *then* $(\zeta_{\mathfrak{n}})$ *is* $\mathfrak{SCa}$ *with respect to the* $\mathfrak{NM}$.

Proof: If $(\zeta_{\mathfrak{n}})$ is statistically convergent to ζ_0 and for given $\mathfrak{v} \in (0,1)$ and $\varpi > 0$, there exists $\mathfrak{v}_1 \in (0,1)$ such that $(1-\mathfrak{v}_1) \odot (1-\mathfrak{v}_1) > 1-\mathfrak{v}$ and $\mathfrak{v}_1 \oplus \mathfrak{v}_1 < \mathfrak{v}$. We have $\Delta(\{\mathfrak{n} \in \mathfrak{I}N : \kappa(\zeta_{\mathfrak{n}}, \zeta_0, \varpi) > 1-\mathfrak{v}, \varrho(\zeta_{\mathfrak{n}}, \zeta_0, \varpi) < \mathfrak{v}, \varphi(\zeta_{\mathfrak{n}}, \zeta_0, \varpi) < \mathfrak{v}\}) = 1$. Refer Theorem (3.4), there exists an increasing index sequence $\{\mathfrak{n}_i\}_{i\in\mathfrak{I}N}$ so that $(\zeta_{\mathfrak{n}_i})$ is convergent to ζ_0. Hence, there exists $\mathfrak{n}_{i_0} \in \{\mathfrak{n}_i\}_{i\in\mathfrak{I}N} : \kappa\left(\zeta_{\mathfrak{n}_i}, \zeta_0, \frac{\varpi}{2}\right) > 1-\mathfrak{v}_1$, $\varrho\left(\zeta_{\mathfrak{n}_i}, \zeta_0, \frac{\varpi}{2}\right) < \mathfrak{v}_1$ and $\varphi\left(\zeta_{\mathfrak{n}_i}, \zeta_0, \frac{\varpi}{2}\right) < \mathfrak{v}_1$ for all $\mathfrak{n}_i \geq \mathfrak{n}_{i_0}$.
Since $\kappa\left(\zeta_{\mathfrak{n}}, \zeta_{\mathfrak{n}_{i_0}}, \varpi\right) \geq \kappa\left(\zeta_{\mathfrak{n}}, \zeta_0, \frac{\varpi}{2}\right) \odot \kappa\left(\zeta_0, \zeta_{\mathfrak{n}_{i_0}}, \frac{\varpi}{2}\right) \geq (1-\mathfrak{v}_1) \odot (1-\mathfrak{v}_1) > 1-\mathfrak{v}$,
$\varrho\left(\zeta_{\mathfrak{n}}, \zeta_{\mathfrak{n}_{i_0}}, \varpi\right) \leq \varrho\left(\zeta_{\mathfrak{n}}, \zeta_0, \frac{\varpi}{2}\right) \oplus \varrho\left(\zeta_0, \zeta_{\mathfrak{n}_{i_0}}, \frac{\varpi}{2}\right) < \mathfrak{v}_1 \oplus \mathfrak{v}_1 < \mathfrak{v}$ and
$\varphi\left(\zeta_{\mathfrak{n}}, \zeta_{\mathfrak{n}_{i_0}}, \varpi\right) \leq \varphi\left(\zeta_{\mathfrak{n}}, \zeta_0, \frac{\varpi}{2}\right) \oplus \varphi\left(\zeta_0, \zeta_{\mathfrak{n}_{i_0}}, \frac{\varpi}{2}\right) < \mathfrak{v}_1 \oplus \mathfrak{v}_1 < \mathfrak{v}$, we have $\Delta(\{\mathfrak{n} \in \mathfrak{I}N : \kappa(\zeta_{\mathfrak{n}}, \zeta_{\mathfrak{n}_{i_0}}, \varpi) > 1-\mathfrak{v}, \varrho(\zeta_{\mathfrak{n}}, \zeta_{\mathfrak{n}_{i_0}}, \varpi) < \mathfrak{v}, \varphi(\zeta_{\mathfrak{n}}, \zeta_{\mathfrak{n}_{i_0}}, \varpi) < \mathfrak{v}\}) = 1$. Therefore, $(\zeta_{\mathfrak{n}})$ is statistically Cauchy with respect to the $\mathfrak{NM}$.

Remark 4.5. Given that a sequence in a $\mathfrak{NMS}$ is Cauchy, it consequently meets the criteria to be classified as $\mathfrak{SCa}$.

Definition 4.6. The $\mathfrak{NMS}$ $(\Omega, \kappa, \varrho, \varphi, \odot, \oplus)$ is termed statistically complete if every $\mathfrak{SCa}$ sequence in Ω is also $\mathfrak{SC}$.

Theorem 4.7. *If* $(\Omega, \kappa, \varrho, \varphi, \odot, \oplus)$ *is a* $\mathfrak{NMS}$ *where* Ω *is statistically complete, then it is also complete according to the* $\mathfrak{NM}$.

Proof: The proof follows a similar approach to Theorem (4.4).

5. Conclusion

This paper has discussed and proved some results of $\mathfrak{SC}$ and $\mathfrak{SCa}$ on $\mathfrak{NMS}$. Additionally looked at the attributes of statistical completeness on $\mathfrak{NMS}$. Our results can be extended to other spaces and be used to arrive at more results in fixed point theory.

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Received: March 1, 2024. Accepted: July 28, 2024

NSS

Neutrosophic Sets and Systems, Vol. 70, 2024

University of New Mexico

Total Domination in Uniform Single Valued Neutrosophic Graph

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Abstract. In graph theory, the neutrosophic graph is a new extension of the intuitionistic fuzzy graph and fuzzy graph. It offers greater accuracy and flexibility in scheduling and implementing numerous real-life issues. The introduction of neutrosophic graphs has opened up an excellent opportunity to portray real-world situations with incomplete and ambiguous information. These neutrosophic graph models are used in many industrial and scientific domains to simulate various problems. This article offers a new approach for calculating the total dominating set of a given single-valued neutrosophic graph using neutrosophic strong arcs. An example is illustrated for calculating the total dominating sets of single-valued neutrosophic graphs. Moreover, certain observations related to uniform single-valued neutrosophic graphs are covered and we also suggest an application of a total dominating set in single-valued neutrosophic graphs in the real world.

Keywords: Domination; Total domination; Single-valued neutrosophic graphs; Neutrosophic strong arcs.

1. Introduction

While dealing with uncertain situations and imperfect knowledge, certain powerful mathematical techniques are required. Graph theory is a mathematical method that successfully manages massive amounts of data. If there is any ambiguity, fuzzy graphs are the ideal tool to use. Graph theory is particularly intriguing because of its potential applications in management, social, computer and information sciences, and data handling. Zadeh's [13] fuzzy set is one of the best solutions for dealing with uncertain conditions and unknown data. A fuzzy set consists of objects defined by a membership function that allots each element a degree of membership ranging from 0 to 1. Rosenfeld [9] examined fuzzy graphs (FG) in-depth, revealing some of their core properties. FG are essential because they are better at dealing with ambiguity; nevertheless, Atanassov [1] noted that fuzzy sets can only handle one-sided uncertainty, which is unsatisfactory since human nature is not limited to yes or no answers, which

is why he proposed the idea of intuitionistic fuzzy set. An element's membership function and non-membership function characterize this intuitionistic fuzzy set. As per Atanassov, the sum of the membership and non-membership functions does not exceed one. Later, F. Smarandache [11] introduced the neutrosophic set. The neutrosophic set helps to handle vague, inconsistent, and uncertain data in real-world situations. H. Wang [12] introduced single-valued neutrosophic sets (S-VNS) to simplify their use in real-world situations. The S-VNS broadens intuitionistic fuzzy sets by employing three separate membership functions, each with values between 0 and 1. Later, Broumi [2] et al. proposed the idea of single-valued neutrosophic graph (S-VNG).

Among various dominations, E J Cockayne [4] proposed total domination in graph theory. Parvati [8] et al. presented the concept of domination in intuitionistic fuzzy graphs (IFG) using strong arcs, and M. Mullai [7] put forward the idea of domination in neutrosophic graphs using a different method. Until now, no studies have been carried out on total domination in a S-VNG using the idea of neutrosophic-strong arcs.

In this paper, we implemented the strong arc concept of Parvati et al. to determine the dominating set (D-Set) and total dominating set (TD-Set) of a S-VNG. The paper is structured as follows:

In Section 2, We discussed fundamental definitions pertinent to our research. In Section 3, A new idea called neutrosophic-strong arc is defined and we developed a new method for determining TD-Set of a S-VNG. Some theorems related to TD-Set in a uniform S-VNG are also discussed in this section. Section 4 contains an application of TD-Set of a S-VNG and the paper's conclusion is finally provided in section 5.

2. Preliminaries

This section will cover the fundamental concepts of S-VNG essential for our study. In this paper, $G^* = (V, E)$ denotes a simple graph.

Definition 2.1. [5] A FG on G^* is denoted as $G = (\delta, \eta)$; where δ is a fuzzy subset of V and η is a symmetric fuzzy relation on δ.

i.e., $\delta : V \to [0, 1]$ and $\eta : E \to [0, 1]$ such that

$\eta(h, l) \leq \delta(h) \wedge \delta(l)$ $\forall h, l \in V$ where hl is an arc betwixt the vertices h and l.

Definition 2.2. [2] An IFG is of the form $G = (V, E)$ where

(1) $V = \{h_1, h_2, ..., h_n\}$ such that $\tau_1 : V \to [0, 1]$ and $\nu_1 : V \to [0, 1]$ denotes the degree of membership and degree of non-membership of an element $h_i \in V$ respectively, and
$0 \leq \tau_1(h_i) + \nu_1(h_i) \leq 1$ $\forall h_i \in V (i = 1, 2, ..., n)$.

(2) $E \subseteq V \times V$ where $\tau_2 : E \to [0, 1]$ and $\nu_2 : E \to [0, 1]$ are such that
$\tau_2(h_i, h_j) \leq \tau_1(h_i) \wedge \tau_1(h_j)$

$\nu_2(h_i, h_j) \geq \nu_1(h_i) \vee \nu_1(h_j)$
$0 \leq \tau_2(h_i, h_j) + \nu_2(h_i, h_j) \leq 1 \ \forall (h_i, h_j) \in E (i = 1, 2, ..., n).$

Definition 2.3. [2] A S-VNG on $G^* = (V, E)$ is defined to be $G = (\phi, \psi)$ where

(1) The functions $T_\phi : V \to [0, 1]$, $I_\phi : V \to [0, 1]$ and $F_\phi : V \to [0, 1]$ indicates the degree of truth-membership function, indeterminacy-membership function and falsity-membership function of an element $h_j \in V$ and
$0 \leq T_\phi(h_j) + I_\phi(h_j) + F_\phi(h_j) \leq 3 \ \forall h_j \in V.$

(2) The functions $T_\psi : E \to [0, 1]$, $I_\psi : E \to [0, 1]$ and $F_\psi : E \to [0, 1]$ are defined by
$T_\psi(h_j, h_k) \leq T_\phi(h_j) \wedge T_\phi(h_k)$
$I_\psi(h_j, h_k) \geq I_\phi(h_j) \vee I_\phi(h_k)$
$F_\psi(h_j, h_k) \geq F_\phi(h_j) \vee F_\phi(h_k)$ and
$0 \leq T_\psi(h_j, h_k) + I_\psi(h_j, h_k) + F_\psi(h_j, h_k) \leq 3 \ \forall (h_j, h_k) \in E; (j, k = 1, 2, ..., n).$

Definition 2.4. [2] A S-VNG $G = (\phi, \psi)$ is a Complete S-VNG if
$T_\psi(h_j, h_k) = T_\phi(h_j) \wedge T_\phi(h_k)$
$I_\psi(h_j, h_k) = I_\phi(h_j) \vee I_\phi(h_k)$
$F_\psi(h_j, h_k) = F_\phi(h_j) \vee F_\phi(h_k) \ \forall h_j, h_k \in V.$

Definition 2.5. [6] A S-VNG $G = (\phi, \psi)$ is a bi-partite if ϕ can be partitioned into two non-empty sets ϕ_1 and ϕ_2 such that

(1) $T_\psi(h_i, l_j) = 0$, $I_\psi(h_i, l_j) = 0$, $F_\psi(h_i, l_j) = 0$ if $(h_i, l_j) \in \phi_1$ or $(h_i, l_j) \in \phi_2$
(2) $T_\psi(h_i, l_j) > 0$, $I_\psi(h_i, l_j) > 0$, $F_\psi(h_i, l_j) > 0$ if $h_i \in \phi_1$ and $l_j \in \phi_2$ (OR)
$T_\psi(h_i, l_j) = 0$, $I_\psi(h_i, l_j) > 0$, $F_\psi(h_i, l_j) > 0$ if $h_i \in \phi_1$ and $l_j \in \phi_2$ (OR)
$T_\psi(h_i, l_j) > 0$, $I_\psi(h_i, l_j) = 0$, $F_\psi(h_i, l_j) > 0$ if $h_i \in \phi_1$ and $l_j \in \phi_2$ (OR)
$T_\psi(h_i, l_j) > 0$, $I_\psi(h_i, l_j) > 0$, $F_\psi(h_i, l_j) = 0$ if $h_i \in \phi_1$ and $l_j \in \phi_2$ (OR)
$T_\psi(h_i, l_j) = 0$, $I_\psi(h_i, l_j) = 0$, $F_\psi(h_i, l_j) > 0$ if $h_i \in \phi_1$ and $l_j \in \phi_2$ (OR)
$T_\psi(h_i, l_j) = 0$, $I_\psi(h_i, l_j) > 0$, $F_\psi(h_i, l_j) = 0$ if $h_i \in \phi_1$ and $l_j \in \phi_2$ (OR)
$T_\psi(h_i, l_j) > 0$, $I_\psi(h_i, l_j) = 0$, $F_\psi(h_i, l_j) = 0$ if $h_i \in \phi_1$ and $l_j \in \phi_2$.

Definition 2.6. [6] A bipartite S-VNG is said to be Complete if it satisfies all the conditions of a Complete S-VNG.

Definition 2.7. [3] Let $G = (\phi, \psi)$ be a S-VNG, $\phi = (T_\phi, I_\phi, F_\phi)$ is a single valued neutrosophic vertex set of G and $\psi = (T_\psi, I_\psi, F_\psi)$ is a single valued neutrosophic edge set of G. Let $\phi = \{h \in V | T_\phi(h) > 0, I_\phi(h) > 0, F_\phi(h) > 0\}$. Then G is a uniform-single valued neutrosophic graph (U-SVNG) of level (k_1, k_2, k_3) if

(1) $T_\psi(h_i, l_j) = k_1$, $I_\psi(h_i, l_j) = k_2$, $F_\psi(h_i, l_j) = k_3 \ \forall (h_i, l_j) \in E$
(2) $T_\phi(h_i) = k_1$, $I_\phi(h_i) = k_2$, $F_\phi(h_i) = k_3$ where $0 < k_1, k_2, k_3 \leq 1$.

Definition 2.8. [6] Let $G = (\phi, \psi)$ be a S-VNG and $h, l \in \phi$. Then,

(1) $T - strength$ of connectedness among h and l is termed as
$T^{\infty}_{\psi}(h,l) = \vee\{T^{n}_{\psi}(h,l) | n = 1,2,...,k\}$; where
$T^{n}_{\psi}(h,l) = T_{\psi}(h,a_1) \wedge T_{\psi}(a_1,a_2), .. \wedge T_{\psi}(a_{n-1},l)$; $h, a_1, a_2, ..., a_{n-1}, l \in V$ and $n = \{1,2,...,k\}$.

(2) $I - strength$ of connectedness among h and l is termed as
$I^{\infty}_{\psi}(h,l) = \wedge\{I^{n}_{\psi}(h,l) | n = 1,2,...,k\}$; where
$I^{n}_{\psi}(h,l) = I_{\psi}(h,a_1) \vee I_{\psi}(a_1,a_2), .. \vee I_{\psi}(a_{n-1},l)$; $h, a_1, a_2, ..., a_{n-1}, l \in V$ and $n = \{1,2,...,k\}$.

(3) $F - strength$ of connectedness among h and l is termed as
$F^{\infty}_{\psi}(h,l) = \wedge\{F^{n}_{\psi}(h,l) | n = 1,2,...,k\}$; where
$F^{n}_{\psi}(h,l) = F_{\psi}(h,a_1) \vee F_{\psi}(a_1,a_2), .. \vee F_{\psi}(a_{n-1},l)$; $h, a_1, a_2, ..., a_{n-1}, l \in V$ and $n = \{1,2,...,k\}$.

Definition 2.9. [10] Let $G = (\phi, \psi)$ be a S-VNG. The vertex cardinality of G is defined as: $|V| = \sum \left(\frac{1+T_{\phi}(h_i)+I_{\phi}(h_i)-F_{\phi}(h_i)}{2} \right)$.

Definition 2.10. [10] Let $G = (\phi, \psi)$ be a S-VNG. The edge cardinality of G is defined as: $|E| = \sum \left(\frac{1+T_{\psi}(h_i,l_j)+I_{\psi}(h_i,l_j)-F_{\psi}(h_i,l_j)}{2} \right)$.

3. Total Domination in a S-VNG

Definition 3.1. An arc $h_i l_j$ of a S-VNG $G = (\phi, \psi)$ is a neutrosophic-strong arcs (N_{SA}) if
$T_{\psi}(h_i, l_j) \geq T^{\infty}_{\psi}(h_i, l_j)$
$I_{\psi}(h_i, l_j) \leq I^{\infty}_{\psi}(h_i, l_j)$ and
$F_{\psi}(h_i, l_j) \leq F^{\infty}_{\psi}(h_i, l_j)$ where $h_i, l_j \in V$.

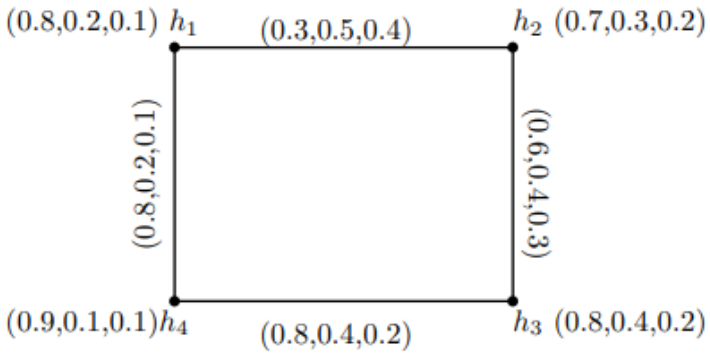
Definition 3.2. Let $G = (\phi, \psi)$ be a S-VNG and $h, l \in V$, then the vertex h dominates l if hl is a N_{SA} in G.

Definition 3.3. Let $G = (\phi, \psi)$ be a S-VNG and $h \in V$. Then, the neutrosophic-strong arc neighborhood vertices of the vertex h ($N_{SA}(h)$) is defined as:
$N_{SA}(h) = \{l \in V : hl$ is a neutrosophic-strong arc in G$\}$.

Definition 3.4. Let $G = (\phi, \psi)$ be a S-VNG. $S \subseteq V$ is said to be a D-Set in G if $\forall h \in V - S$, $\exists$ a $l \in S$ such that h and l are neutrosophic-strong arc neighborhood vertices in G.

Definition 3.5. A D-Set S of a S-VNG is said to be a minimal D-Set if no proper subset of S will form a D-Set in G.

The lower domination number, denoted by $\gamma_d^{'}$ is the minimum cardinality among all the minimal D-Set of G and the upper domination number, denoted by $\gamma_D^{'}$ is the maximum cardinality among all the minimal D-Set of G.



S-VNG

FIGURE 1. h_1h_4, h_2h_3, h_3h_4 are the neutrosophic-strong arcs and $N_{SA}(h_1) = \{h_4\}$, $N_{SA}(h_2) = \{h_3\}$, $N_{SA}(h_3) = \{h_2, h_4\}$, $N_{SA}(h_4) = \{h_1, h_3\}$. $A = \{h_1, h_3\}$, $B = \{h_3, h_4\}$, $C = \{h_1, h_2\}$ and $D = \{h_2, h_4\}$ are the minimal D-set and $|A| = 1.95$, $|B| = 1.95$, $|C| = 1.85$ and $|D| = 1.85$ are their corresponding cardinalities. $\gamma_d^{'} = 1.85$ and $\gamma_D^{'} = 1.95$. $B = \{h_3, h_4\}$ is the minimal TD-Set of fig:1 with $\gamma_{td}^{'} = \gamma_{TD}^{'} = 1.95$.

Definition 3.6. $S \subseteq V$ is called a TD-Set of G if every vertices in V is dominated by at least one vertex of S. Note that TD-Set is defined solely for graphs that do not contain isolated vertices..

Lower total domination number is denoted by $\gamma_{td}^{'}$ and upper total domination number is denoted by $\gamma_{TD}^{'}$.

Example 3.7. Consider a S-VNG without isolated vertices $G = (V, E)$ as shown in fig.1

Theorem 3.8. *Let $G = (V, E)$ be a complete bi-partite S-VNG without isolated vertices. Then $\gamma_{td}^{'} = |h| + |l|$ where h and l are two vertices with minimum cardinality among the vertex sets V_1 and V_2.*

Proof. Let $G = (V, E)$ be a complete bi-partite S-VNG without isolated vertices and also let h be the vertex with minimum cardinality in V_1 and l represent the vertex with minimum cardinality in V_2.

Since all of the vertices in the provided S-VNG meets the conditions for strong arcs, every vertices in V_2 will be dominated by a vertex in V_1.

Let $T = \{h, l\}$ is not a TD-Set of $G = (V, E)$.

Then $z \in V - \{h, l\}$ is adjacent to more than one vertex in G.

That is, z is adjacent to both h and l. This contradicts the claim that $G = (V, E)$ is a complete bi-partite S-VNG.

Thus $T = \{h, l\}$ will form a TD-Set of $G = (V, E)$ and $\gamma_{td}^{'} = |h| + |l|$. □

Definition 3.9. A U-SVNG $G = (\phi, \psi)$ is a complete U-SVNG if

$T_\psi(h_j, h_k) = T_\phi(h_j) \wedge T_\phi(h_k)$

$I_\psi(h_j, h_k) = I_\phi(h_j) \vee I_\phi(h_k)$

$F_\psi(h_j, h_k) = F_\phi(h_j) \vee F_\phi(h_k)\ \forall h_j, h_k \in V.$

Theorem 3.10. *Let $G = (\phi, \psi)$ be a U-SVNG without isolated vertices, then for a n-cycle (C_n) graph of G with $n \geq 4$, $\gamma_{td}^{'} = (n-2)|h_i|$; $h_i \in V$.*

Proof. Assume that the U-SVNG $G = (\phi, \psi)$ has no isolated vertices. By the definition of U-SVNG, all the edges and vertices of C_n graph will have same values. Let $S \subseteq V$ be a TD-Set C_n graph, and let $V = \{h_1, h_2, ..., h_n\}$ be the graph's vertices. Neutrosophic vertex cardinality of S in a S-VNG is given by

$|S| = \sum \left(\frac{1+T_\phi(h_i)+I_\phi(h_i)-F_\phi(h_i)}{2} \right);\ h_i \in V$

Claim: $\gamma_{td}^{'} = |S| = (n-2)|h_i|; h_i \in V.$

Here, Two cases will arrive.

Case:1 $|S| \leq (n-2)|h_i|$

Let $S = \{h_1, h_2, ..., h_{n-2}\}$ be the TD-Set of the C_n graph. If S is not a minimal TD-Set, then there exist a subset of S which is able to dominate all the vertices of C_n graph. This is not possible because one vertex will remain undominated by any of the vertices in S if less than $(n-2)$ vertices are used. This implies S is a TD-Set.

Therefore $|S| = (n-2)|h_i|$.

Case:2 $|S| \geq (n-2)|h_i|$.

If possible,

When a new vertex l is added to S, $S = \{h_1, h_2, ..., h_{n-2}, l\}$ becomes the new set S. Now S has $(n-1)$ elements and it will form a TD-Set. Upon evaluating the set S, a subset of S is found that has the potential to dominate every vertices of C_n graph. Thus S is not a minimal TD-Set.

Therefore, $\gamma_{td}^{'} \not> (n-2)|h_i|$.

From the two cases, we can conclude that the set S needs only $(n-2)$ vertices to dominate all the vertices of the given C_n graph.

Thus, $\gamma_{td}^{'} = |S| = (n-2)|h_i|; h_i \in V.$ □

Theorem 3.11. *If $G = (\phi, \psi)$ is a complete U-SVNG without isolated vertices, then $\gamma_{td}^{'} = 2|h_i|$; $h_i \in V$.*

Proof. Let $G = (\phi, \psi)$ be a complete U-SVNG without isolated vertices. By the definition of U-SVNG, G's vertices and edges will have same values.

i.e,

$T_\psi(h_j, h_k) = k_1 = T_\phi(h_j) = T_\phi(h_k)$,
$I_\psi(h_j, h_k) = k_2 = I_\phi(h_j) = I_\phi(h_k)$ and
$F_\psi(h_j, h_k) = k_3 = F_\phi(h_j) = F_\phi(h_k)$ $\forall (h_j, h_k) \in E$ and $0 < k_1, k_2, k_3 \leq 1$.

A TD-Set needs minimum of 2 elements to dominate all the vertices of any graph. As there is a neutrosophic-strong arc connecting each pair of vertices of G, all of its vertices will dominate one another. Since G is a complete U-SVNG, every vertices in it will have the same values and its TD-Set only needs two elements in order to dominate the other vertices of G. As a result, the cardinality of one of its vertices is always twice the overall domination number. $\gamma_{td}^{'} = 2|h_i|$; $h_i \in V$. □

3.1. *Observations on total domination number in a U-SVNG*

Let $G = (\phi, \psi)$ be a simple U-SVNG without isolated vertices. In this section, we will show the value of $\gamma_{td}^{'}(G)$ for different graph classes. We denote P_n, W_n, L_n, S_n, F_n, $L_{m,n}$, $T_{m,n}$ and B_n for the Path, Wheel graph, Ladder graph, Star graph, Friendship graph, Lollipop graph, Tadpole graph and n-Book graph.

- $\gamma_{td}^{'}(P_n) = (n-1)|h_i|$, $n \geq 3$ and $h_i \in V$.
- $\gamma_{td}^{'}(W_n) = 2|h_i|$, $h_i \in V$.
- $\gamma_{td}^{'}(L_n) = (2n-4)|h_i|$, $n \geq 3$ and $h_i \in V$.
- $\gamma_{td}^{'}(S_n) = 2|h_i|$, $h_i \in V$.
- $\gamma_{td}^{'}(F_n) = 2|h_i|$, $h_i \in V$.
- $\gamma_{td}^{'}(L_{m,n}) = n|h_i|$, $n \geq 2$ and $h_i \in V$.
- $\gamma_{td}^{'}(T_{m,n}) = \gamma_{td}^{'}(C_m) + \gamma_{td}^{'}(P_n)$, $m \geq 4$ and $n \geq 3$.
- $\gamma_{td}^{'}(B_n) = 2|h_i|$, $n \geq 3$ and $h_i \in V$.

4. **Applications**

The reliability of interactions might be uncertain or not fully understood in biological networks such as protein - protein interaction. Researchers study the interaction between proteins in order to better understand living organism's functions, signaling systems and general behaviors. A variety of biological activities depend heavily on interactions between proteins, which frequently cooperate in complex networks. These interactions can include physical bindings and regulatory relationships. The nature and power of these protein-protein interactions, however, are not entirely clear. Due to experimental limitations, environmental variables, or the complexity of biological systems, certain interactions may be widely reported and understood, while others may be unidentified. neutrosophic graph provide a way to represent the uncertainty in protein - protein interaction by assigning T, I and F values to both vertices and edges of the graph. The edge's T value shows "well established interactions", I denote

"interactions with some ambiguity", and F defines "interactions that have not been proven yet". Total domination in neutrosophic graphs applied to protein - protein interaction networks aims to identify a subset of proteins that collectively dominate or influence the entire network taking into account the uncertain and varying strength of interactions. Identifying a subset of proteins that collectively dominate the network either directly/ indirectly. This subset should have a significant impact on the overall stability and behavior of the network. When we apply neutrosophic graph for finding the group of proteins the result will be more precise since they are taking into account the interactions with ambiguity which is not present in intuitionistic fuzzy graph.

Below are the benefits of applying total domination in Neutrosophic Graphs for the study of protein - protein interaction

- Identify a group of proteins that collectively play a crucial role in preserving the network's stability and functionality.
- Understand possible therapeutic targets or genetic alterations that might be used to affect the behavior of the entire network.
- Prioritize experimental validation efforts on interactions involving proteins in TD-Set, helping to clarify and refine the understanding of network.

5. Conclusions

The S-VNG, an extension of FG and IFG, increases the system's accuracy, flexibility, and compatibility.

In this research, we suggested a novel idea for identifying the D-Set and TD-Set of a S-VNG using neutrosophic-strong arcs, as well as we defined notions such as neutrosophic-strong arc neighborhood vertices, complete U-SVNG and discussed a few theorems related to total domination in a U-VNG. Furthermore, an application of TD-set in a S-VNG is also discussed. We would like to broaden our research to uncover even more properties on TD-Set in a S-VNG and also want to develop an algorithm for finding the TD-Set of a given S-VNG .

Funding:

This research did not receive any outside funding.

Conflicts of Interest:

The authors have no conflicts of interest to declare.

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Received: March 3, 2024. Accepted: July 29, 2024

Neutrosophic Sets and Systems, Vol. 70, 2024
University of New Mexico

Kalman Bucy Filtered Neuro Fuzzy Image Denoising for Medical Image Processing

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Abstract: Neutrosophic sets (NS) have referred to as interval fuzzy sets applied in minimizing the uncertainty and fuzziness in computer-vision and machine-learning communities and hence employed for several applications. As far as medical image processing applications are concerned NSs are obtained as an important technique for de-noising. Also, fuzzy segmentation with machine and deep learning is determined as a familiar procedure that splits input image into distinct regions for precise learning. Several research works conducted in different image-processing domains. However, less works was focused on denoising and segmentation of medical image processing with minimal time complexity and accuracy. In this work we plan to develop a Kalman–Bucy Filtered Neutrosophic Neuro Fuzzy Image Denoising (KBF-NNFID) method with the objective of reducing the noisy artifacts with higher peak signal-to-noise ratio in a computationally efficient manner. First, medical images obtained from Brain MRI LGG segmentation dataset are subjected to filtering employing Kalman Bucy Filtering algorithm with series of measurements examined. Second with the filtered medical images provided as input, uncertainty is handled by utilizing Neutrosophic Neuro Fuzzy set (NNFS) with help of the membership grade. With the aid of three membership grades, i.e., truth, indeterminacy and falsity, uncertainty involved in noisy image are said to be handled in a time efficient manner. By this way, an efficient image denoising process is performed with better PSNR. Experimental evaluation is carried out using medical images with different performance metrics such as enhanced PSNR and true positive rate up to 13%, 14% as well minimum execution time by 38% using medical images.

Keywords: Neutrosophic Sets, Kalman–Bucy, Fuzzy Segmentation, Image Processing, Neutrosophic Neuro Fuzzy, Image Denoising

1. Introduction

Diagnosis and forecasting is one of the most laborious and demanding job owing to confined distinction of experts as well as fuzziness in medical science. And medical image processing applications are concerned NSs are achieved as main technique for de-noising. Medical image procedure has comprised in image segmentation, registration, three dimensional rebuilding as well as motion analysis. Additional study direction is depended on precise image segmentation.

In discovering the numerous diseases intensity, discrete professional specialists may result in incorrect diagnosis. With the objective of performing diagnosis instinct in the medical images, several image processing techniques have been investigated in Neutrosophic theory to interpreting intrinsic insecurity, as well as imprecision.

Purpose of speckle noise involved in Optimal Coherence Tomographic (OCT) images was eliminated [1] via Iterative contraction algorithm based on Chi-square similarity and fuzzy logic. Here the OCT image was split to image blocks with low rank group matrix. During fuzzy logic, singular values were obtained via dissimilar weights. Also, denoising effect was enhanced. With this the PSNR was said to be improved. However, the running time required in denoising was said to be compromised due to lack in optimization policy.

Deviations involved in the imaging system were focused in [2] Saliency Map and Neutrosophic Set Theory (SMNS). First, guided filter was employed in image using different channels to focus on the weak edges. Image foreground information was highlighted to generate saliency map. Followed by Neutrosophic Set area with which the results were interpreted as true, indeterminate and false. As a result, SMNS method saw an improvement in precision, recall and running time. However, the noise ratio (i.e., Peak Signal-to Noise Ratio – PSNR) was not focused.

Noise data was determined in [3] during mathematical model. Owing to reason that noise has a great influence on uncertainty of gray stage. Direct method was employed for determining noise variance. Here, the image noise variance was determined from fuzziness of noisy image employing polynomial model. However, the noise affected uncertainty of gray step as well as fuzziness of image. In addition, peak signal-to-noise ratio was not improved.

An improved FCM algorithm was introduced in [4] with anti-noise capability. In this work, noise reduction was performed in novel image segmentation algorithm with dictionary learning. Grayscale features was extracted in an accurate manner. However, computational cost involved in denoising was not reduced by employing improved FCM algorithm.

1.1 Objective and Contributions

Our study aims to accurately perform image denoising even in the presence of uncertainty from brain MRI images. We develop an end-to-end image denoising model based on Kalman Bucy Filtering algorithm. With this, denoised images, uncertainty issue is handled by employing Neutrosophic Neuro Fuzzy set, which are fed to image processing for producing finally detected output. Following the research line of previous works Chi-square similarity and fuzzy logic [1] and Saliency Map and Neutrosophic Set Theory (SMNS) [2], mathematical model in MR images [3] and improved FCM algorithm [4], several modifications are introduced to such a method to further enhance its performance and applicability for robust medical image denoising.

The key contributions of our paper are the following:

- To design a Kalman–Bucy Filtered Neutrosophic Neuro Fuzzy Image Denoising (KBF-NNFID) method for reducing noisy artifacts by higher PSNR in a computationally efficient fashion.

- A novelty of Kalman Bucy Filtering-based Image Denoising algorithmis examined in denoised images is achieved in a computationally efficient manner by means of Kalman Bucy Gain function.
- A new algorithm called Neutrosophic Neuro Fuzzy set in an accurate manner which help of Neutrosophic Fuzzy set subjected to deep neural network also addressed uncertainty involved through denoising.
- Simulation outcome of KBF-NNFID achieves better medical diagnosis in terms of PSNR, execution time as well as true positive rate in comparison with the other medical diagnosis methods.

1.2 Road map

Article planned as: Related works concerning image denoising introduced in Section 2. Thereafter in the Section 3, the proposed KBF-NNFID is designed. Section 4 presents Experiments. Section 5 explains discussion via table and graphical representations. Lastly, Section 6 demonstrates summary.

2. Literature Survey

With contemporary evolution in technology, several healthcare techniques escalate and filtrate in each and every medical diagnosis therefore paving new mechanisms for disease diagnosis. As a result, electronic healthcare has made an appearance as a state-of-the-art tendency in our society. Moreover, these mechanisms are fundamentally a guideline to the evolution of medical diagnosis. But the presence of noise makes the medical images indistinguishable and may confuse disease identification and analysis. Therefore, medical image denoising is a mandatory technique for medical image processing.

Deep learning is the expeditious growing area in artificial intelligence and is significant used in several domains, including medication. In [5], an extensive profile was provided on studies accomplished on the region of medical image analysis. However, yet it remains a challenging issue for segmenting tumor accurately owing to the speckle noise and inconsistency observed in the tissue background image. To overcome these issues, breast tumor segmentation employing neutrosophic set (NS) was proposed in [6]. However, it failed to identify noisy images. Most of gradient basis of edge detection techniques is highly susceptible to image noise that in turn results in less efficiency.

Some of challenges, issues and advantages of applying deep learning for edge detection were investigated in [7]. Neutrosophic set theory [8] was employed that in turn preserved image characteristics as well as minimization of false occurring due to noise. However, the object quality was not focused. To address on this aspect, a Modified Neutrosophic Set Segmentation method was proposed in [9] that with the aid of 3D modeling not only improved the image quality but also the PSNR.

Neutrosophic set has preferable course of action for image quantifying due to the reason that indeterminacy. With example, medical diagnosis images obtained via CCTV footage can included several blur pixels. These characteristic can be addressed via neutrosophic sets where each pixel is classification based on three factors, extent of truth, extent of falsity and extent of indeterminacy.

In [10], Pythagorean fuzzy set was introduced as an innovative model that with the aid of thresholding function obtained best results on separating of the object into its variants accurately. In [11], Linguistic Neutrosophic Cubic Set (LNCS) was proposed to handle data uncertainty via aggregation operators. Yet another convolutional neural network was applied in [12] that with the aid of sparseness factor not only ensured security but also denoised efficiently.

An enhanced self-learning weighted fuzzy algorithm that included distinct weights for calculating distance via continuous iterative self-learning was proposed in [13]. Followed by which, distance evaluated in addition to the weights obtained were utilized based on self-learning with the purpose of enhancing segmentation performance and robustness. DL was investigated [14]. An in-depth analysis of conventional ML, DL and certain potential methods were designed in [15].

Both one as well as two-sample hypothesis testing issue concerned in [16] by Neutrosophic fuzzy Sign set. In [17], DCNN and whale optimizer were presented by early medical diagnosis. The integration model along with the gradient descent-based model also enhanced the accuracy rate involved in the medical diagnosis. Yet another Jarque-Bera test was applied in [18] under the fuzzy environment for obtaining accurate inter valued data. A comprehensive approach for detection and classification of tumors was designed in [19] to not only reduce the noise but also resulted in accuracy via morphological operators. Identical lesion region was determined via hybrid fuzzy c means with more similarity in an accurate manner was presented in [20].

In [21], trigonometric similarity measures of NHSSs are performed. But, cosine and cotangent similarities were not discovered. Neutrosophic Dombi fuzzy graph is carried in [22] as well as Advanced Multi-Criteria Decision Making (MCDM) techniques designed in [23] to applications of multiple recent and innovative healthcare analytics problems. Emergency branch obtained in [24] via plithogenic set method. Several optimization systems designed in [25] to decision analytics, and data science in applying defined set of smart environments.

2.1 Research gap

In this work, a novel method called, Kalman–Bucy Filtered Neutrosophic Neuro Fuzzy Image Denoising (KBF-NNFID) for medical image denoising is proposed. Disadvantages of traditional techniques were obvious. Simple as well as invariant features formerly used for designing novel image denoising. This schemes performance plateaus that create them critical to improve.

3. KBF-NNFID Method

Owing to the presence of noise, uncertainty and fuzziness compute assisted diagnosis for medical images are said to be confined. Such constrains may influence decisions involving disease diagnosis while ascertaining the type of the disease and the grade of the disease as well. Fuzzy sets are considerably utilized in minimizing the uncertainty and fuzziness in diverse applications. However, those models disregard the spatial aspects of the pixels owing to noise. To control such disadvantages of the fuzzy-based methods, neutrosophic set is utilized alternatively.

As a consequence, different types of image denoising methods were explored on the basis of the neutrosophic theory for throwing light on the ambiguity and vagueness in medical image denoising. In this section a method called, Kalman–Bucy Filtered Neutrosophic Neuro Fuzzy Image Denoising (KBF-NNFID) for reducing the noisy artifacts with higher PSNR at computationally efficient manner is designed. KBF-NNFID diagram is given in Figure 1. .

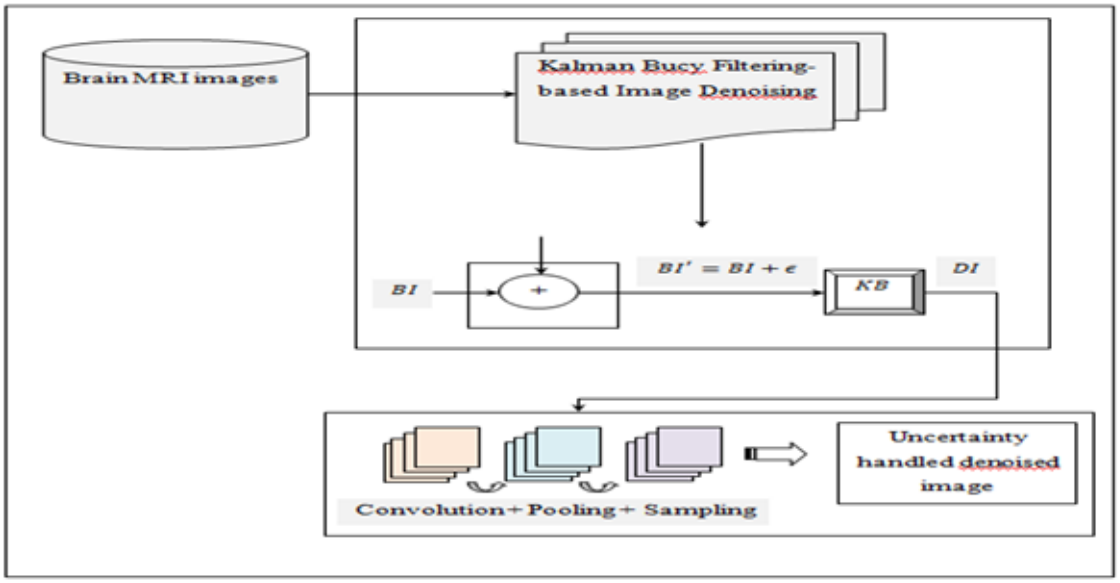

Figure 1 Block diagram of Kalman–Bucy Filtered Neutrosophic Neuro Fuzzy Image Denoising

As shown in the above figure with the Brain MRI LGG segmentation dataset provided as input, the overall KBF-NNFID method is split into two sections. They are image denoising model to reduce the noise in Brain MRI images by employing Kalman Bucy Filtering algorithm and handling uncertainty of noisy image by means of Neutrosopic Neuro Fuzzy set.

3.1 Dataset description

This Brain MRI LGG segmentation dataset consists of brain MR images as well as taken from The Cancer Imaging Archive (TCIA). First, patients who did not have preoperative FLAIR were excluded. As of 120 cases remained, furthermore 10 patients were excluded for whom pertinent genomic information are not obtainable. Final examined cohort of 110 patient's data included as five institutions. They correspond to 110 patients included least FLAIR sequence as well as genomic cluster data. Each images offered in tif format by 3 channels per image. In 101 cases, 3 sequences were found to be accessible as well as dataset is said to be prepared in 110 folders. Every folder have MR images by naming convention: `TCGA_<institution-code>_<patient-id>_<slice-number>.tif`. In 9 cases, post-contrast sequence has absent as well as pre-contrast sequence has lost in 6 cases. Via FLAIR, absent sequences were restored for constructing each images 3-channel.

3.2 Kalman Bucy Filtering-based Image Denoising model

Over the recent few years, image prior learning has materialized as an efficient instrument for image denoising that makes the most of prior knowledge to obtain sparse coding and employ them in reconstructing the clean image from noisy one. However, these image prior learning models are not said to be free from limitations and therefore failed attempts in enhancing performance and efficiency concurrently. In this section, objective of noise in Brain MRI images minimized via Kalman Bucy Filtering. Formulate the estimate of unknown variables depending on the single

measurement using joint probability distribution over variables for every timeframe. Kalman Bucy Filtering-based Image Denoising model plan is shown in Figure 2.

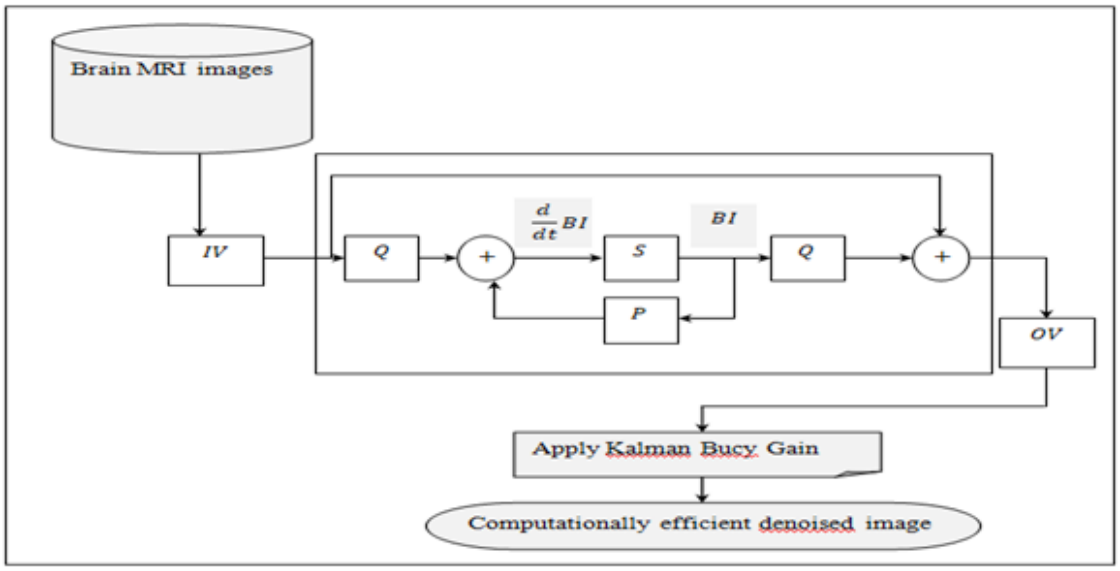

Figure 2 Block diagram of Kalman Bucy Filtering-based Image Denoising model

In figure, input of Brain MRI images assume and stored in form of vector represented as input vector '*IV*' is subjected to state space model for obtaining time series images. Followed by which a Kalman Bucy Gain function is applied to generated denoised images. Let us consider a Brain MRI images as input '*BI*' be a two dimensional function '*BI(a,b)*', 'a' and 'b' are spatial coordinates with amplitude of ' *BI*' at any location '(a,b)' refers to the intensity of the image at that point. As illustrated in the above figure, the degradation process is modeled with the aid of a noise term, '€', that functions on an input Brain MRI image 'BI' to produce a degraded image, '*BI*'. Given this noisy inference, with certain perception of the noise term, the restoration process capitulates an estimate, '*BI*', of the original image '*BI*'. Then, a Brain MRI input image '*BI*' of size 'a*b' pixels with gray level function is mathematically formulated as given below.

$$BI = \left(BI_{a,b}\right), where \;\; a, b = 1,2, \ldots., A, B \quad (1)$$

The noisy Brain MRI input image 'BI''' with gray level function is then defined as given below.

$$BI' = \left(BI'_{a,b}\right) + \epsilon_{a,b}, where \;\; a, b = 1,2, \ldots, A, B \quad (2)$$

Then, with the above two formulations as given in (1) for original image and (2) for noisy image, the objective of applying the filtering during denoise process remains in estimating the signal or pixels at each time 't' making entire utilization of the information (i.e., feature information like, patient ID, gender, age, race, ethnicity and so on) furnished by the observations up to that specific time. The Kalman Bucy filtering applied in our work is said to be recursive in 't' and hence said to be Gaussian. The Kalman Bucy filtering for denosing Brain MRI images with 'i' inputs, 'j' outputs and 's' state variables (i.e., probable subset of image pixels) is then represented in the form of state space model and is mathematically formulated as given below.

$$\frac{d}{dt}BI(t) = P(t)BI(t) + Q(t)IV(t) + \epsilon_1(t) \quad (3)$$

$$OV(t) = R(t)BI(t) + \epsilon_1(t) \quad (4)$$

From the above equations (3) and (4), at each time 't', '$P(t)$' denotes the state matrix (i.e., '$\dim[P(.)] = s * s$'), '$Q(t)$' denotes the input matrix (i.e., '$\dim[Q(.)] = s * i$'), '$R(t)$' denotes the output matrix (i.e., '$\dim[R(.)] = j * s$'), '$BI(.)$', '$IV(.)$' and '$OV(.)$' representing the state vector, input vector and output vector respectively. Then, the Kalman Bucy filter with series of measurement observed over a time period 't' comprises of two differential equations. The state estimate series of measurement observed over a time period 't' to denoise Brain MRI image 'BI''' is then mathematically expressed as given below.

$$\frac{d}{dt}BI'(t) = P(t)BI'(t) + Q(t)IV(t) + KB(t)[OV(t) - R(t)BI'(t)] \quad (5)$$

In a similar manner, the covariance series of measurement observed over a time period 't' to denoise Brain MRI image 'BI''' with intensities '$Int_1(t)$' (i.e., the tumor location in our work) is then mathematically expressed as given below.

$$\frac{d}{dt}CV(t) = P(t)CV(t) + CV(t)P^T(t) + Int_1(t) - KB(t) \quad (6)$$

From the above two equations (5) and (6), the Kalman Bucy gain to denoise the Brain MRI image 'BI''' using joint probability distribution is mathematically formulated as given below.

$$DI(t) = KB(t) = CV(t)R^T(t)Int_1^{-1}(t) \quad (7)$$

From the above equation (7), Kalman Bucy gain '$KB(t)$' at time 't' is evaluated based on the intensity of the noise '$\epsilon_1(t)$', therefore finally producing the denoised images. The pseudo code representation of Kalman Bucy Filtering-based Image Denoising is given below.

Input: Dataset 'DS', Features '$F = F_1, F_2, \dots, F_n$', Brain MRI Images '$BI = BI_1, BI_2, \dots, BI_m$'
Output: PNSR-improved Brain MRI Images '$DI(t)$'
1: Initialize 'n', 'm', time 't'
2: Begin
3: For each Dataset 'DS' with Features 'F' and Brain MRI Images 'BI'
4: Formulate Brain MRI input image with gray level function as in equation (1)
5: Formulate Brain MRI noisy image with gray level function as in equation (2)
6: For 'i' inputs, 'j' outputs and 's' state variables
7: Formulate the state space representation for the corresponding Brain MRI Images 'BI' as in equations (3) and (4)
8: Estimate state estimate series of measurement observed over a time period 't' as in equation (5)
9: Evaluate covariance series of measurement observed over a time period 't' as in equation (6)
10: Obtain Kalman Bucy gain as in equation (7)
11: Return denoised images 'DI'
12: End for
13: End for
14: End

Algorithm 1 Kalman Bucy Filtering-based Image Denoising

As given in the above Kalman Bucy Filtering-based Image Denoising algorithm, initially, the gray level functions are obtained separately for Brain MRI input and Brain MRI noisy images. Second, a state space representation with separate matrix for input, output and state is obtained. Third, over a succession of time period, state and covariance series are estimated. Finally, denoised images are obtained via Kalman Bucy gain. With this, the PSNR is improved in computationally efficient manner.

3.3 Neutrosopic Neuro Fuzzy set (NNFS)

Indeterminacy of all elements cannot measure as well as simply evaluated in traditional set. On other hand, fuzzy set occupied its application for handling uncertainty. In certain applications not only true membership must be considered. In addition, inclusion of false as well as indeterminacy of membership plays a major role.

To name a few applications being the processing of medical images that in turn would possess certain amount of indeterminacy value. As a result it is found to be laborious and cumbersome process to be addressed by employing the traditional fuzzy set. Neutrosopic Neuro Fuzzy set (NNFS) is used to tackle the uncertainty with help of the membership grade.

In our work, via truth, indeterminacy as well as falsity membership grades, neutrosophic set (NS) is utilized for handling the uncertainty of noisy image. An efficient image denoising process is carried out with higher PSNR. Figure 3 shows the block diagram of Neutrosopic Neuro Fuzzy set model.

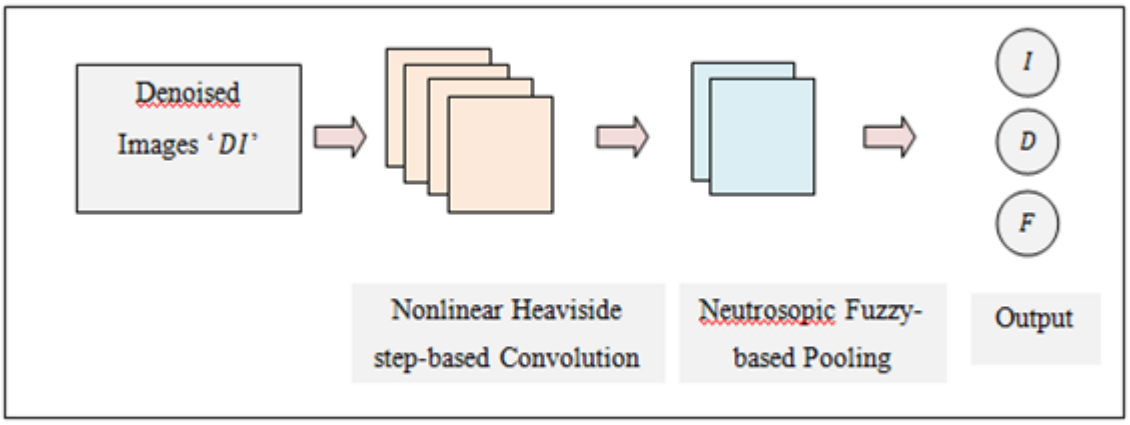

Figure 3 Structure of Neutrosopic Neuro Fuzzy set model

As shown in the above figure, in our work, deep neural network is integrated with neutrosophic fuzzy set Neutrosopic Neuro Fuzzy set for handling uncertainty issue in denoised brain MRI images. The neutrosophic denoised image is characterized by three subsets 'T', 'I' and 'F' respectively. With the aid of the above sets, '$T(m,n)$', '$I(m,n)$' and '$F(m,n)$' in neutrosophic are network's weights are updated. Finally, the updated weights were reused for new sets of images. Let us consider a pixel 'Pi' in neutrosophic denoised image described as '$Pi(m,n)\{T(m,n), I\ (m,n), F(m,n)\}$'. As a result, for each pixel in the neutrosophic denoised

image, the truth set 'T', false set 'F' and indeterminacy set 'I' has to be evaluated. Therefore, a pixel in the neutrosophic denoised image is represented as given below.

$$DI_{NS}(m,n) = \{T(m,n), I(m,n), F(m,n)\} \tag{8}$$

From the above formulation, '$t\%$' represents true (i.e., set of white pixels), '$i\%$'represents indeterminate (i.e., set of boundaries) and '$f\%$' represents false (i.e., set of non-white pixels), where '$t \in T$', '$i \in I$', and '$f \in F$' respectively. Let us further assume layer '$l-1$', '$m-th$' input denoised image as '$DI^{l-1}(m)$' and '$W^l(m,n)$' represents the weight that associates '$n-th$' feature of the output layer to the '$m-th$' feature of the input layer. Then, the convolution layer for the corresponding input denoised image is mathematically stated as given below.

$$DI^l(n) = HF\ \left[\sum_m \left(DI^{l-1}(m) * W^l(m,n) + B^l(n)\right)\right] \tag{9}$$

From the above equation (9), '$B^l(n)$' represents the bias of the convolution layer with '$*$' denoting the convolution operation and 'f' is a nonlinear Heaviside step function which is acquired by

$$HF(DI) = \frac{d}{dDI} MAX\{EI, 0\} \tag{10}$$

From the above equation (10), Heaviside step function 'HF' is evaluated for each denoised image 'DI' based on the derivative results. Followed by which in the pooling layer, '$DI^l(n)$' is obtained as given below.

$$DI^l(n) = POOL\ \ [DI^{l-1}(n)] \tag{11}$$

From the above equation (11), '$POOL$' denotes the sampling function applied to the denoised image 'DI' at the '$l-1$' layer and during the training phase, three functions '$T(m,n)$', '$I(m,n)$' and '$F(m,n)$' are evolved as given below.

$$T(m,n) = \frac{LMV(m,n) - LMV_{min}}{LMV_{max} - LMV_{min}} \tag{12}$$

$$LMV(m,n) = \frac{1}{(w*w)} DI(m+u, n+v) \tag{13}$$

$$I(m,n) = \frac{\delta(m,n) - \delta_{min}}{\delta_{max} - \delta_{min}} \tag{14}$$

$$\delta(m,n) = ABS[LMV(m,n)] \tag{15}$$

$$F(m,n) = 1 - T(m,n) \tag{16}$$

From the above equations (12) and (13), '$LMV(m,n)$', 'LMV_{min}' and 'LMV_{max}' denotes the local mean value, minimum local mean value and the maximum local mean value of the pixel 'm,n' respectively. The local mean value is obtained by utilizing neighboring pixels possessing similar intensity value with 'w' representing the window size of the 'DI'. Also from (14) and (15), the indeterminate resultant value '$I(m,n)$' is arrived at by means of the absolute value of local mean value '$\delta(m,n)$', average minimum value 'δ_{min}' and average maximum value 'δ_{max}' respectively. Finally, the falsified resultant value is obtained in equation (16). With this uncertainty was removed from denoised brain MRI images. With the design of this novel neutrosophic nonlinear Heaviside

step function which was based on the local mean value utilizing neighboring pixels this deep neural network model considered uncertainty with the aid of membership grade. This in turn handled uncertainty by means of additional domain 'I' that increases the efficiency of dealing with uncertainty in case of denoised images. The pseudo code representation of Neutrosopic Neuro Fuzzy set is given below.

Input: Dataset 'DS', Features '$F = F_1, F_2, \dots, F_n$', Brain MRI Images '$BI = BI_1, BI_2, \dots, BI_m$'
Output: Uncertainty resolved denoised image
1: Initialize denoised images 'DI'
2: Begin
3: For each Dataset 'DS' with Features 'F' and denoised images 'DI'
4: Formulate neutrosophic denoised image as in equation (8)
5: Formulate convolutional layer as in equations (9) and (10)
6: Evaluate '$DI^l(n)$' in the pooling layer as in equation (11)
7: For each denoised image 'DI' at the '$l-1$' layer
8: Formulate true (i.e., set of white pixels) set as in equations (12) and (13)
9: Formulate indeterminate (i.e., set of boundaries) set as in equations (14) and (15)
10: Formulate false (i.e., set of non-white pixels) set as in equation (16)
11: Return
12: End for
13: End for
14: End

Algorithm 2 Neutrosopic Neuro Fuzzy set

As given in the above Neutrosopic Neuro Fuzzy set algorithm, the application of Neutrosopic Fuzzy set for denoised brain MRI images are applied in the deep neural network framework. With the denoised images obtained as input, the pixel representation of the neutrosophic denoised image is first obtained. Second, convolution layer for the corresponding input denoised image is performed using nonlinear Heaviside step function. Finally, in the pooling layer, three functions are evolved and network's weights are updated. Finally, the updated weights are utilized for the other sets of images.

4. Experimental setup

Quantitative outcomes for estimating the image denoising based on neutrosophic fuzzy set methods with simulations performed in Matlab. In experiments, performance of the image denoising based on neutrosophic fuzzy set method called, Kalman–Bucy Filtered Neutrosophic Neuro Fuzzy Image Denoising (KBF-NNFID) is compared with [1] and SMNS [2]. Several factors such as PSNR, execution time and true positive rate have been used for the evaluation of aforementioned methods using Brain MRI LGG segmentation dataset.

5. Results

KBF-NNFID for medical image denoising in the presence of uncertainty using three different performance metrics by making detailed comparison with [1], [2] are provided.

5.1 PSNR

It has refers to percentage among highest feasible importance of a brain MRI as well as potentiality of corrupting noise to influences precision of its categorization. It is utilized in quantifying rebuilding quality during denoising process. The PSNR value is mathematically stated as given below.

$$PSNR(dB) = 10log_{10}\frac{255^2}{MSE} \tag{17}$$

Where, '$PSNR$' measured with mean square error 'MSE'.

$$MSE = \frac{1}{MN}\sum_{i=1}^{M}\sum_{j=1}^{N}[BI(m,n) - DI(m,n)]^2 \tag{18}$$

Where, 'MSE' evaluated depended on square dissimilarity pixel values among processed denoised image size '$DI(m,n)$' as well as original brain MRI image '$BI(m,n)$' by higher probable pixel value as 255. The PSNR is estimated in terms of decibel (dB). KBF-NNFID method on medical image denoising with [1] [2] using similar brain MRI image dataset is compared in Table 1.

Table 1 PSNR measure of the proposed KBF-NNFID method and other state-of-the-art methods

Image size (KB)	PSNR (dB)		
	KBF-NNFID	Iterative contraction algorithm based on Chi-square similarity and fuzzy logic	SMNS
28.7	64.65	58.25	55.32
36.5	62.15	56.66	53.14
27.7	60.22	56.81	55.32
52.3	58.63	52.25	48.70
56.4	64.95	56.66	51.40
33.9	63.31	58.25	53.14
51.3	66.24	62.68	58.25
46.5	61.73	56.66	54.72
40.7	57.63	53.14	50.64
31.7	60.22	55.32	54.16

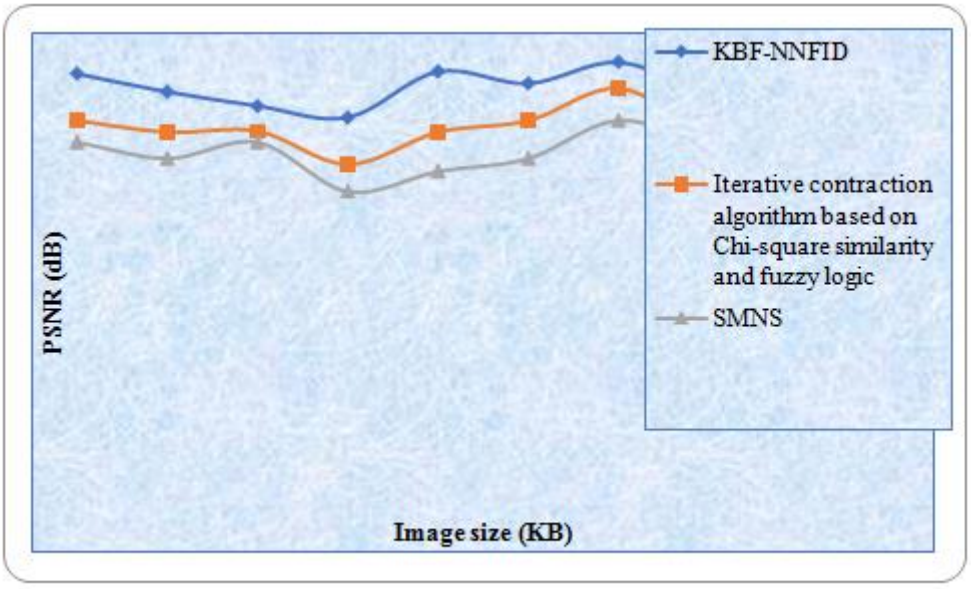

Figure 4 PSNR versus image size

PSNR demonstrated in Figure 4. From the above figure, by dissimilar image sizes concerned in medial image denoising even in the presence of uncertainty, PSNR as well be different. But, performance result of PSNR using the KBF-NNFID than [1] and [2]. Image size of 28.7KB considered, KBF-NNFID of PSNR is 64.65dB, and 58.25dB and 55.32 dB using [1] and [2]. On contrary to existing, KBF-NNFID better. Neutrosopic Neuro Fuzzy set algorithm in Neutrosopic Fuzzy set for denoised brain MRI images in deep neural network framework. Here, first the pixel representation of the neutrosophic denoised image was first obtained. Next, with the aid of nonlinear Heaviside step function convolution was performed for respective input denoised image. Finally with the three functions obtained in the pooling layer, network's weights were updated. As a result, the PSNR rate using KBF-NNFID method was said to be improved by 9% compared to [1] and 16% compared to [2].

5.2 Execution time

Image denoising execution time is second factor. Image denoising execution time is mathematically formulated by.

$$ET = \sum_{i=1}^{n} BI_i * Time\ \left(KB(t)\right) \tag{19}$$

From the above equation (19), the image denoising execution time 'ET' is measured based on the brain images images 'BI_i' as well as '$Time\ \left(KB(t)\right)$' is time needed at image denoising. It has estimated in milliseconds (ms). Performance of the proposed KBF-NNFID method on image denoising with previous work Iterative contraction algorithm based on Chi-square similarity and fuzzy logic [1] and SMNS [2] conducted on medical images using similar Brain MRI image dataset in Table 2.

Table 2 Execution time measure of the proposed KBF-NNFID method and other state-of-the-art methods

Images	Execution time (ms)		
	KBF-NNFID	Iterative contraction algorithm based on Chi-square similarity and fuzzy logic	SMNS
10	0.35	0.41	0.46
20	0.42	0.55	0.7
30	0.48	0.65	0.85
40	0.53	0.68	0.95
50	0.55	0.74	1.25
60	0.59	0.8	1.4
70	0.65	0.85	1.55
80	0.68	0.92	1.7
90	0.75	1.15	1.85
100	0.82	1.35	2.05

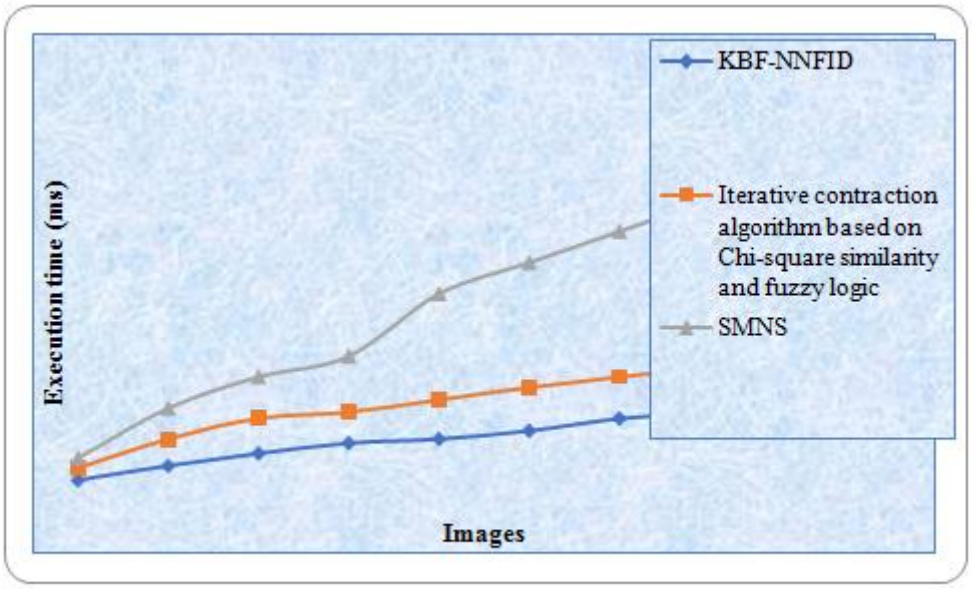

Figure 5 Execution times versus images

Execution time illustrated in Figure 5. From figure, horizontal axis is number of sample images as well as vertical axis is execution time evaluated in milliseconds (ms). Image denoising execution time has straightly enhancing to samples. Nevertheless, experiments carried out by 10 samples demonstrated KBF-NNFID, [1] [2] execution time is 0.35ms, 0.41ms and 0.46ms. On contrary to conventional, ET of KBF-NNFID is lesser. Kalman Bucy Filtering-based Image Denoising algorithm utilized. initially gray level functions were evaluated distinctly for Brain MRI input and Brain MRI noisy images. Moreover, a state space representation with separate matrix for input, output and state were obtained. With this the execution time involved using KBF-NNFID method were found to be comparatively lesser by 26%, as well as 50% than [1] [2].

5.3 True positive rate

True positive rate plays main task. Maximum values of true positive rate establish consistency of image denoising scheme employed. True positive rate is for denoise image even in presence of uncertainty in a successful manner with real positive range, i.e., to properly discover denoised image as denoised and vice versa. This is mathematically expressed as given below.

$$TPR = \frac{TP}{TP+FN} * 100 \tag{20}$$

From the above equation (20), true positive rate 'TPR', true positive is 'TP', false negative is 'FN'. TPR comparison is given in Table 3 in KBF-NNFID [1] [2] .

Table 3 True positive rate measure of the proposed KBF-NNFID method and other state-of-the-art methods

Images	True positive rate (%)		
	KBF-NNFID	Iterative contraction algorithm based on Chi-square similarity and fuzzy logic	SMNS
10	90	80	70
20	88.35	79.35	68.45
30	88	79	65.35
40	87.45	78.15	65
50	87	78	64.15
60	86.55	77.34	64
70	86.25	77	63.85
80	86	75.35	63.25
90	84.35	75	63
100	84	73.15	61

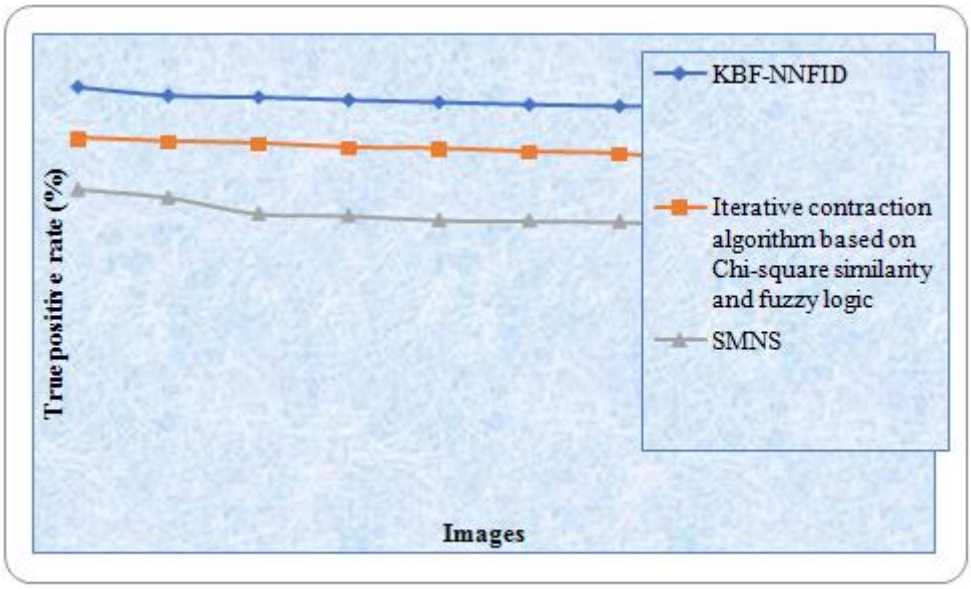

Figure 6 True positive rate time versus images

TPR proved in Figure 6.In figure, TPR for three techniques are found to be inversely relative to number of images. But, 10 images are taken for measuring simulation then TPR is 9% by KBF-NNFID, 80% and 70% by [1] and [2]. KBF-NNFID of TPR has better than existing [1], [2]. KBF-NNFID of TPR has better than [1], [2]. Symmetric Convolute layer employed for extract features. TPR enhanced by 5%, 11% than 1], [2].

6. Summary

Novel Kalman–Bucy Filtered Neutrosophic Neuro Fuzzy Image Denoising (KBF-NNFID) method is developed for medical image denoising even under uncertainty from brain MRI images. We developed the KBF-NNFID method by integrating Kalman Bucy Filtering and Neutrosopic Neuro Fuzzy set for minimum noisy artifacts with enhanced PSNR in computationally efficient fashion. A novelty of Kalman Bucy Filtering-based Image Denoising algorithmis determined in denoised images is achieved in a computationally efficient manner by means of Kalman Bucy Gain function. The state space model and series of measurement were observed over a time period by means of Kalman Bucy Filtering. Next, by employing nonlinear Heaviside step function that initially convoluted the denoise images and then performed pooling via local mean value.A new algorithm called Neutrosophic Neuro Fuzzy set in an accurate manner which help of Neutrosophic Fuzzy set subjected to deep neural network also addressed uncertainty involved during the denoising process. In this manner the brain MRI images were denoised even under uncertainty in a timely way. An image denoising was performed for preserving the geometrical features by means of Kalman Bucy Filtering-based denoising model for removing the noise. Second with the denoised brain MRI images as input, Neutrosopic Neuro Fuzzy set algorithm was applied that handled uncertainty via indeterminate functions. We used the Brain MRI Images Dataset for medical image denoising. Experimental evaluation is carried out using medical images with different performance metrics such as 13% enhanced PSNR, 38% minimum ET and 14% improved TPR.

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Received: March 3, 2024. Accepted: July 30, 2024

NSS

Neutrosophic Sets and Systems, Vol. 70, 2024

University of New Mexico

Identification of most impact factors of CIBIL score using Fermatean Neutrosophic Dombi Fuzzy Graphs

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Abstract. Graph theory plays a vital role in modeling real-world scenarios like network security and expert systems. Various extensions of graph theoretical conceptions have been designed for addressing uncertainty in graphical network scenarios. The concept of Fermatean Neutrosophic Dombi Fuzzy Graphs ($FNDF_graphs$) represents a novel and innovative extension in graph theory, combining the principles of Fermatean Neutrosophic fuzzy graphs and the Dombi operator. FNDFGs enhances the representation and analysis of uncertain relationships in a graph also it offers a more comprehensive and flexible approach to modeling uncertainty in graph structures. The main objective of this present research study focused on FNDFGs and their operations. At the end, an algorithm for Fermatean Neutrosophic Dombi fuzzy multi-criteria decision-making is given, which incorporate the concepts of Fermatean Neutrosophic sets and Dombi operations. Furthermore, a numerical example based on the selection of the most suitable CIBIL score application is put forward to illuminate the aptness of the proposed research work.

Keywords: Fermatean Neutrosophic sets; Fermatean Neutrosophic Dombi fuzzy graphs; Dombi; Fermatean fuzzy graphs.

1. Introduction

Fuzzy set (F_{set}) theory [1] is a mathematical framework for handling uncertainty and partial information, allowing elements to have varying degrees of membership between 0 and 1. This is in contrast to classical set (C_{set}) theory, where membership is binary (0 *or* 1). The F_{set} theory is particularly useful for improving decision-making processes and adapting to real-world scenarios where traditional binary logic falls short. Intuitionistic Fuzzy sets (IF_{set}), introduced

by Atanassov [2], extend (F_{set}) by incorporating both membership and non-membership degrees for each element, with their sum not exceeding 1. Neutrosophic sets (N_{set}), developed by Smarandache [4], include truth, falsity, and indeterminacy degrees, allowing their sum to range from 0 to 3, which is an extension of the F_{set} and IF_{set} theory [1, 4, 5]. These extensions enhance applications in artificial intelligence, decision-making, and image analysis by providing more comprehensive measures of uncertainty.

IF_{set} and N_{set} provide robust frameworks for managing imprecision and uncertainty in complex scenarios. Yager [68] created Pythagorean fuzzy sets (PF_{set}) as an extension of IF_{set} by allowing the sum of the squares of membership and non-membership degrees to be less than or equal to 1. This extension provides greater flexibility and better handles uncertainty. Because of this, PF_{set} explain more uncertainty than IF_{set}. Smarandache [4] introduced and subsequently expanded the degree of dependence between components of F_{set} and N_{set}. One special caseknown as the Pythagorean Neutrosophic set (PN_{set}) with independent indeterminacy and dependent truth and falsity is selected from among the three membership functions of N_{set}, subject to the requirement that the sum of the squares for membership, indeterminacy, and non-membership fall between 0 and 2 [28]. Graphs are visuals that show the relationships between objects. Because relationships between objects are more ambiguous, fuzzy graph (F_{graph}) models must be framed rather than regular graphs (R_{graph}), which have the same structure. Kaufmann [9] introduced fuzzy graphs (IF_{graph}) using Zadeh's fuzzy relation. Rosenfeld defined and developed a number of basic and theoretical concepts, including cycles, connectedness, and bridges [10].

The IF_{graph} were introduced by Karunambigai and Parvathi [11], and their further development into the intuitionistic fuzzy hypergraph ($IF_{hypergraph}$)and subsequent exploration of its uses [12] are noteworthy. The qualities of degree and regular SVN_{graph} were also studied [14], while Broumi et al. provided instances and properties of SVN_{graph} [13]. Pythagorean fuzzy graphs PF_{graph} were introduced to the F_{graph} notion in [15]. F_{graph} and PN_{set} were combined to create the recently developed idea of Pythagorean neutrosophic fuzzy graph (PNF_{set}) [1622]. Dombi fuzzy graphs (DF_{graph}) was first introduced by Ashraf et al. [23].

The DF_{graph} were thereafter the subject of much investigation, leading to the creation of the interval valued Dombi fuzzy neutrosophic graph ($IVDFN_{graph}$)[24], the DF_{graph} [19, 25], the PDF_{graph} [25], and the Dombi bipolar fuzzy graph (DBF_{graph}) [27]. Fermatean fuzzy sets (FF_{set}) were first introduced in [36] by allowing them the sum of the cubes of membership and non-membership degrees is less than or equal to 1. Their features were further explored

and extended to Fermatean fuzzy graphs (FF_{graph}) in [37, 38, 39]. Fermatean Neutrosophic fuzzy sets (FNF_{set}) were first described in [40] and combine the principles of FNF_{set} with fuzzy graphs, as discussed in [41], providing even more nuanced handling of uncertainty. The study in [37] proposes a new interpretation of the Fermatean neutrosophic Dombi fuzzy network and identifies several by products of its direct, cartesian construction. The Applications of Complex N_{graph} Structures are discussed in [38].

CIBIL, or the Credit Information Bureau (India) Limited, is a prominent credit information company in India, known for providing credit scores, commonly referred to as CIBIL scores. These scores are crucial for lenders in determining an individual's creditworthiness and influence lending decisions. The CIBIL score, ranging from 300 to 900, reflects factors like credit behavior, repayment history, debt-to-income ratio, and credit card usage. A higher score enhances the likelihood of loan approval at favorable terms. CIBIL plays a pivotal role in the Indian financial sector by aiding individuals and businesses in managing their credit health through regular checks and awareness of their credit reports. Maintaining a good CIBIL score is essential for financial well-being and favorable access to credit. Key practices include timely repayment of bills, keeping credit utilization low, having a diverse credit portfolio, limiting new credit applications, regularly checking your credit report for accuracy, maintaining a long credit history, being cautious with joint accounts, managing debt responsibly, updating personal information, using credit wisely, and avoiding settlements or write-offs. Consistent financial discipline and responsible credit management contribute to a positive credit profile over time. In this proposed work, the algorithm of financial decision making problem was developed using the concept of presents FND_{graphs}, which was implemented in the selection of best cibil score application.

This research proposal is organized as follows: Section 2 describes some basic prerequisite material on Pythagorean fuzzy graph (PF_{graph}), Pythagorean neutrosophic fuzzy graph (PNF_{graph}), Dombi fuzzy graph (DF_{graph}), and Pythagorean Dombi fuzzy graph (PDF_{graph}),theory. Section 3 proposes the concept of Fermatean neutrosophic Dombi fuzzy graph ($FNDF_{graph}$) and some basic operations with illustrative numerical examples. In Section 4 we discussed decision-making problems using $FNDF_{graph}$. Section 6 concludes the proposed work with some future directions.

2. **Preliminaries**

This section describes a brief review of PF_{graph}, PNF_{graph}, and DF_{graph} graph theory, which will be utilized for further development $FNDF_{graph}$.

Table 1. Literature review of basic Preliminaries

Typical Reference	Definition
S.Naz et aS.[15]	$\mathbf{PF_{graph}}$: $G = (\mathscr{M}, \mathscr{N})$ with $\mathscr{M} = \left\langle \tilde{\lambda}_{\mathscr{M}}, \tilde{\beta}_{\mathscr{M}} \right\rangle$ is a PF_{set} in $\mathscr{Z}$ with $0 \leq \tilde{\alpha}^2_{\mathscr{N}}(s^*) + \tilde{\beta}^2_{\mathscr{N}}(s^*) \leq 1 \ \forall \ s \in \mathscr{Z}$ and $\mathscr{N} = \left\langle \tilde{\lambda}_{\mathscr{N}}, \tilde{\beta}_{\mathscr{N}} \right\rangle$ is a PF_{set} in $\mathscr{Q} : \mathscr{Z} \times \mathscr{Z}$ such that $\tilde{\lambda}_{\mathscr{M}}(s^*, t^*) \leq (\tilde{\lambda}_{\mathscr{M}}(s^*) \wedge \tilde{\lambda}_{\mathscr{M}}(t^*))$, $\tilde{\beta}_{\mathscr{N}}(s^*, t^*) \geq (\tilde{\beta}_{\mathscr{M}}(s^*) \vee \tilde{\beta}_{\mathscr{N}}(t^*))$ and $0 \leq \tilde{\alpha}^2_{\mathscr{N}}(s^*, t^*) + \tilde{\beta}^2_{\mathscr{N}}(s^*, t^*) \leq 1$ $\forall \ s^*, t^* \in \mathscr{Z}$.
D.Ajay et al.[19]	**Dombi's family** : $\mathbf{t-norm}$: $\frac{1}{1+[(\frac{1-g_1}{g_1})^{\tilde{\gamma}}+(\frac{1-g_2}{g_2})^{\tilde{\gamma}}]^{\frac{1}{\tilde{\gamma}}}}, \tilde{\gamma} > 0.$ $\mathbf{t-conorm}$: $\frac{1}{1+[(\frac{1-g_1}{g_1})^{-\tilde{\gamma}}+(\frac{1-g_2}{g_2})^{-\tilde{\gamma}}]^{\frac{1}{-\tilde{\gamma}}}}, \tilde{\gamma} > 0.$ $\mathbf{negation}$: $1 - g_1$. $\mathbf{T-operators}$: $\mathscr{T}(g_1, g_2) = \frac{g_1 g_2}{g_1 + g_2 - g_1 g_2}$ and $\mathscr{S}(g_1, g_2) = \frac{g_1 + g_2 - 2g_1 g_2}{1 - g_1 g_2}$. which is obtained by taking $\tilde{\gamma} = 1$, in Dombi family of t-norms and t-conorms.
D.Ajay et al.[16]	$\mathbf{PNF_{graph}}$:$G = (\mathscr{M}, \mathscr{N})$, where $\mathscr{M} = \{a_1, a_2, \cdots, a_n\}$ such that $\tilde{\lambda}_{\mathscr{M}}, \tilde{\beta}_{\mathscr{M}}, \tilde{\gamma}_{\mathscr{M}}$ are from $\mathscr{M}$ to $[0,1]$ with $0 \leq \tilde{\lambda}^2_{\mathscr{M}}(a_i) + \tilde{\beta}^2_{\mathscr{M}}(a_i) + \tilde{\gamma}^2_{\mathscr{M}}(a_i) \leq 2 \ \forall a_i \in \mathscr{M}$ and $\tilde{\lambda}_{\mathscr{N}}, \tilde{\beta}_{\mathscr{N}}, \tilde{\gamma}_{\mathscr{N}}$ are from $\mathscr{A} \times \mathscr{A}$ to $[0,1]$ such that $\tilde{\lambda}_{\mathscr{N}}(a_i, a_j) \leq (\tilde{\lambda}_{\mathscr{M}}(a_i) \wedge \tilde{\lambda}_{\mathscr{M}}(a_j))$,$\tilde{\beta}_{\mathscr{N}}(a_i, a_j) \leq (\tilde{\beta}_{\mathscr{M}}(a_i) \wedge \tilde{\beta}_{\mathscr{M}}(a_j))$ and $\tilde{\gamma}_{\mathscr{N}}(a_i, a_j) \leq (\tilde{\gamma}_{\mathscr{M}}(a_i) \vee \tilde{\gamma}_{\mathscr{M}}(a_j))$ with $0 \leq \tilde{\lambda}^2_{\mathscr{N}}(a_i, a_j) + \tilde{\beta}^2_{\mathscr{N}}(a_i, a_j) + \tilde{\gamma}^2_{\mathscr{N}}(a_i, a_j) \leq 2 \ \forall a_i, a_j \in \mathscr{M} \times \mathscr{M}$.
P.Chellamani et al.[17]	$\mathbf{DF_{graph}}$: $G = (\mathscr{M}, \mathscr{N})$, where $\mathscr{M} : \mathscr{Z} \to [0,1]$ is contained in $\mathscr{Z}$ and $\mathscr{N} : \mathscr{Z} \times \mathscr{Z} \to [0,1]$ is a symmetric fuzzy relation on $\mathscr{M}$ such that $\tilde{\lambda}_{\mathscr{N}}(s^*, t^*) \leq \frac{\tilde{\lambda}_{\mathscr{M}}(s^*)\tilde{\lambda}_{\mathscr{M}}(t^*)}{\tilde{\lambda}_{\mathscr{M}}(s^*) + \tilde{\lambda}_{\mathscr{M}}(t^*) - \tilde{\lambda}_{\mathscr{M}}(s^*)\tilde{\lambda}_{\mathscr{M}}(t^*)} \ \forall \ s^*, t^* \in \mathscr{Z}$.
Akram et al. [39]	$\mathbf{PDF_{graph}}$: $\tilde{\lambda}_{\mathscr{N}}(s^*, t^*) \leq \frac{\tilde{\lambda}_{\mathscr{M}}(s^*)\tilde{\lambda}_{\mathscr{M}}(t^*)}{\tilde{\lambda}_{\mathscr{M}}(s^*) + \tilde{\lambda}_{\mathscr{M}}(t^*) - \tilde{\lambda}_{\mathscr{M}}(s^*)\tilde{\lambda}_{\mathscr{M}}(t^*)}$, $\tilde{\beta}_{\mathscr{N}}(s^*, t^*) \leq \frac{\tilde{\beta}_{\mathscr{M}}(s^*) + \tilde{\beta}_{\mathscr{M}}(t^*) - 2\tilde{\beta}_{\mathscr{M}}(s^*)\tilde{\beta}_{\mathscr{M}}(t^*)}{1 - \tilde{\beta}_{\mathscr{M}}(s^*)\tilde{\beta}_{\mathscr{M}}(t^*)}$ and $0 \leq \tilde{\lambda}^2_{\mathscr{N}}(s^*, t^*) + \tilde{\beta}^2_{\mathscr{N}}(s^*, t^*) \leq 1 \ \forall \ s^*, t^* \in \mathscr{Z}$.

3. **Fermatean Neutrosophic Dombi Fuzzy Graphs**

$FNDF_{graph}$ is a sophisticated mathematical framework that combines Fermatean fuzzy sets, Dombi operators, and neutrosophic logic. They provide a flexible and nuanced way to model complex systems with uncertain, vague, and contradictory information. This approach is

valuable in fields such as decision-making and artificial intelligence, where precise handling of uncertain data is crucial.

Definition 3.1. $FNDF_{graph}$ is an ordered pair $G = (\mathscr{M}, \mathscr{N})$ with $\mathscr{M} = \left\langle \tilde{\lambda}_{\mathscr{M}}, \tilde{\beta}_{\mathscr{M}}, \tilde{\gamma}_{\mathscr{M}} \right\rangle$ is a FN_{set} in $\mathscr{Z}$ and $\mathscr{N} = \left\langle \tilde{\lambda}_{\mathscr{N}}, \tilde{\beta}_{\mathscr{N}}, \tilde{\gamma}_{\mathscr{N}} \right\rangle$ is a FN_{set} in $\mathscr{Q} : \mathscr{Z} \times \mathscr{Z}$ such that

$$\tilde{\lambda}_{\mathscr{N}}(s^*, t^*) \leq \frac{\tilde{\lambda}_{\mathscr{M}}(s^*)\tilde{\lambda}_{\mathscr{M}}(t^*)}{\tilde{\lambda}_{\mathscr{M}}(s^*) + \tilde{\lambda}_{\mathscr{M}}(t^*) - \tilde{\lambda}_{\mathscr{M}}(s^*) \cdot \tilde{\lambda}_{\mathscr{M}}(s^*, t^*)},$$

$$\tilde{\beta}_{\mathscr{N}}(s^*, t^*) \leq \frac{\tilde{\beta}_{\mathscr{M}}(s^*)\tilde{\beta}_{\mathscr{M}}(t^*)}{\tilde{\beta}_{\mathscr{M}}(s^*) + \tilde{\beta}_{\mathscr{M}}(t^*) - \tilde{\beta}_{\mathscr{M}}(s^*) \cdot \tilde{\beta}_{\mathscr{M}}(s^*, t^*)},$$

$$\tilde{\gamma}_{\mathscr{N}}(s^*, t^*) \leq \frac{\tilde{\gamma}_{\mathscr{M}}(s^*) + \tilde{\gamma}_{\mathscr{M}}(t^*) - 2\ \tilde{\gamma}_{\mathscr{M}}(s^*)\ \tilde{\gamma}_{\mathscr{M}}(t^*)}{1 - \tilde{\gamma}_{\mathscr{M}}(s^*)\ \tilde{\gamma}_{\mathscr{M}}(t^*)},$$

and $0 \leq \tilde{\alpha}^3_{\mathscr{N}}(s^*, t^*) + \tilde{\beta}^3_{\mathscr{N}}(s^*, t^*) + \tilde{\gamma}^3_{\mathscr{N}}(s^*, t^*) \leq 3$ for all $s^*, t^* \in \mathscr{Z}$.

Definition 3.2. Let $\bar{\mathscr{N}} = \left\{ \left((s^*, t^*), \tilde{\lambda}_{\mathscr{N}}(s^*, t^*), \tilde{\beta}_{\mathscr{N}}(s^*, t^*), \tilde{\gamma}_{\mathscr{N}}(s^*, t^*) \right) / s^*, t^* \in \mathscr{E} \right\}$ be a $FNDF_{edge\ set}$ in $FNDF_{graph}$ then the order and size of $FNDF_{graph}$ is defined by

$(i).\bar{O}(FNDF_{graph}) = \left(\sum_{s^* \mathscr{Z}} \tilde{\lambda}_{\mathscr{M}}(s^*), \sum_{s^* \in \mathscr{Z}} \tilde{\beta}_{\mathscr{M}}(s^*), \sum_{s^* \in \mathscr{Z}} \tilde{\gamma}_{\mathscr{M}}(s^*) \right),$

$(ii).\bar{S}(FNDF_{graph}) = \left(\sum_{s^*, t^* \in \mathscr{E}} \tilde{\lambda}_{\mathscr{N}}(s^*, t^*), \sum_{s^*, t^* \in \mathscr{E}} \tilde{\beta}_{\mathscr{N}}(s^*, t^*), \sum_{s^*, t^* \in \mathscr{E}} \tilde{\gamma}_{\mathscr{N}}(s^*, t^*) \right)$

Definition 3.3. Let $\bar{\mathscr{N}} = \left\{ \left((s^*, t^*), \tilde{\lambda}_{\mathscr{N}}(s^*, t^*), \tilde{\beta}_{\mathscr{N}}(s^*, t^*), \tilde{\gamma}_{\mathscr{N}}(s^*, t^*) \right) / s^*, t^* \in \mathscr{E} \right\}$ be a $FNDF_{edge\ set}$ in $FNDF_{graph}$ then the degree and total degree of vertex $s^* \in \mathscr{Z}$ is defined by $\bar{D}_{FNDF_{graph}}(s^*) = \left(\bar{D}_{\tilde{\alpha}}(s^*), \bar{D}_{\tilde{\beta}}(s^*), \bar{D}_{\tilde{\gamma}}(s^*) \right)$, and $(\bar{T}\bar{D})_{FNDF_{graph}}(s^*) = \left((\bar{T}\bar{D})_{\tilde{\alpha}}(s^*), (\bar{T}\bar{D})_{\tilde{\beta}}(s^*), (\bar{T}\bar{D})_{\tilde{\gamma}}(s^*) \right)$, where

$$(i).\bar{D}_{\tilde{\alpha}}(s^*) = \sum_{s^*, t^* \neq s^* \in \mathscr{Z}} \tilde{\lambda}_{\mathscr{N}}(s^*, t^*) = \sum_{s^*, t^* \neq s \in \mathscr{Z}} \frac{\tilde{\lambda}_{\mathscr{M}}(s^*)\ \tilde{\lambda}_{\mathscr{M}}(t^*)}{\tilde{\lambda}_{\mathscr{M}}(s^*) + \tilde{\lambda}_{\mathscr{M}}(t^*) - \tilde{\lambda}_{\mathscr{M}}(s^*)\ \tilde{\lambda}_{\mathscr{M}}(t^*)},$$

$$\bar{D}_{\tilde{\beta}}(s^*) = \sum_{s^*, t^* \neq s^* \in \mathscr{Z}} \tilde{\beta}_{\mathscr{N}}(s^*, t^*) = \sum_{s^*, t^* \neq s \in \mathscr{Z}} \frac{\tilde{\beta}_{\mathscr{M}}(s^*)\ \tilde{\beta}_{\mathscr{M}}(t^*)}{\tilde{\beta}_{\mathscr{M}}(s^*) + \tilde{\beta}_{\mathscr{M}}(t^*) - \tilde{\beta}_{\mathscr{M}}(s^*)\ \tilde{\beta}_{\mathscr{M}}(t^*)},$$

$$\bar{D}_{\tilde{\gamma}}(s^*) = \sum_{s^*, t^* \neq s^* \in \mathscr{Z}} \tilde{\gamma}_{\mathscr{N}}(s^*, t^*) = \sum_{s^*, t^* \neq s \in \mathscr{Z}} \frac{\tilde{\gamma}_{\mathscr{M}}(s^*) + \tilde{\gamma}_{\mathscr{M}}(t^*) - 2\ \tilde{\gamma}_{\mathscr{M}}(s^*)\ \tilde{\gamma}_{\mathscr{M}}(t^*)}{1 - \tilde{\gamma}_{\mathscr{M}}(s^*)\ \tilde{\gamma}_{\mathscr{M}}(t^*)}$$

$(ii).(\bar{T}\bar{D})_{\tilde{\alpha}}(s^*) = \sum_{s^*, t^* \neq s^* \in \mathscr{Z}} \tilde{\lambda}_{\mathscr{N}}(s^*, t^*) + \tilde{\lambda}_{\mathscr{M}}(s^*) =$

$$\sum_{s^*, t^* \neq s^* \in \mathscr{Z}} \frac{\tilde{\lambda}_{\mathscr{M}}(s^*)\ \tilde{\lambda}_{\mathscr{M}}(t^*)}{\tilde{\lambda}_{\mathscr{M}}(s^*) + \tilde{\lambda}_{\mathscr{M}}(t^*) - \tilde{\lambda}_{\mathscr{M}}(s^*)\ \tilde{\lambda}_{\mathscr{M}}(t^*)} + \tilde{\lambda}_{\mathscr{M}}(s^*),$$

$(\bar{T}\bar{D})_{\tilde{\beta}}(s^*) = \sum_{s^*, t^* \neq s^* \in \mathscr{Z}} \tilde{\beta}_{\mathscr{N}}(s^*, t^*) + \tilde{\beta}_{\mathscr{M}}(s^*) =$

$$\sum_{s^*, t^* \neq s^* \in \mathscr{Z}} \frac{\tilde{\beta}_{\mathscr{M}}(s^*)\ \tilde{\beta}_{\mathscr{M}}(t^*)}{\tilde{\beta}_{\mathscr{M}}(s^*) + \tilde{\beta}_{\mathscr{M}}(t^*) - \tilde{\beta}_{\mathscr{M}}(s^*)\ \tilde{\beta}_{\mathscr{M}}(t^*)} + \tilde{\beta}_{\mathscr{M}}(s^*),$$

$(\bar{T}\bar{D})_{\tilde{\gamma}}(s^*) = \sum_{s^*,t^* \neq s^* \in \mathscr{Z}} \tilde{\gamma}_{\mathscr{N}}(s^*,t^*) + \tilde{\gamma}_{\mathscr{M}}(s^*) =$
$\sum_{s^*,t^* \neq s^* \in \mathscr{Z}} \dfrac{\tilde{\gamma}_{\mathscr{M}}(s^*) + \tilde{\gamma}_{\mathscr{M}}(t^*) - 2\ \tilde{\gamma}_{\mathscr{M}}(s^*)\ \tilde{\gamma}_{\mathscr{M}}(t^*)}{1 - \tilde{\gamma}_{\mathscr{M}}(s^*)\ \tilde{\gamma}_{\mathscr{M}}(t^*)} + \tilde{\gamma}_{\mathscr{M}}(s^*).$

Definition 3.4. The complement of a $FNDF_{graph}$ $G = (\mathscr{M}, \mathscr{N})$ is a $FNDF_{graph}$ $G^C = (\mathscr{M}^C, \mathscr{N}^C)$ which is defined by
$(i).\tilde{\lambda}^C_{\mathscr{M}}(s^*) = \tilde{\lambda}_{\mathscr{M}}(s^*)$, $\tilde{\beta}^C_{\mathscr{M}}(s^*) = \tilde{\beta}_{\mathscr{M}}(s^*)$ and $\tilde{\gamma}^C_{\mathscr{M}}(s^*) = \tilde{\gamma}_{\mathscr{M}}(s^*)$.
$(ii).\tilde{\lambda}^C_{\mathscr{N}}(s^*,t^*) =$
$$\begin{cases} \dfrac{\tilde{\lambda}_{\mathscr{M}}(s^*)\tilde{\lambda}_{\mathscr{M}}(t^*)}{\tilde{\lambda}_{\mathscr{M}}(s^*) + \tilde{\lambda}_{\mathscr{M}}(t^*) - \tilde{\lambda}_{\mathscr{M}}(s^*)\tilde{\lambda}_{\mathscr{M}}(t^*)} & if\ \tilde{\lambda}_{\mathscr{N}}(s^*,t^*) = 0, \\ \dfrac{\tilde{\lambda}_{\mathscr{M}}(s^*)\tilde{\lambda}_{\mathscr{M}}(t^*)}{\tilde{\lambda}_{\mathscr{M}}(s^*) + \tilde{\lambda}_{\mathscr{M}}(t^*) - \tilde{\lambda}_{\mathscr{M}}(s^*)\tilde{\lambda}_{\mathscr{M}}(t^*)} - \tilde{\lambda}_{\mathscr{N}}(s^*,t^*) & if\ 0 < \tilde{\lambda}_{\mathscr{N}}(s^*,t^*) \leq 1 \end{cases}$$
$(iii).\tilde{\beta}^C_{\mathscr{N}}(s^*,t^*) =$
$$\begin{cases} \dfrac{\tilde{\beta}_{\mathscr{M}}(s^*)\tilde{\beta}_{\mathscr{M}}(t^*)}{\tilde{\beta}_{\mathscr{M}}(s^*) + \tilde{\beta}_{\mathscr{M}}(t^*) - \tilde{\beta}_{\mathscr{M}}(s^*)\tilde{\beta}_{\mathscr{M}}(t^*)} & if\ \tilde{\beta}_{\mathscr{N}}(s^*,t^*) = 0, \\ \dfrac{\tilde{\beta}_{\mathscr{M}}(s^*)\tilde{\beta}_{\mathscr{M}}(t^*)}{\tilde{\beta}_{\mathscr{M}}(s^*) + \tilde{\beta}_{\mathscr{M}}(t^*) - \tilde{\beta}_{\mathscr{M}}(s^*)\tilde{\beta}_{\mathscr{M}}(t^*)} - \tilde{\beta}_{\mathscr{N}}(s^*,t^*) & if\ 0 < \tilde{\beta}_{\mathscr{N}}(s^*,t^*) \leq 1 \end{cases}$$
$(iv).\tilde{\gamma}^C_{\mathscr{N}}(s^*,t^*) =$
$$\begin{cases} \dfrac{\tilde{\gamma}_{\mathscr{M}}(s^*) + \tilde{\gamma}_{\mathscr{M}}(t^*) - 2\tilde{\gamma}_{\mathscr{M}}(s^*)\tilde{\gamma}_{\mathscr{M}}(t^*)}{1 - \tilde{\gamma}_{\mathscr{M}}(s^*)\tilde{\gamma}_{\mathscr{M}}(t^*)} & if\ \tilde{\gamma}_{\mathscr{N}}(s^*,t^*) = 0, \\ \dfrac{\tilde{\gamma}_{\mathscr{M}}(s^*) + \tilde{\gamma}_{\mathscr{M}}(t^*) - 2\tilde{\gamma}_{\mathscr{M}}(s^*)\tilde{\gamma}_{\mathscr{M}}(t^*)}{1 - \tilde{\gamma}_{\mathscr{M}}(s^*)\tilde{\gamma}_{\mathscr{M}}(t^*)} - \tilde{\gamma}_{\mathscr{N}}(s^*,t^*) & if\ 0 < \tilde{\gamma}_{\mathscr{N}}(s^*,t^*) \leq 1 \end{cases}$$

Theorem 3.5. *If $G = (\mathscr{M}, \mathscr{N})$ is a $FNDF_{graph}$, then $(G^C)^C = G$.*

Proof. Consider G as a $FNDF_{graph}$. By definition, $(FNDF_{graph})^C$, we have $(\tilde{\lambda}^C_{\mathscr{M}})^C(s^*) = \tilde{\lambda}^C_{\mathscr{M}}(s^*) = \tilde{\lambda}_{\mathscr{M}}(s^*)$, $(\tilde{\beta}^C_{\mathscr{M}})^C(s^*) = \tilde{\beta}^C_{\mathscr{M}}(s^*) = \tilde{\beta}_{\mathscr{M}}(s^*)$, $(\tilde{\gamma}^C_{\mathscr{M}})^C(s^*) = \tilde{\gamma}^C_{\mathscr{M}}(s^*) = \tilde{\gamma}_{\mathscr{M}}(s^*)$, for all $s^* \in \mathscr{Z}$.

If $\tilde{\lambda}_{\mathscr{N}}(s^*,t^*) = 0$, $\tilde{\beta}_{\mathscr{N}}(s^*,t^*) = 0$, $\tilde{\gamma}_{\mathscr{N}}(s^*,t^*) = 0$, then
$(\tilde{\lambda}^C_{\mathscr{M}})^C(s^*) =$
$$\frac{(\tilde{\lambda}_{\mathscr{M}}(s^*)\ \tilde{\lambda}_{\mathscr{M}}(t^*))^C}{\tilde{\lambda}^C_{\mathscr{M}}(s^*) + \tilde{\lambda}^C_{\mathscr{M}}(t^*) - (\tilde{\lambda}_{\mathscr{M}}(s^*)\ \tilde{\lambda}_{\mathscr{M}}(t^*))^C} = \frac{\tilde{\lambda}_{\mathscr{M}}(s^*)\ \tilde{\lambda}_{\mathscr{M}}(t^*)}{\tilde{\lambda}_{\mathscr{M}}(s^*) + \tilde{\lambda}_{\mathscr{M}}(t^*) - \tilde{\lambda}_{\mathscr{M}}(s^*)\ \tilde{\lambda}_{\mathscr{M}}(t^*)} = \tilde{\lambda}_{\mathscr{N}}(s^*,t^*),$$
$(\tilde{\beta}^C_{\mathscr{M}})^C(s^*) =$
$$\frac{(\tilde{\beta}_{\mathscr{M}}(s^*)\ \tilde{\beta}_{\mathscr{M}}(t^*))^C}{\tilde{\beta}^C_{\mathscr{M}}(s^*) + \tilde{\beta}^C_{\mathscr{M}}(t^*) - (\tilde{\beta}_{\mathscr{M}}(s^*)\ \tilde{\beta}_{\mathscr{M}}(t^*))^C} = \frac{\tilde{\beta}_{\mathscr{M}}(s^*)\ \tilde{\beta}_{\mathscr{M}}(t^*)}{\tilde{\beta}_{\mathscr{M}}(s^*) + \tilde{\beta}_{\mathscr{M}}(t^*) - \tilde{\beta}_{\mathscr{M}}(s^*)\ \tilde{\beta}_{\mathscr{M}}(t^*)} = \tilde{\beta}_{\mathscr{N}}(s^*,t^*),$$
$(\tilde{\gamma}^C_{\mathscr{M}})^C(s^*) =$
$$\frac{\tilde{\gamma}^C_{\mathscr{M}}(s^*) + \tilde{\gamma}^C_{\mathscr{M}}(t^*) - 2(\tilde{\gamma}_{\mathscr{M}}(s^*)\ \tilde{\gamma}_{\mathscr{M}}(t^*))^C}{1 - (\tilde{\gamma}_{\mathscr{M}}(s^*)\ \tilde{\gamma}_{\mathscr{M}}(t^*))^C} = \frac{\tilde{\gamma}_{\mathscr{M}}(s^*) + \tilde{\gamma}_{\mathscr{M}}(t^*) - 2\ \tilde{\gamma}_{\mathscr{M}}(s^*)\ \tilde{\gamma}_{\mathscr{M}}(t^*)}{1 - \tilde{\gamma}_{\mathscr{M}}(s^*)\ \tilde{\gamma}_{\mathscr{M}}(t^*)} =$$
$\tilde{\gamma}_{\mathscr{N}}(s^*,t^*)$.

If $0 < \tilde{\lambda}_{\mathscr{N}}(s^*,t^*), \tilde{\beta}_{\mathscr{N}}(s^*,t^*), \tilde{\gamma}_{\mathscr{N}}(s^*,t^*) \leq 1$, then

$$(\tilde{\lambda}^C_{\mathscr{N}})^C(s^*,t^*) = \frac{(\tilde{\lambda}_{\mathscr{M}}(s^*)\ \tilde{\lambda}_{\mathscr{M}}(t^*))^C}{\tilde{\lambda}^C_{\mathscr{M}}(s^*)+\tilde{\lambda}^C_{\mathscr{M}}(t^*)-(\tilde{\lambda}_{\mathscr{M}}(s^*)\ \tilde{\lambda}_{\mathscr{M}}(t^*))^C} - \tilde{\lambda}^C_{\mathscr{N}}(s^*,t^*)$$

$$= \frac{\tilde{\lambda}_{\mathscr{M}}(s^*)\ \tilde{\lambda}_{\mathscr{M}}(t^*)}{\tilde{\lambda}_{\mathscr{M}}(s^*)+\tilde{\lambda}_{\mathscr{M}}(t^*)-\tilde{\lambda}_{\mathscr{M}}(s^*)\ \tilde{\lambda}_{\mathscr{M}}(t^*)} - \left[\frac{\tilde{\lambda}_{\mathscr{M}}(s^*)\ \tilde{\lambda}_{\mathscr{M}}(t^*)}{\tilde{\lambda}_{\mathscr{M}}(s^*)+\tilde{\lambda}_{\mathscr{M}}(t^*)-\tilde{\lambda}_{\mathscr{M}}(s^*)\ \tilde{\lambda}_{\mathscr{M}}(t^*)} - \tilde{\lambda}_{\mathscr{N}}(s^*,t^*)\right] = \tilde{\lambda}_{\mathscr{N}}(s^*,t^*),$$

$$(\tilde{\beta}^C_{\mathscr{N}})^C(s^*,t^*) = \frac{(\tilde{\beta}_{\mathscr{M}}(s^*)\ \tilde{\beta}_{\mathscr{M}}(t^*))^C}{\tilde{\mathscr{M}}^C(s^*)+\tilde{\lambda}^C_{\mathscr{M}}(t^*)-(\tilde{\lambda}_{\mathscr{M}}(s^*)\ \tilde{\beta}_{\mathscr{M}}(t^*))^C} - \tilde{\beta}^C_{\mathscr{N}}(s^*,t^*)$$

$$= \frac{\tilde{\beta}_{\mathscr{M}}(s^*)\ \tilde{\beta}_{\mathscr{M}}(t^*)}{\tilde{\beta}_{\mathscr{M}}(s^*)+\tilde{\beta}_{\mathscr{M}}(t^*)-\tilde{\beta}_{\mathscr{M}}(s^*)\ \tilde{\beta}_{\mathscr{M}}(t^*)} - \left[\frac{\tilde{\beta}_{\mathscr{M}}(s^*)\ \tilde{\beta}_{\mathscr{M}}(t^*)}{\tilde{\beta}_{\mathscr{M}}(s^*)+\tilde{\beta}_{\mathscr{M}}(t^*)-\tilde{\beta}_{\mathscr{M}}(s^*)\ \tilde{\beta}_{\mathscr{M}}(t^*)} - \tilde{\beta}_{\mathscr{N}}(s^*,t^*)\right] = \tilde{\beta}_{\mathscr{N}}(s^*,t^*),$$

$$(\tilde{\gamma}^C_{\mathscr{M}})^C(s^*) = \frac{\tilde{\gamma}^C_{\mathscr{M}}(s^*)+\tilde{\gamma}^C_{\mathscr{M}}(t^*)-2(\tilde{\gamma}_{\mathscr{M}}(s^*)\ \tilde{\gamma}_{\mathscr{M}}(t^*))^C}{1-(\tilde{\gamma}_{\mathscr{M}}(s^*)\ \tilde{\gamma}_{\mathscr{M}}(t^*))^C} - \tilde{\gamma}^C_{\mathscr{N}}(s^*,t^*)$$

$$= \frac{\tilde{\gamma}_{\mathscr{M}}(s^*)+\tilde{\gamma}_{\mathscr{M}}(t^*)-2\ \tilde{\gamma}_{\mathscr{M}}(s^*)\ \tilde{\gamma}_{\mathscr{M}}(t^*)}{1-\tilde{\gamma}_{\mathscr{M}}(s^*)\ \tilde{\gamma}_{\mathscr{M}}(t^*)} - \left[\frac{\tilde{\gamma}_{\mathscr{M}}(s^*)+\tilde{\gamma}_{\mathscr{M}}(t^*)-2\ \tilde{\gamma}_{\mathscr{M}}(s^*)\ \tilde{\gamma}_{\mathscr{M}}(t^*)}{1-\tilde{\gamma}_{\mathscr{M}}(s^*)\ \tilde{\gamma}_{\mathscr{M}}(t^*)} - \tilde{\gamma}_{\mathscr{N}}(s^*,t^*)\right] = \tilde{\gamma}_{\mathscr{N}}(s^*,t^*)$$

$\forall s^*, t^* \in \mathscr{Z}$.

Hence, the $((FNDF_{graph})^C)^C$ is a $FNDF_{graph}$ itself. □

Definition 3.6. A homomorphism of $FNDF_{graph}$ $h : G_1 \to G_2$ with $G_1 = (\mathscr{M}_1, \mathscr{N}_1)$ and $G_2 = (\mathscr{M}_2, \mathscr{N}_2)$ is a map $h : \mathscr{Z}_1 \to \mathscr{Z}_2$ satisfying

$(i).\tilde{\lambda}_{\mathscr{M}_1}(s^*) \leq \tilde{\lambda}_{\mathscr{M}_2}(h(s^*))$,

$\tilde{\beta}_{\mathscr{M}_1}(s^*) \leq \tilde{\beta}_{\mathscr{M}_2}(h(s^*))$,

$\tilde{\gamma}_{\mathscr{M}_1}(s^*) \leq \tilde{\gamma}_{\mathscr{M}_2}(h(s^*))$.

$(ii).\tilde{\lambda}_{\mathscr{N}_1}(s^*,t^*) \leq \tilde{\lambda}_{\mathscr{N}_2}(h(s^*)h(t^*))$,

$\tilde{\beta}_{\mathscr{N}_1}(s^*,t^*) \leq \tilde{\beta}_{\mathscr{N}_2}(h(s^*)h(t^*))$,

$\tilde{\gamma}_{\mathscr{N}_1}(s^*,t^*) \leq \tilde{\gamma}_{\mathscr{N}_2}(h(s^*)h(t^*))\ \forall\ s^* \in \mathscr{Z}_1, (s^*,t^*) \in \mathscr{E}_1$.

Definition 3.7. An isomorphism of $FNDF_{graph}$ $h : G_1 \to G_2$ with $G_1 = (\mathscr{M}_1, \mathscr{N}_1)$ and $G_2 = (\mathscr{M}_2, \mathscr{N}_2)$ is a bijective mapping $h : \mathscr{Z}_1 \to \mathscr{Z}_2$ satisfying

$(i).\tilde{\lambda}_{\mathscr{M}_1}(s^*) = \tilde{\lambda}_{\mathscr{M}_2}(h(s^*))$,

$\tilde{\beta}_{\mathscr{M}_1}(s^*) = \tilde{\beta}_{\mathscr{M}_2}(h(s^*))$,

$\tilde{\gamma}_{\mathscr{M}_1}(s^*) = \tilde{\gamma}_{\mathscr{M}_2}(h(s^*))$.

$(ii).\tilde{\lambda}_{\mathscr{N}_1}(st) = \tilde{\lambda}_{\mathscr{N}_2}(h(s^*)h(t^*))$,

$\tilde{\beta}_{\mathscr{N}_1}(s^*,t^*) = \tilde{\beta}_{\mathscr{N}_2}(h(s^*)h(t^*))$,

$\tilde{\gamma}_{\mathscr{N}_1}(s^*,t^*) = \tilde{\gamma}_{\mathscr{N}_2}(h(s^*)h(t^*))\ \forall\ s^* \in \mathscr{Z}_1, s^*,t^* \in \mathscr{E}_1$.

Definition 3.8. A weak isomorphism $FNDF_{graph}$ $h : G_1 \to G_2$ with $G_1 = (\mathscr{M}_1, \mathscr{N}_1)$ and $G_2 = (\mathscr{M}_2, \mathscr{N}_2)$ is a bijective homomorphism $h : \mathscr{Z}_1 \to \mathscr{Z}_2$ satisfying
$(i).h$ is a homomorphism.
$(ii).\tilde{\lambda}_{\mathscr{M}_1}(s^*) = \tilde{\lambda}_{\mathscr{M}_2}(h(s^*))$,

$\tilde{\beta}_{\mathscr{M}_1}(s^*) = \tilde{\beta}_{\mathscr{M}_2}(h(s^*))$,

$\tilde{\gamma}_{\mathscr{M}_1}(s^*) = \tilde{\gamma}_{\mathscr{M}_2}(h(s^*))$, $s^* \in \mathscr{Z}_1$.

Definition 3.9. A co-weak isomorphism $FNDF_{graph}$ $h : G_1 \to G_2$ with $G_1 = (\mathscr{M}_1, \mathscr{N}_1)$ and $G_2 = (\mathscr{M}_2, \mathscr{N}_2)$ is a bijective homomorphism $h : \mathscr{Z}_1 \to \mathscr{Z}_2$ satisfying
$(i).h$ is a homomorphism.
$(ii).\tilde{\lambda}_{\mathscr{N}_1}(s^*,t^*) = \tilde{\lambda}_{\mathscr{N}_2}(h(s^*)h(t^*))$,

$\tilde{\beta}_{\mathscr{N}_1}(s^*,t^*) = \tilde{\beta}_{\mathscr{N}_2}(h(s^*)h(t^*))$,

$\tilde{\gamma}_{\mathscr{N}_1}(s^*,t^*) = \tilde{\gamma}_{\mathscr{N}_2}(h(s^*)h(t^*))\ \forall\ s^* \in \mathscr{Z}_1, (s^*,t^*) \in \mathscr{E}_1$.

Definition 3.10. A $FNDF_{graph}$ $G = (\mathscr{M}, \mathscr{N})$ is said to be self-complement if $G^C \cong G$.

Proposition 3.11. *If $G = (\mathscr{M}, \mathscr{N})$ is a self-complementary $FNDF_{graph}$, then*

$$\sum\nolimits_{s^* \neq t} \tilde{\lambda}_{\bar{\mathscr{N}}}(s^*,t^*) = \frac{1}{2}\sum\nolimits_{s^* \neq t^*} \frac{\tilde{\lambda}_{\bar{\mathscr{M}}}(s^*)\ \tilde{\lambda}_{\bar{\mathscr{M}}}(t^*)}{\tilde{\lambda}_{\bar{\mathscr{M}}}(s^*) + \tilde{\lambda}_{\bar{\mathscr{M}}}(t^*) - \tilde{\lambda}_{\bar{\mathscr{M}}}(s^*)\ \tilde{\lambda}_{\bar{\mathscr{M}}}(s^*)},$$

$$\sum\nolimits_{s \neq t^*} \tilde{\beta}_{\bar{\mathscr{N}}}(s^*,t^*) = \frac{1}{2}\sum\nolimits_{s^* \neq t^*} \frac{\tilde{\beta}_{\bar{\mathscr{M}}}(s^*)\ \tilde{\beta}_{\bar{\mathscr{M}}}(t^*)}{\tilde{\beta}_{\bar{\mathscr{M}}}(s^*) + \tilde{\beta}_{\bar{\mathscr{M}}}(t^*) - \tilde{\beta}_{\bar{\mathscr{M}}}(s^*)\ \tilde{\beta}_{\bar{\mathscr{M}}}(t^*)},$$

$$\sum\nolimits_{s^* \neq t} \tilde{\gamma}_{\bar{\mathscr{N}}}(s^*,t^*) = \frac{1}{2}\sum\nolimits_{s^* \neq t^*} \frac{\tilde{\gamma}_{\bar{\mathscr{M}}}(s^*) + \tilde{\gamma}_{\bar{\mathscr{M}}}(t^*) - 2\ \tilde{\gamma}_{\bar{\mathscr{M}}}(s^*)\ \tilde{\gamma}_{\bar{\mathscr{M}}}(t^*)}{1 - \tilde{\gamma}_{\bar{\mathscr{M}}}(s^*)\ \tilde{\gamma}_{\bar{\mathscr{M}}}(t^*)}.$$

Proof. Let G be a self-complementary $FNDF_{graph}$. Then there exists an isomorphism $h : \mathscr{Z}_1 \to \mathscr{Z}_2$ such that

$\tilde{\lambda}^C_{\mathscr{M}}(h(s^*)) = \tilde{\lambda}_{\mathscr{M}}(s^*)$, $\tilde{\beta}^C_{\mathscr{M}}(h(s^*)) = \tilde{\beta}_{\mathscr{M}}(s^*)$, $\tilde{\gamma}^C_{\mathscr{M}}(h(s^*)) = \tilde{\gamma}_{\mathscr{M}}(s^*)$. $\forall\ s^* \in \mathscr{Z}_1$

$\tilde{\lambda}^C_{\mathscr{M}}(h(s^*))(h(t^*)) = \tilde{\lambda}_{\mathscr{M}}(s^*)$, $\tilde{\beta}^C_{\mathscr{M}}(h(s^*))(h(t^*)) = \tilde{\beta}_{\mathscr{M}}(s^*)$, $\tilde{\gamma}^C_{\mathscr{M}}(h(s^*))(h(t^*)) = \tilde{\gamma}_{\mathscr{M}}(s^*,t^*)$. $\forall$ $(s^*,t^*) \in \mathscr{E}_1$.

By definition of complement of G, we have

$$\tilde{\lambda}^C_{\mathscr{M}}(h(s^*))(h(t^*)) = \frac{(\tilde{\lambda}_{\mathscr{M}}(h(s^*))\ \tilde{\lambda}_{\mathscr{M}}(h(t^*)))^C}{\tilde{\lambda}^C_{\mathscr{M}}(h(s^*)) + \tilde{\lambda}^C_{\mathscr{M}}(h(t^*)) - (\tilde{\lambda}_{\mathscr{M}}(h(s^*))\ \tilde{\lambda}_{\mathscr{M}}(h(t^*)))^C} - \tilde{\lambda}_{\mathscr{N}}(h(s^*)h(t^*))$$

$$\tilde{\lambda}_{\mathscr{M}}(s^*,t^*) = \frac{\tilde{\lambda}_{\mathscr{M}}(s^*)\ \tilde{\lambda}_{\mathscr{M}}(t^*)}{\tilde{\lambda}_{\mathscr{M}}(s^*) + \tilde{\lambda}_{\mathscr{M}}(t^*) - \tilde{\lambda}_{\mathscr{M}}(s^*)\ \tilde{\lambda}_{\mathscr{M}}(t^*)} - \tilde{\lambda}_{\mathscr{N}}(h(s^*)h(t^*))$$

$$\sum_{s^*\neq t^*}\tilde{\lambda}_{\mathscr{M}}(s^*,t^*)=\sum_{s^*\neq t^*}\frac{\tilde{\lambda}_{\mathscr{M}}(s^*)\ \tilde{\lambda}_{\mathscr{M}}(t^*)}{\tilde{\lambda}_{\mathscr{M}}(s^*)+\tilde{\lambda}_{\mathscr{M}}(t^*)-\tilde{\lambda}_{\mathscr{M}}(s^*)\ \tilde{\lambda}_{\mathscr{M}}(t^*)}-\sum_{s\neq t}\tilde{\lambda}_{\mathscr{N}}(h(s^*)h(t^*))$$

$$\sum_{s^*\neq t^*}\tilde{\lambda}_{\mathscr{M}}(s^*,t^*)+\sum_{s^*\neq t^*}\tilde{\lambda}_{\mathscr{N}}(h(s^*)h(t^*))=\sum_{s^*\neq t^*}\frac{\tilde{\lambda}_{\mathscr{M}}(s^*)\ \tilde{\lambda}_{\mathscr{M}}(t^*)}{\tilde{\lambda}_{\mathscr{M}}(s^*)+\tilde{\lambda}_{\mathscr{M}}(t^*)-\tilde{\lambda}_{\mathscr{M}}(s^*)\ \tilde{\lambda}_{\mathscr{M}}(t^*)}$$

$$2\sum_{s^*\neq t^*}\tilde{\lambda}_{\mathscr{M}}(s^*,t^*)=\sum_{s^*\neq t^*}\frac{\tilde{\lambda}_{\mathscr{M}}(s^*)\ \tilde{\lambda}_{\mathscr{M}}(t^*)}{\tilde{\lambda}_{\mathscr{M}}(s^*)+\tilde{\lambda}_{\mathscr{M}}(t^*)-\tilde{\lambda}_{\mathscr{M}}(s^*)\ \tilde{\lambda}_{\mathscr{M}}(t^*)}$$

$$\sum_{s^*\neq t^*}\tilde{\lambda}_{\mathscr{M}}(s^*,t^*)=\frac{1}{2}\sum_{s^*\neq t^*}\frac{\tilde{\lambda}_{\mathscr{M}}(s^*)\ \tilde{\lambda}_{\mathscr{M}}(t^*)}{\tilde{\lambda}_{\mathscr{M}}(s^*)+\tilde{\lambda}_{\mathscr{M}}(t^*)-\tilde{\lambda}_{\mathscr{M}}(s^*)\ \tilde{\lambda}_{\mathscr{M}}(t^*)}$$

Similarly,

$$\tilde{\beta}^C_{\mathscr{M}}(h(s^*))(h(t^*))=\frac{(\tilde{\beta}_{\mathscr{M}}(h(s^*))\ \tilde{\beta}_{\mathscr{M}}(h(t^*)))^C}{\tilde{\beta}^C_{\mathscr{M}}(h(s^*))+\tilde{\beta}^C_{\mathscr{M}}(h(t^*))-(\tilde{\beta}_{\mathscr{M}}(h(s^*))\ \tilde{\beta}_{\mathscr{M}}(h(t^*)))^C}-\tilde{\beta}_{\mathscr{N}}(h(s^*)h(t^*)),$$

$$\tilde{\beta}_{\mathscr{M}}(s^*,t^*)=\frac{\tilde{\beta}_{\mathscr{M}}(s^*)\ \tilde{\beta}_{\mathscr{M}}(t^*)}{\tilde{\beta}_{\mathscr{M}}(s^*)+\tilde{\beta}_{\mathscr{M}}(t^*)-\tilde{\beta}_{\mathscr{M}}(s^*)\ \tilde{\beta}_{\mathscr{M}}(t^*)}-\tilde{\beta}_{\mathscr{N}}(h(s^*)h(t^*))$$

$$\sum_{s^*\neq t^*}\tilde{\beta}_{\mathscr{M}}(s^*,t^*)=\sum_{s^*\neq t^*}\frac{\tilde{\beta}_{\mathscr{M}}(s^*)\ \tilde{\beta}_{\mathscr{M}}(t^*)}{\tilde{\beta}_{\mathscr{M}}(s^*)+\tilde{\beta}_{\mathscr{M}}(t^*)-\tilde{\beta}_{\mathscr{M}}(s^*)\ \tilde{\beta}_{\mathscr{M}}(t^*)}-\sum_{s\neq t}\tilde{\beta}_{\mathscr{N}}(h(s^*)h(t^*))$$

$$\sum_{s\neq t}\tilde{\beta}_{\mathscr{M}}(s^*,t^*)+\sum_{s^*\neq t^*}\tilde{\beta}_{\mathscr{N}}(h(s^*)h(t^*))=\sum_{s^*\neq t^*}\frac{\tilde{\beta}_{\mathscr{M}}(s^*)\ \tilde{\beta}_{\mathscr{M}}(t^*)}{\tilde{\beta}_{\mathscr{M}}(s^*)+\tilde{\beta}_{\mathscr{M}}(t^*)-\tilde{\beta}_{\mathscr{M}}(s^*)\ \tilde{\beta}_{\mathscr{M}}(t^*)}$$

$$2\sum_{s^*\neq t^*}\tilde{\beta}_{\mathscr{M}}(s^*,t^*)=\sum_{s^*\neq t^*}\frac{\tilde{\beta}_{\mathscr{M}}(s^*)\ \tilde{\beta}_{\mathscr{M}}(t^*)}{\tilde{\beta}_{\mathscr{M}}(s^*)+\tilde{\beta}_{\mathscr{M}}(t^*)-\tilde{\beta}_{\mathscr{M}}(s^*)\ \tilde{\beta}_{\mathscr{M}}(t^*)}$$

$$\sum_{s^*\neq t^*}\tilde{\beta}_{\mathscr{M}}(s^*,t^*)=\tfrac{1}{2}\sum_{s^*\neq t^*}\frac{\tilde{\beta}_{\mathscr{M}}(s^*)\ \tilde{\beta}_{\mathscr{M}}(t^*)}{\tilde{\beta}_{\mathscr{M}}(s^*)+\tilde{\beta}_{\mathscr{M}}(t^*)-\tilde{\beta}_{\mathscr{M}}(s^*)\ \tilde{\beta}_{\mathscr{M}}(t^*)}$$

$$\tilde{\gamma}^C_{\mathscr{M}}(h(s^*)h(t^*))=\frac{\tilde{\gamma}^C_{\mathscr{M}}(h(s^*))+\tilde{\gamma}^C_{\mathscr{M}}(h(t^*))-2\tilde{\gamma}^C_{\mathscr{M}}(h(s^*))\ \tilde{\gamma}^C_{\mathscr{M}}(h(t^*))}{1-\tilde{\gamma}^C_{\mathscr{M}}(h(s^*))\ \tilde{\gamma}^C_{\mathscr{M}}(h(t^*))}-\tilde{\gamma}^C_{\mathscr{M}}(h(s^*)h(t^*))$$

$$\tilde{\gamma}_{\mathscr{N}}(s^*,t^*)=\frac{\tilde{\gamma}_{\mathscr{M}}(s^*)+\tilde{\gamma}_{\mathscr{M}}(t^*)-2\tilde{\gamma}_{\mathscr{M}}(s^*)\ \tilde{\gamma}_{\mathscr{M}}(t^*)}{1-\tilde{\gamma}_{\mathscr{M}}(s^*)\ \tilde{\gamma}_{\mathscr{M}}(t^*)}-\tilde{\gamma}_{\mathscr{N}}(h(s^*)h(t^*))$$

$$\sum_{s\neq t}\tilde{\gamma}_{\mathscr{N}}(s^*,t^*)=\sum_{s^*\neq t^*}\frac{\tilde{\gamma}_{\mathscr{M}}(s^*)+\tilde{\gamma}_{\mathscr{M}}(t^*)-2\tilde{\gamma}_{\mathscr{M}}(s^*)\ \tilde{\gamma}_{\mathscr{M}}(t^*)}{1-\tilde{\gamma}_{\mathscr{M}}(s^*)\ \tilde{\gamma}_{\mathscr{M}}(t^*)}-\sum_{s\neq t}\tilde{\gamma}_{\mathscr{N}}(h(s^*)h(t^*))$$

$$\sum_{s\neq t^*}\tilde{\gamma}_{\mathscr{N}}(s^*,t^*)+\sum_{s\neq t}\tilde{\gamma}_{\mathscr{N}}(h(s^*)h(t^*))=\sum_{s^*\neq t^*}\frac{\tilde{\gamma}_{\mathscr{M}}(s^*)+\tilde{\gamma}_{\mathscr{M}}(t^*)-2\tilde{\gamma}_{\mathscr{M}}(s^*)\ \tilde{\gamma}_{\mathscr{M}}(t^*)}{1-\tilde{\gamma}_{\mathscr{M}}(s^*)\ \tilde{\gamma}_{\mathscr{M}}(t^*)}$$

$$2\sum_{s^*\neq t^*}\tilde{\gamma}_{\mathscr{N}}(s^*,t^*)=\sum_{s\neq t}\frac{\tilde{\gamma}_{\mathscr{M}}(s^*)+\tilde{\gamma}_{\mathscr{M}}(t^*)-2\tilde{\gamma}_{\mathscr{M}}(s^*)\ \tilde{\gamma}_{\mathscr{M}}(t^*)}{1-\tilde{\gamma}_{\mathscr{M}}(s^*)\ \tilde{\gamma}_{\mathscr{M}}(t^*)}$$

$$\sum_{s^*\neq t^*}\tilde{\gamma}_{\mathscr{N}}(s^*,t^*)=\frac{1}{2}\sum_{s\neq t}\frac{\tilde{\gamma}_{\mathscr{M}}(s^*)+\tilde{\gamma}_{\mathscr{M}}(t^*)-2\tilde{\gamma}_{\mathscr{M}}(s^*)\ \tilde{\gamma}_{\mathscr{M}}(t^*)}{1-\tilde{\gamma}_{\mathscr{M}}(s^*)\ \tilde{\gamma}_{\mathscr{M}}(t^*)}.\ \square$$

Proposition 3.12. *Let $G=(\mathscr{M},\mathscr{N})$ be a $FNDF_{graph}$. If*

$$\tilde{\lambda}_{\mathscr{N}}(s^*,t^*)=\frac{1}{2}\left(\frac{\tilde{\lambda}_{\mathscr{M}}(s^*)\ \tilde{\lambda}_{\mathscr{M}}(t^*)}{\tilde{\lambda}_{\mathscr{M}}(s^*)+\tilde{\lambda}_{\mathscr{M}}(t^*)-\tilde{\lambda}_{\mathscr{M}}(s^*)\ \tilde{\lambda}_{\mathscr{M}}(t^*)}\right),$$

$$\tilde{\beta}_{\mathscr{N}}(s^*,t^*)=\frac{1}{2}\left(\frac{\tilde{\beta}_{\mathscr{M}}(s^*)\ \tilde{\beta}_{\mathscr{M}}(t^*)}{\tilde{\beta}_{\mathscr{M}}(s^*)+\tilde{\beta}_{\mathscr{M}}(t^*)-\tilde{\beta}_{\mathscr{M}}(s^*)\ \tilde{\beta}_{\mathscr{M}}(t^*)}\right),$$

$\tilde{\gamma}_{\mathscr{N}}(s^*,t^*) = \frac{1}{2}\left(\frac{\tilde{\gamma}_{\mathscr{M}}(s^*)+\tilde{\gamma}_{\mathscr{M}}(t^*)-2\tilde{\gamma}_{\mathscr{M}}(s^*)\ \tilde{\gamma}_{\mathscr{M}}(t^*)}{1-\tilde{\gamma}_{\mathscr{M}}(s^*)\ \tilde{\gamma}_{\mathscr{M}}(t^*)}\right)\ \forall\ s^*,t^*\in\mathscr{Z}$ *then* G *is self-complementary.*

Proof. Let G be a $FNDF_{graph}$ that satisfies

$$\tilde{\lambda}_{\mathscr{N}}(s^*,t^*) = \frac{1}{2}\left(\frac{\tilde{\lambda}_{\mathscr{M}}(s^*)\ \tilde{\lambda}_{\mathscr{M}}(t^*)}{\tilde{\lambda}_{\mathscr{M}}(s^*)+\tilde{\lambda}_{\mathscr{M}}(t^*)-\tilde{\lambda}_{\mathscr{M}}(s^*)\ \tilde{\lambda}_{\mathscr{M}}(t^*)}\right),$$
$$\tilde{\beta}_{\mathscr{N}}(s^*,t^*) = \frac{1}{2}\left(\frac{\tilde{\beta}_{\mathscr{M}}(s^*)\ \tilde{\beta}_{\mathscr{M}}(t^*)}{\tilde{\beta}_{\mathscr{M}}(s^*)+\tilde{\beta}_{\mathscr{M}}(t^*)-\tilde{\beta}_{\mathscr{M}}(s^*)\ \tilde{\beta}_{\mathscr{M}}(t^*)}\right),$$
$$\tilde{\gamma}_{\mathscr{N}}(s^*,t^*) = \frac{1}{2}\left(\frac{\tilde{\gamma}_{\mathscr{M}}(s^*)+\tilde{\gamma}_{\mathscr{M}}(t^*)-2\tilde{\gamma}_{\mathscr{M}}(s^*)\ \tilde{\gamma}_{\mathscr{M}}(t^*)}{1-\tilde{\gamma}_{\mathscr{M}}(s^*)\ \tilde{\gamma}_{\mathscr{M}}(t^*)}\right)\ \forall\ s^*,t^*\in\mathscr{Z}.$$

Then clearly, $I:\mathscr{Z}\to\mathscr{Z}$ is the identify mapping represents an isomorphism from G to G^C satisfying the condition:
$\tilde{\lambda}^C_{\mathscr{M}}(s^*)=\tilde{\lambda}^C_{\mathscr{M}}(I(s^*))$, $\tilde{\beta}^C_{\mathscr{M}}(s^*)=\tilde{\beta}^C_{\mathscr{M}}(I(s^*))$, and $\tilde{\gamma}^C_{\mathscr{M}}()=\tilde{\gamma}^C_{\mathscr{M}}(I(s^*))\ \forall\ s\in\mathscr{Z}$.

The membership grade of an edge (s^*,t^*) is given by

$$\tilde{\lambda}_{\mathscr{N}}(s^*,t^*) = \frac{1}{2}\left(\frac{\tilde{\lambda}_{\mathscr{M}}(s^*)\ \tilde{\lambda}_{\mathscr{M}}(t^*)}{\tilde{\lambda}_{\mathscr{M}}(s^*)+\tilde{\lambda}_{\mathscr{M}}(t^*)-\tilde{\lambda}_{\mathscr{M}}(s^*)\ \tilde{\lambda}_{\mathscr{M}}(t^*)}\right)\ \forall\ s^*,t^*\in\mathscr{Z}.$$

we have $\tilde{\lambda}^C_{\mathscr{N}}(I(s^*)I(t^*)) = \tilde{\lambda}^C_{\mathscr{N}}(s^*,t^*)$

$$= \frac{(\tilde{\lambda}_{\mathscr{M}}(s^*)\ \tilde{\lambda}_{\mathscr{M}}(t^*))^C}{\tilde{\lambda}^C_{\mathscr{M}}(s^*)+\tilde{\lambda}^C_{\mathscr{M}}(t^*)-(\tilde{\lambda}_{\mathscr{M}}(s^*)\ \tilde{\lambda}_{\mathscr{M}}(t^*))^C} - \tilde{\lambda}_{\mathscr{N}}(s^*,t^*)$$
$$= \frac{\tilde{\lambda}_{\mathscr{M}}(s^*)\ \tilde{\lambda}_{\mathscr{M}}(t^*)}{\tilde{\lambda}_{\mathscr{M}}(s^*)+\tilde{\lambda}_{\mathscr{M}}(t^*)-\tilde{\lambda}_{\mathscr{M}}(s^*)\ \tilde{\lambda}_{\mathscr{M}}(t^*)} - \frac{1}{2}\left(\frac{\tilde{\lambda}_{\mathscr{M}}(s^*)\ \tilde{\lambda}_{\mathscr{M}}(t^*)}{\tilde{\lambda}_{\mathscr{M}}(s^*)+\tilde{\lambda}_{\mathscr{M}}(t^*)-\tilde{\lambda}_{\mathscr{M}}(s^*)\ \tilde{\lambda}_{\mathscr{M}}(t^*)}\right)$$
$$= \frac{1}{2}\left(\frac{\tilde{\lambda}_{\mathscr{M}}(s^*)\ \tilde{\lambda}_{\mathscr{M}}(t^*)}{\tilde{\lambda}_{\mathscr{M}}(s^*)+\tilde{\lambda}_{\mathscr{M}}(t^*)-\tilde{\lambda}_{\mathscr{M}}(s^*)\ \tilde{\lambda}_{\mathscr{M}}(t^*)}\right) = \tilde{\lambda}_{\mathscr{N}}(s^*,t^*).$$

In similar way, the indeterminacy grade of an edge (s^*,t^*) is

$$\tilde{\beta}_{\mathscr{N}}(s^*,t^*) = \frac{1}{2}\left(\frac{\tilde{\beta}_{\mathscr{M}}(s^*)\ \tilde{\beta}_{\mathscr{M}}(t^*)}{\tilde{\beta}_{\mathscr{M}}(s^*)+\tilde{\beta}_{\mathscr{M}}(t^*)-\tilde{\beta}_{\mathscr{M}}(s^*)\ \tilde{\beta}_{\mathscr{M}}(t^*)}\right)\ \forall\ s^*,t^*\in\mathscr{Z}.$$

we have $\tilde{\beta}^C_{\mathscr{N}}(I(s^*)I(t^*)) = \tilde{\beta}^C_{\mathscr{N}}(s^*,t^*)$

$$= \frac{(\tilde{\beta}_{\mathscr{M}}(s^*)\ \tilde{\beta}_{\mathscr{M}}(t^*))^C}{\tilde{\beta}^C_{\mathscr{M}}(s^*)+\tilde{\beta}^C_{\mathscr{M}}(t^*)-(\tilde{\beta}_{\mathscr{M}}(s^*)\ \tilde{\lambda}_{\mathscr{M}}(t^*))^C} - \tilde{\lambda}_{\mathscr{N}}(s^*,t^*)$$
$$= \frac{\tilde{\beta}_{\mathscr{M}}(s^*)\ \tilde{\beta}_{\mathscr{M}}(t^*)}{\tilde{\beta}_{\mathscr{M}}(s^*)+\tilde{\beta}_{\mathscr{M}}(t^*)-\tilde{\beta}_{\mathscr{M}}(s^*)\ \tilde{\beta}_{\mathscr{M}}(t^*)} - \frac{1}{2}\left(\frac{\tilde{\beta}_{\mathscr{M}}(s^*)\ \tilde{\beta}_{\mathscr{M}}(t^*)}{\tilde{\beta}_{\mathscr{M}}(s^*)+\tilde{\beta}_{\mathscr{M}}(t^*)-\tilde{\beta}_{\mathscr{M}}(s^*)\ \tilde{\beta}_{\mathscr{M}}(t^*)}\right)$$
$$= \frac{1}{2}\left(\frac{\tilde{\beta}_{\mathscr{M}}(s^*)\ \tilde{\beta}_{\mathscr{M}}(t^*)}{\tilde{\beta}_{\mathscr{M}}(s^*)+\tilde{\beta}_{\mathscr{M}}(t^*)-\tilde{\beta}_{\mathscr{M}}(s^*)\ \tilde{\beta}_{\mathscr{M}}(t^*)}\right) = \tilde{\beta}_{\mathscr{N}}(s^*,t^*).$$

$$\tilde{\gamma}_{\mathscr{N}}(s^*,t^*) = \frac{1}{2}\left(\frac{\tilde{\gamma}_{\mathscr{M}}(s^*)+\tilde{\gamma}_{\mathscr{M}}(t^*)-2\tilde{\gamma}_{\mathscr{M}}(s^*)\ \tilde{\gamma}_{\mathscr{M}}(t^*)}{1-\tilde{\gamma}_{\mathscr{M}}(s^*)\ \tilde{\gamma}_{\mathscr{M}}(t^*)}\right)\ \forall\ s^*,t^*\in\mathscr{Z}.$$

So, we have $\tilde{\gamma}^C_{\mathscr{M}}(I(s^*)I(t^*)) = \tilde{\gamma}^C_{\mathscr{N}}(s^*,t^*) = \frac{\tilde{\gamma}^C_{\mathscr{M}}(s^*)+\tilde{\gamma}^C_{\mathscr{M}}(t^*)-2\tilde{\gamma}^C_{\mathscr{M}}(s^*)\ \tilde{\gamma}^C_{\mathscr{M}}(t^*)}{1-\tilde{\gamma}^C_{\mathscr{M}}(s^*)\ \tilde{\gamma}^C_{\mathscr{M}}(t^*)} - \tilde{\gamma}^C_{\mathscr{M}}(s^*,t^*)$

$$= \frac{\tilde{\gamma}_{\mathscr{M}}(s^*) + \tilde{\gamma}_{\mathscr{M}}(t^*) - 2\ \tilde{\gamma}_{\mathscr{M}}(s^*)\ \tilde{\gamma}_{\mathscr{M}}(t^*)}{1 - \tilde{\gamma}_{\mathscr{M}}(s^*)\ \tilde{\gamma}_{\mathscr{M}}(t^*)} - \frac{1}{2}\left(\frac{\tilde{\gamma}_{\mathscr{M}}(s^*) + \tilde{\gamma}_{\mathscr{M}}(t^*) - 2\ \tilde{\gamma}_{\mathscr{M}}(s^*)\ \tilde{\gamma}_{\mathscr{M}}(t^*)}{1 - \tilde{\gamma}_{\mathscr{M}}(s^*)\ \tilde{\gamma}_{\mathscr{M}}(t^*)}\right)$$
$$= \frac{1}{2}\left(\frac{\tilde{\gamma}_{\mathscr{M}}(s^*) + \tilde{\gamma}_{\mathscr{M}}(t^*) - 2\ \tilde{\gamma}_{\mathscr{M}}(s^*)\ \tilde{\gamma}_{\mathscr{M}}(s^*)}{1 - \tilde{\gamma}_{\mathscr{M}}(s^*)\ \tilde{\gamma}_{\mathscr{M}}(t^*)}\right) = \tilde{\gamma}_{\mathscr{N}}(s^*, t^*)$$

Since the conditions of isomorphism $\tilde{\lambda}^C_{\mathscr{N}}(I(s^*)I(s^*)) = \tilde{\lambda}_{\mathscr{N}}(s^*, t^*), \tilde{\beta}^C_{\mathscr{N}}(I(s^*)I(t^*)) = \tilde{\beta}_{\mathscr{N}}(s^*, t^*)$ and $\tilde{\gamma}^C_{\mathscr{N}}(I(s^*)I(t^*)) = \tilde{\gamma}_{\mathscr{N}}(s^*, t^*)$ are satisfied by I, $G = (\mathscr{M}, \mathscr{N})$ is self-complementary. □

Proposition 3.13. *If $G_1 = (\mathscr{M}_1, \mathscr{N}_1)$ and $G_2 = (\mathscr{M}_2, \mathscr{N}_2)$ are two isomorphic $FNDF_{graphs}$, then their complements are also isomorphic.*

Proof. Suppose G_1 and G_2 are two isomorphic $FNDF_{graphs}$. Then by definition of isomorphism, there exists a bijective mapping $h : \mathscr{Z}_1 \to \mathscr{Z}_2$ that satisfies
$\tilde{\lambda}_{\mathscr{M}_1}(s^*) = \tilde{\lambda}_{\mathscr{M}_2}(h(s^*))$, $\tilde{\beta}_{\mathscr{M}_1}(s^*) = \tilde{\beta}_{\mathscr{M}_2}(h(s^*))$ and $\tilde{\gamma}_{\mathscr{M}_1}(s^*) = \tilde{\gamma}_{\mathscr{M}_2}(h(s^*))\ \forall\ s \in \mathscr{Z}_1$,
$\tilde{\lambda}_{\mathscr{N}_1}(s^*, t^*) = \tilde{\lambda}_{\mathscr{N}_2}(h(s^*)h(t^*))$, $\tilde{\beta}_{\mathscr{N}_1}(s^*, t^*) = \tilde{\beta}_{\mathscr{N}_2}(h(s^*)h(t^*))$ and $\tilde{\gamma}_{\mathscr{N}_1}(s^*, t^*) = \tilde{\gamma}_{\mathscr{N}_2}(h(s^*)h(t^*))$. $\forall\ s^*, t^* \in \mathscr{E}_1$.
From definition of $FNDF'_{graph}s$ complement, (s^*, t^*) is

$$\tilde{\lambda}^C_{\mathscr{N}_1}(s^*, t^*) = \frac{\tilde{\lambda}_{\mathscr{M}_1}(s^*)\ \tilde{\lambda}_{\mathscr{M}_1}(t^*)}{\tilde{\lambda}_{\mathscr{M}_1}(s^*) + \tilde{\lambda}_{\mathscr{M}_1}(t^*) - \tilde{\lambda}_{\mathscr{M}_1}(s^*)\ \tilde{\lambda}_{\mathscr{M}_1}(t^*)} - \tilde{\lambda}_{\mathscr{N}_1}(s^*, t^*)$$
$$= \frac{\tilde{\lambda}_{\mathscr{M}_2}(h(s^*))\ \tilde{\lambda}_{\mathscr{M}_2}(h(t^*))}{\tilde{\lambda}_{\mathscr{M}_2}(h(s^*)) + \tilde{\lambda}_{\mathscr{M}_2}(h(t^*)) - \tilde{\lambda}_{\mathscr{M}_2}(h(s^*))\ \tilde{\lambda}_{\mathscr{M}_2}(h(t^*))} - \tilde{\lambda}_{\mathscr{N}_2}(h(s^*)h(t^*))$$
$$= \tilde{\lambda}^C_{\mathscr{N}_1}(h(s^*)h(t^*)).$$

$$\text{Similarly, } \tilde{\beta}^C_{\mathscr{N}_1}(s^*, t^*) = \frac{\tilde{\beta}_{\mathscr{M}_1}(s^*)\ \tilde{\beta}_{\mathscr{M}_1}(t^*)}{\tilde{\beta}_{\mathscr{M}_1}(s^*) + \tilde{\beta}_{\mathscr{M}_1}(t^*) - \tilde{\beta}_{\mathscr{M}_1}(s^*)\ \tilde{\beta}_{\mathscr{M}_1}(t^*)} - \tilde{\beta}_{\mathscr{N}_1}(s^*, t^*)$$
$$= \frac{\tilde{\beta}_{\mathscr{M}_2}(h(s^*))\ \tilde{\beta}_{\mathscr{M}_2}(h(t^*))}{\tilde{\beta}_{\mathscr{M}_2}(h(s^*)) + \tilde{\beta}_{\mathscr{M}_2}(h(t^*)) - \tilde{\beta}_{\mathscr{M}_2}(h(s^*))\ \tilde{\beta}_{\mathscr{M}_2}(h(t^*))} - \tilde{\beta}_{\mathscr{N}_2}(h(s^*)h(t^*))$$
$$= \tilde{\beta}^C_{\mathscr{N}_1}(h(s^*)h(t^*)).$$

Also, is,

$$\tilde{\gamma}^C_{\mathscr{N}_1}(s^*, t^*) = \frac{\tilde{\gamma}_{\mathscr{M}_1}(s^*) + \tilde{\gamma}_{\mathscr{M}_1}(t^*) - 2\tilde{\gamma}_{\mathscr{M}_1}(s^*)\ \tilde{\gamma}_{\mathscr{M}_1}(t^*)}{1 - \tilde{\gamma}_{\mathscr{M}_1}(s^*)\ \tilde{\gamma}_{\mathscr{M}_1}(t^*)} - \tilde{\gamma}_{\mathscr{N}_1}(s^*, t^*)$$
$$= \frac{\tilde{\gamma}_{\mathscr{M}_2}(h(s^*)) + \tilde{\gamma}_{\mathscr{M}_2}(h(t^*)) - 2\ \tilde{\gamma}_{\mathscr{M}_2}(h(s^*))\ \tilde{\gamma}_{\mathscr{M}_2}(h(t^*))}{1 - \tilde{\gamma}_{\mathscr{M}_2}(h(s^*))\ \tilde{\gamma}_{\mathscr{M}_2}(h(t^*))} - \tilde{\gamma}_{\mathscr{N}_2}(h(s^*)h(t^*))$$
$$= \tilde{\gamma}^C_{\mathscr{N}_1}(h(s^*)h(t^*)).$$

Hence, the complements are isomporhic to each other and the converse also true.□

Definition 3.14. A $FNDF_{graph}$ is complete if

$$\tilde{\lambda}_{\mathscr{N}}(s^*, t^*) = \frac{\tilde{\lambda}_{\mathscr{M}}(s^*)\ \tilde{\lambda}_{\mathscr{M}}(t^*)}{\tilde{\lambda}_{\mathscr{M}}(s^*) + \tilde{\lambda}_{\mathscr{M}}(t^*) - \tilde{\lambda}_{\mathscr{M}}(s^*)\ \tilde{\lambda}_{\mathscr{M}}(t^*)},$$

$$\tilde{\beta}_{\mathscr{N}}(s^*,t^*)=\frac{\tilde{\beta}_{\mathscr{M}}(s^*)\ \tilde{\beta}_{\mathscr{M}}(t^*)}{\tilde{\beta}_{\mathscr{M}}(s^*)+\tilde{\beta}_{\mathscr{M}}(t^*)-\tilde{\beta}_{\mathscr{M}}(s^*)\ \tilde{\beta}_{\mathscr{M}}(t^*)},$$
$$\tilde{\gamma}_{\mathscr{N}}(s^*,t^*)=\frac{\tilde{\gamma}_{\mathscr{M}}(s^*)+\tilde{\gamma}_{\mathscr{M}}(t^*)-2\tilde{\gamma}_{\mathscr{M}}(s^*)\ \tilde{\gamma}_{\mathscr{M}}(t^*)}{1-\tilde{\gamma}_{\mathscr{M}}(s^*)\ \tilde{\gamma}_{\mathscr{M}}(t^*)}\ \forall\ s^*,t^*\in\mathscr{Z}.$$

The above mentioned properties are satisfied for the $FNDF_{graph}$ in Example 1, thus the $FNDF_{graph}$ is a complete $FNDF_{graph}$.

Definition 3.15. A $FNDF_{graph}$ is said to be strong if

$$\tilde{\lambda}_{\mathscr{N}}(s^*,t^*)=\frac{\tilde{\lambda}_{\mathscr{M}}(s^*)\ \tilde{\lambda}_{\mathscr{M}}(t^*)}{\tilde{\lambda}_{\mathscr{M}}(s^*)+\tilde{\lambda}_{\mathscr{M}}(t^*)-\tilde{\lambda}_{\mathscr{M}}(s^*)\ \tilde{\lambda}_{\mathscr{M}}(t^*)},$$
$$\tilde{\beta}_{\mathscr{N}}(s^*,t^*)=\frac{\tilde{\beta}_{\mathscr{M}}(s^*)\ \tilde{\beta}_{\mathscr{M}}(t^*)}{\tilde{\beta}_{\mathscr{M}}(s^*)+\tilde{\beta}_{\mathscr{M}}(t^*)-\tilde{\beta}_{\mathscr{M}}(s^*)\ \tilde{\beta}_{\mathscr{M}}(t^*)},$$
$$\tilde{\gamma}_{\mathscr{N}}(s^*,t^*)=\frac{\tilde{\gamma}_{\mathscr{M}}(s^*)+\tilde{\gamma}_{\mathscr{M}}(t^*)-2\tilde{\gamma}_{\mathscr{M}}(s^*)\ \tilde{\gamma}_{\mathscr{M}}(t^*)}{1-\tilde{\gamma}_{\mathscr{M}}(s^*)\ \tilde{\gamma}_{\mathscr{M}}(t^*)}\ \forall\ s^*,t^*\in\mathscr{E}.$$

4. Numerical Approach

The concept of $FNDF_{graphs}$ as a novel approach to decision-making which is suitable for handling uncertainties and imprecise information, from the real-world scenarios. The selection of the most suitable CIBIL score (Credit Information Bureau (India) Limited) applications is crucial for individuals and businesses seeking financial products. In this section, we discussed how the proposed methodology was applied to the selection of CIBIL score applications.

4.1 Algorithm for CIBIL Score Application Selection:

Below is a algorithm for MCDM aimed at selecting the most suitable CIBIL score application.

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S1: Input attributes $A=\{a_1,a_2,\cdots,a_k\}$ and factors $F=\{f_1,f_2,\cdots,f_n\}$ with weight vector $W=\{w_1,w_2,\cdots,w_n\}$ and construct $FF_{relation}$ $L^{(g)}=(l_{pq}^{(g)})_{k\times k}$ to each criterion.

S2: Aggregate all $l_{pq}^{(g)}=(\tilde{\alpha}_{pq}^{(g)},\tilde{\beta}_{pq}^{(g)},\tilde{\gamma}_{pq}^{(g)})$ $(p,q=1,2,\cdots,k)$ regarding criteria $F_p(p=1,2,3,4)$ and get $L=(l_{pq})_{k\times k}$, where $l_{pq}=(\tilde{\lambda}_{pq},\tilde{\beta}_{pq},\tilde{\gamma}_{pq})$ is the value assigned for the alternative a_p over a_q with respect to all the considered criteria F_l by using Fermatean Neutrosophic Dombi fuzzy weighted arithmetic averaging $(FNDFWAA)$ operator given by

$$l_{pq}=FNDFWAA(l_{pq}^{(1)},l_{pq}^{(2)},\cdots,l_{pq}^{(n)},)=$$
$$\sqrt{1-\frac{1}{1+\left[\sum_{j=1}^{n}w_j\left(\frac{(\tilde{\beta}_{pq}^{j})^3}{1-(\tilde{\beta}_{pq}^{g})^3}\right)^{\rho}\right]^{\frac{1}{\rho}}}},\sqrt{1-\frac{1}{1+\left[\sum_{j=1}^{n}w_j\left(\frac{(\tilde{\alpha}_{pq}^{j})^3}{1-(\tilde{\beta}_{pq}^{g})^3}\right)^{\rho}\right]^{\frac{1}{\rho}}}},\frac{1}{1+\left[\sum_{j=1}^{n}w_j\left(\frac{1-(\tilde{\gamma}_{pq}^{j})}{(\tilde{\gamma}_{pq}^{g})}\right)^{\rho}\right]^{\frac{1}{\rho}}}$$

S3: Draw $FNFD_{graphs}$ based on L.

S4: Draw the $FNFPD_{graphs}$ based on the condition $\tilde{\lambda}_{pq}\geq 0.5$ $(l,p=1,2,\cdots,k)$.

S5: Compute $out-d(A_i)$ $(i=1,2,\cdots,k)$ for each A_i in the $FNFPD_{graphs}$.

S6: Arrangement of the alternatives based on the diminishing value of the membership degrees of $out-d(A_i)$.

S7: The optimal alternative is the alternative with the maximum membership degree of $out - d(A_i)$.

4.2 Choosing the most suitable CIBIL score application:

Any individual's creditworthiness can be represented numerically through their CIBIL scores. Its importance lies in its impact on various financial aspects, including loan approval, interest rates, credit card offers, negotiating power, rental approvals, employment opportunities, insurance premiums, and access to financial products. A good CIBIL score reflects financial discipline and enhances an individual's ability to secure favorable terms in financial transactions. It serves as a tool for lenders to assess credit risk and is a crucial element in financial planning and responsible financial behavior. Regular monitoring and maintenance of a healthy CIBIL score are essential for achieving financial stability and flexibility.

Selecting the most suitable CIBIL score application involves considering several factors to ensure it meets a person needs and provides accurate and valuable credit information. CRISIL (Credit Rating Information Services of India Limited) (A_1), ICRA (Investment Information and Credit Rating Agency) Limited(A_2), CARE (Credit Analysis and Research Limited) (A_3), India Ratings and Research Pvt. Ltd.(A_4), and Brickwork Ratings India Pvt Ltd. (A_5), are indeed the five major credit rating agencies or credit bureaus operating in India. These agencies play a crucial role in providing credit reports and credit scores, which are used by lenders to assess the creditworthiness of individuals and businesses.

Several factors contribute to the calculation of the CIBIL score, and understanding these factors is crucial for maintaining a healthy credit profile. The five major factors that influence the CIBIL score are Payment HistoryF_1, Credit Utilization Ratio F_2, Length of Credit History F_3, Types of Credit F_4, and New Credit and Inquiries F_5.

$W = (0.3, 0.1, 0.1, 0.3, 0.2)$presents preferable information $L^{(g)} = l_{pq}^{(g)})_{5\times5}$ $(g = 1, 2, 3, 4, 5)$. Here $l_{pq}^{(g)} = (\tilde{\alpha}_{pq}^{(g)}, \tilde{\beta}_{pq}^{(g)}, \tilde{\gamma}_{pq}^{(g)})$ is the Fermatean Neutrosophic number assigned by decision-making expert. Also, The degree to each CIBIL score application A_l are $\tilde{\alpha}_{pq}^{(g)}, \tilde{\beta}_{pq}^{(g)}$ and $\tilde{\gamma}_{pq}^{(g)}$ by either preferred or not preferred over the application A_p regarding the given criteria. $L^{(g)} = (l_{pq}^{(g)})_{5\times5}$ are given tables (I - V).

Table 1. Comparision for Factor I

$L^{(1)}$	A_1	A_2	A_3	A_4	A_5
A_1	(.5,.4,.5)	(.7,.4,.2)	(.7,.5,.1)	(.4,.2,.5)	(.4,.2,.4)
A_2	(.2,.4,.7)	(.5,.4,.5)	(.6,.5,.4)	(.7,.2,.6)	(.5,.2,.5)
A_3	(.1,.5,.7)	(.4,.5,.6)	(.5,.4,.5)	(.7,.4,.3)	(.4,.1,.5)
$A4$	(.5,.2,.4)	(.6,.2,.7)	(.3,.4,.7)	(.5,.4,.5)	(.4,.2,.5)
$A5$	(.4,.2,.4)	(.5,.2,.5)	(.5,.1,.4)	(.5,.2,.4)	(.5,.4,.5)

Table 2. Comparision for Factor II

$L^{(2)}$	A_1	A_2	A_3	A_4	A_5
A_1	(.5,.4,.5)	(.8,.8,.5)	(.7,.3,.4)	(.8,.4,.7)	(.8,.5,.6)
A_2	(.5,.8,.8)	(.5,.4,.5)	(.7,.4,.5)	(.3,.4,.6)	(.7,.6,.5)
A_3	(.4,.3,.7)	(.5,.4,.7)	(.5,.4,.5)	(.7,.7,.6)	(.8,.7,.8)
A_4	(.7,.4,.8)	(.6,.4,.3)	(.6,.7,.7)	(.5,.4,.5)	(.8,.8,.6)
A_5	(.6,.5,.8)	(.5,.6,.7)	(.8,.7,.8)	(.6,.8,.8)	(.5,.4,.5)

Table 3. Comparision for Factor III

$L^{(3)}$	A_1	A_2	A_3	A_4	A_5
A_1	(.5,.4,.5)	(.8,.6,.5)	(.7,.4,.6)	(.9,.7,.4)	(.7,.5,.5)
A_2	(.5,.6,.8)	(.5,.4,.5)	(.8,.6,.6)	(.9,.8,.6)	(.6,.8,.5)
A_3	(.6,.4,.7)	(.6,.6,.8)	(.5,.4,.5)	(.8,.4,.7)	(.7,.8,.9)
A_4	(.4,.7,.9)	(.6,.8,.9)	(.7,.4,.8)	(.5,.4,.5)	(.8,.6,.7)
A_5	(.5,.5,.7)	(.5,.8,.6)	(.9,.8,.7)	(.7,.6,.8)	(.5,.4,.5)

Table 4. Comparision for Factor IV

$L^{(4)}$	A_1	A_2	A_3	A_4	A_5
A_1	(.5,.4,.5)	(.9,.7,.5)	(.6,.3,.4)	(.7,.5,.6)	(.8,.3,.6)
A_2	(.5,.7,.9)	(.5,.4,.5)	(.9,.6,.5)	(.6,.4,.7)	(.7,.6,.6)
A_3	(.4,.3,.6)	(.5,.6,.9)	(.5,.4,.5)	(.9,.6,.4)	(.6,.7,.8)
A_4	(.6,.5,.7)	(.7,.4,.6)	(.4,.6,.9)	(.5,.4,.5)	(.9,.7,.6)
A_5	(.6,.3,.8)	(.6,.6,.7)	(.8,.7,.6)	(.6,.7,.9)	(.5,.4,.5)

Table 5. Comparision for Factor V

$L^{(5)}$	A_1	A_2	A_3	A_4	A_5
A_1	(.5,.4,.5)	(.8,.5,.4)	(.6,.6,.8)	(.6,.7,.8)	(.9,.4,.6)
A_2	(.4,.5,.8)	(.5,.4,.5)	(.8,.7,.7)	(.7,.5,.8)	(.8,.6,.5)
A_3	(.8,.6,.6)	(.7,.7,.8)	(.5,.4,.5)	(.8,.5,.4)	(.8,.6,.7)
A_4	(.8,.7,.6)	(.8,.5,.7)	(.4,.5,.8)	(.5,.4,.5)	(.8,.8,.7)
A_5	(.6,.4,.9)	(.5,.6,.8)	(.7,.6,.8)	(.7,.8,.8)	(.5,.4,.5)

With the purpose to complete the grouped $l_{pq} = (\tilde{\lambda}_{pq}, \tilde{\beta}_{pq}, \tilde{\gamma}_{pq})$ $(p, q = 1, 2, 3, 4, 5)$ of the cibil score application A_l over A_p regarding all considered factors $l^{(g)}(g = 1, 2, 3, 4, 5)$, the FND-FWAA operator is defined as

$$l_{pq} = FNDFWAA(l_{pq}^{(1)}, l_{pq}^{(2)}, \cdots, l_{pq}^{(n)},) =$$
$$\sqrt{1 - \frac{1}{1+\left[\sum_{j=1}^{n} w_j \left(\frac{(\tilde{\beta}_{pq}^{j})^3}{1-(\tilde{\beta}_{pq}^{g})^3}\right)^{\rho}\right]^{\frac{1}{\rho}}}}, \sqrt{1 - \frac{1}{1+\left[\sum_{j=1}^{n} w_j \left(\frac{(\tilde{\alpha}_{pq}^{j})^3}{1-(\tilde{\beta}_{pq}^{g})^3}\right)^{\rho}\right]^{\frac{1}{\rho}}}}, \frac{1}{1+\left[\sum_{j=1}^{n} w_j \left(\frac{1-(\tilde{\gamma}_{pq}^{j})}{(\tilde{\gamma}_{pq}^{g})}\right)^{\rho}\right]^{\frac{1}{\rho}}}$$

Dombi's t-norm and t-conorm are obtained when $\rho = 1$. and the values are shown in Table VI.

Table VI. Combined Fermatean Neutrosophic fuzzy relation

L	A_1	A_2	A_3	A_4	A_5
A_1	(.368, .253, .5)	(.762, .503, .33)	(.534, .323, .226)	(.614, .416, .574)	(.712, .225, .514)
A_2	(.284, .503, .793)	(.368, .253, .5)	(.746, .457, .556)	(.617, .482, .663)	(.573, .463, .527)
A_3	(.457, .323, .648)	(.416, 457, .741)	(.368, .253, .5)	(.756, .4, .391)	(.566, .52, .668)
A_4	(.532, .416, .573)	(.579, .374, .604)	(.332, .4, .782)	(.368, .253, .5)	(.745, .578, .591)
A_5	(.409, .224, .619)	(.394, .463, .63)	(.675, .52, .566)	(.487, .578, .632)	(.368, .253, .5)

TheFND_{graphs} according to L, is in Figure 3.

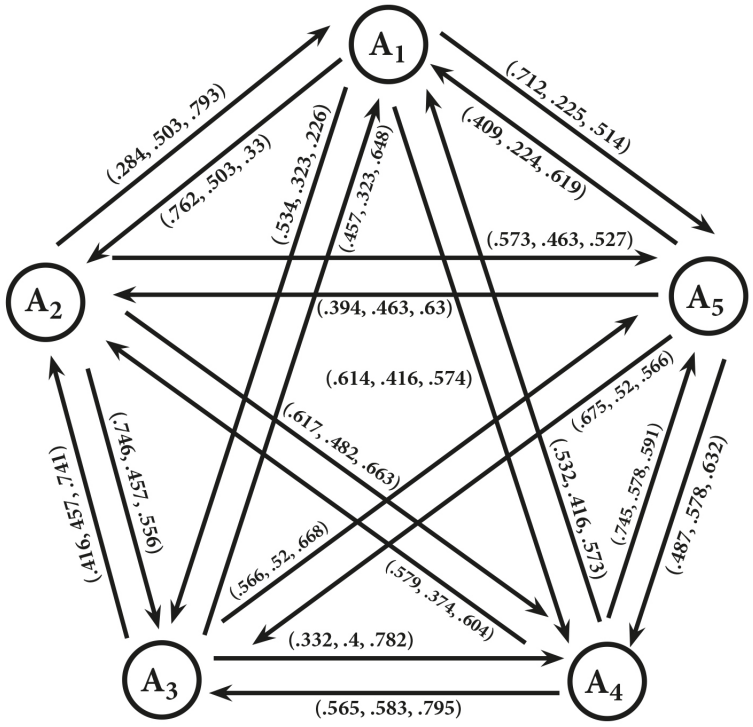

FIGURE 1. PND_{graphs}

We consider the condition of $\tilde{\lambda}_{pq} \geq 0.5$ $(l, p = 1, 2, 3, 4, 5)$ a partial directed graph is drawn in Figure 4.

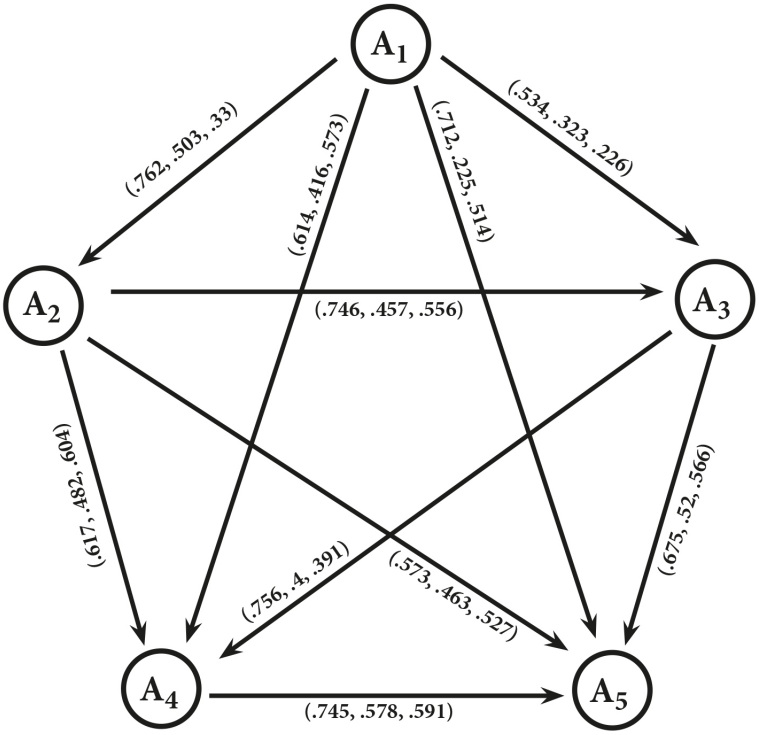

FIGURE 2. $PDPND_{graphs}$

The out-degrees $out - d(A_l)$ $(l = 1, 2, 3, 4, 5)$ are computed as

$out - d(A_1) = (2.622, 1.467, 1.643)$

$out - d(A_2) = (2.698, 1.905, 2.017)$

$out - d(A_3) = (2.711, 1.7, 1.739)$

$out - d(A_4) = (2.732, 1.876, 2.159)$

$out - d(A_5) = (2.705, 1.786, 2.198)$

Based on the above computation, the optimal ranking order is

$$A_4 \succ A_3 \succ A_5 \succ A_2 \succ A_1$$

So, A_4 is the best application for money transferring.

5. **Conclusion**

This work presents a concept of $FNDF_{graph}$. A few definitions and properties of this novel $FNDF_{graph}$ model have been described along with its introduction. The concept of $FNDF_{graph}$ offers a novel way to study and analyze systems with complex relationships and

dynamic properties. The $FNDF_{graph}$ concept can be further developed to study the functioning of both bipolar and $FNDF_{graph}$, as well as some practical applications.

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Received: March 7, 2024. Accepted: July 29, 2024

NSS
Neutrosophic Sets and Systems, Vol. 70, 2024
University of New Mexico

Operations with n-Refined Literal Neutrosophic Numbers using the Identification Method and the n-Refined AH-Isometry

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Abstract

If we know the determinate and sub-indeterminate part(s) of an argument $a_0 + a_1 I_1 + a_2 I_2 + ... + a_n I_n$ how do we similarly find the determinate and sub-indeterminate part(s) of a function (or operation) of this argument $f(a_0 + a_1 I_1 + a_2 I_2 + ... + a_n I_n)$? The AH-Isometry is designed to do just that.

The function *f* may be of one or more variables (arguments), it may also be some unary or n-ary operation. Real examples are presented in this paper of such arguments and functions.
We now present for the first time the most general form, the n-Refined AH-Isometry.

Keywords: Indeterminacy, Sub-Indeterminacies, n-Refined Literal and Numerical Neutrosophic Numbers, Identification Method, AH-Isometry, n-Refined Neutrosophic AH-Isometry

1. Introduction

The literal indeterminacy (I) was for the first time refined/split into literal sub-indeterminacies (I_1, I_2, …, I_n) by Smarandache [1] in 2015, who defined a multiplication law of these literal sub-indeterminacies to be able to build the Refined I-Neutrosophic Algebraic Structures.

The AH-Isometry [2] was firstly introduced by Mohammad Abobala and Ahmad Hatip in 2021, where AH stands for Abobala-Hatip.

The 2-Refined Neutrosophic AH-Isometry [3] was introduced by Celik and Hatip in 2022. Many papers [5-10] were published about this fundamental AH-Isometry in the rings of neutrosophic literal numbers.

We now introduce for the first time the n-Refined Neutrosophic AH-Isometry.

2. Operations with n-Refined Literal Neutrosophic Numbers

Let's have the following two n-Refined Literal Neutrosophic Numbers:

$N_1 = a_0 + a_1 I_1 + a_2 I_2 + ... + a_m I_m$ where $a_0, a_1, a_2, ..., a_m$ are real or complex numbers,

and similarly

$N_2 = b_0 + b_1 I_1 + b_2 I_2 + ... + b_m I_m$ where $b_0, b_1, b_2, ..., b_m$ are real or complex numbers,

The first three are Straightforward Operations:

2.1. Addition:

$$N_1 + N_2 = (a_0 + b_0) + (a_1 + b_1) I_1 + (a_2 + b_2) I_2 + ... + (a_m + b_m) I_m$$

2.2. Subtraction:

$$N_1 - N_2 = (a_0 - b_0) + (a_1 - b_1) I_1 + (a_2 - b_2) I_2 + ... + (a_m - b_m) I_m$$

2.3. Scalar Multiplication:

Let *c* be a real or complex number. Then:

$$c \cdot N_1 = c \cdot a_0 + c \cdot a_1 I_1 + c \cdot a_2 I_2 + ... + c \cdot a_m I_m$$

2.4. Multiplication:

In many previous applications the easiest way of multiplication (intersection) of sub-indeterminacies was defined as:

$I_j \cdot I_k = I_{\min\{j,k\}}$ for all $i, j \in \{1, 2, ..., m\}$.

Consequently, $(I_j)^2 = (I_j)^r = I_j$, for any integer $r \geq 1$,

and

$$I_{j_1} \cdot I_{j_2} \cdot ... \cdot I_{j_p} = I_{\min\{j_1, j_2, ..., j_p\}}$$

also

$\sqrt[s]{I_j} = I_j$, for integer $s \geq 1$, since, raising both sides to the *s*-power, one gets: $(\sqrt[s]{I_j})^s = (I_j)^s = I_j$.

2.5. Negative Exponent:

$(I_j)^{-m}, m > 0,$ is undefined.

2.6. Remark

The multiplication and division of the literal or numerical sub-indeterminacies ($I_a \cdot I_k$ *or* $I_a \div I_k$) are defined by experts depending on each specific application.

These applications on the literal sub-indeterminacies do not, in general, coincide with the same applications on the numerical sub-indeterminacies).

2.7. Division by Identification Method of 2-Refined Neutrosophic Literal Numbers:

Let's divide $N_1 \div N_2$ by the identification method.

The determinate part of the 2-refined literal neutrosophic number $N_1 = a_0 + a_1 I_1 + a_2 I_2$ is a_0, while a_1 is its first sub-indeterminate part, and a_2 is its second sub-indeterminate part.

And, similarly, the determinate part of the 2-refined literal neutrosophic number $N_2 = b_0 + b_1 I_1 + b_2 I_2$ is b_0, while b_1 is its first sub-indeterminate part, and b_2 is its second sub-indeterminate part.

Denote $(a_0 + a_1 I_1 + a_2 I_2) \div (b_0 + b_1 I_1 + b_2 I) = \dfrac{a_0 + a_1 I_1 + a_2 I_2}{b_0 + b_1 I_1 + b_2 I_2} \equiv x + y I_1 + z I_2$

where $a_0, a_1, a_2, b_0, b_1, b_2$ are known real numbers, while x, y, z are unknown real numbers we need to find out.

We know that

$$I_1 \cdot I_1 = I_1$$
$$I_2 \cdot I_2 = I_2$$
$$I_1 . I_2 = I_2 . I_1 = I_1$$

where I_1, I_2 are literal sub-indeterminacies.

By identification ($\equiv$) method, we have:

$$\frac{a_0 + a_1 I_1 + a_2 I_2}{b_0 + b_1 I_1 + b_2 I_2} \equiv x + y I_1 + z I_2$$

or $a_0 + a_1 I_1 + a_2 I_2 \equiv (b_0 + b_1 I_1 + b_2 I_2) \cdot (x + y I_1 + z I_2)$

Whence:

$$a_0 + a_1 I_1 + a_2 I_2 \equiv b_0 x + b_0 y I_1 + b_0 z I_2 + b_1 I_1 x + b_1 I_1 y I_1 + b_1 I_1 z I_2 + b_2 I_2 x + b_2 I_2 y I_1 + b_2 I_2 z I_2$$
$$\equiv b_0 x + b_0 y I_1 + b_0 z I_2 + b_1 I_1 x + + b_1 I_1 y + b_1 I_1 z + b_2 I_2 x + b_2 I_1 y + b_2 I_2 z$$
$$\equiv b_0 x + (b_0 y + b_1 x + + b_1 y + b_1 z) I_1 + (b_0 z + b_2 x + b_2 z) I_2$$

We get a linear system of *3* equations with *3* variables *x, y, z*:

$$a_0 = b_0 x$$
$$a_1 = b_0 y + b_1 x + + b_1 y + b_1 z = b_1 x + (b_0 + b_1) y + b_1 z$$
$$a_2 = b_0 z + b_2 x + b_2 z = b_2 x + (b_0 + b_2) z$$

that we need to solve. It has no solution: if $b_0 = 0$ and $a_0 \neq 0$,

or if $b_0 + b_2 = 0$ and $a_0 + a_2 \neq 0$, or if $b_0 + b_1 + b_2 = 0$ and $a_0 + a_1 + a_2 \neq 0$.

2.8. Division of 2-Refined Literal Neutrosophic Numbers using the AH-Isometry

Let $R[I_1, I_2] = \{a + bI_1 + cI_2; a, b, c \in R\}$, where R is the set of real numbers, be the **2-Refined Literal Neutrosophic Numbers Ring**.

Let $f : R[I_1, I_2] \times R[I_1, I_2] \to R[I_1, I_2]$ be a 2-refined literal neutrosophic function (or operation) of two variables.

Then

$$f(a_0 + a_1 I_1 + a_2 I_2, b_0 + b_1 I_1 + b_2 I_2) =$$
$$= f(a_0, b_0) + [f(a_0 + a_1 + a_2, b_0 + b_1 + b_2) - f(a_0 + a_2, b_0 + b_2)] I_1 + [f(a_0 + a_2, b_0 + b_2) - f(a_0, b_0)] I_2$$

$$= \frac{a_0}{b_0} + [\frac{a_0 + a_1 + a_2}{b_0 + b_1 + b_2} - \frac{a_0 + a_2}{b_0 + b_2}] I_1 + [\frac{a_0 + a_2}{b_0 + b_2} - \frac{a_0}{b_0}] I_2$$

which works when all denominators $b_0, b_0 + b_2, b_0 + b_1 + b_2$ are different from zero.

If $b_0 = 0$ and $a_0 = 0$, or if $b_0 + b_2 = 0$ and $a_0 + a_2 = 0$, or if $b_0 + b_1 + b_2 = 0$ and $a_0 + a_1 + a_2 = 0$,

it may be possible to get some solutions, depending on each application.

Let $g : R[I_1, I_2] \to R[I_1, I_2]$ be a 2-refined literal neutrosophic function (or operation) of one variable. Using the AH-Isometry, we get:

$$g(a_0 + a_1 I_1 + a_2 I_2) = g(a_0) + [g(a_0 + a_1 + a_2) - g(a_0 + a_2)] I_1 + [g(a_0 + a_2) - g(a_0)] I_2 \quad (*)$$

3. **Example of 2-Refined AH-Isometry**:

Calculate the natural logarithm: $L = \ln(5 + 2I_1 - 3I_2)$.

We know about the argument $5 + 2I_1 - 3I_2$ that its determinate part is *5*, its first sub-indeterminate part is $2I_1$, and his second sub-indeterminate part is $-3I_2$.

But we do not know the determinate part, nor the first and second sub-indeterminate parts, of the whole expression *L*. The AH-Isometry helps us to find them.

The function $g = ln$ (natural logarithm), $a_0 = 5$, $a_1 = 2$, $a_2 = -3$ and we substitute them into the above (*) relationship.

$L = ln(5) + [\, ln(5+2+(-3)) - ln(5+(-3))\,]I_1 + [\, ln(5+(-3)) - ln(5)\,]I_2$

$= ln(5) + [\, ln(4) - ln(2)]I_1 + [\, ln(2)-ln(5)\,]I_2$

$= ln(5) + ln(4/2)\cdot I_1 + ln(2/5)\cdot I_2$

$= ln(5) + ln(2)\cdot I_1 + ln(0.4)\cdot I_2$

Therefore, the determinate part of L is $ln(5)$, the first sub-indeterminate part of L is $ln(2)$, and the second sub-indeterminate part of L is $ln(0.4)$.

4. Examples of 2-Refined Numerical Neutrosophic Numbers

First Example:

$$N_1 = 5 + 4I_1 - 2I_2 = \; 5 + 4\cdot(0.\overline{35}) - 2\cdot(0.\overline{127})$$

where the determinate part of N_1 is 5, while the numerical sub-indeterminacies parts are:

$I_1 = 0.\overline{35} = \;$ 0.353535 ... (infinitely many decimals),

$I_2 = 0.\overline{127} = \;$ 0.127127127 ... (infinitely many decimals).

Second Example:

$N_2 = -2 + e + 3\pi = -2 + 1\cdot I_1 - 3I_2$, where

$I_1 = e = 2.7182818$ ... and

$I_2 = \; \pi = 3.1415926$ ...

5. Example of 4-Refined Numerical Neutrosophic Numbers:

$$N_4 = 8 + 2\sqrt[2]{7} - 6\sqrt[3]{111} + \sqrt[4]{54} - \sqrt[5]{4}$$
$$\equiv 8 + 2I_1 - 6I_2 + I_3 - I_4$$

where

$I_1 = \sqrt[2]{7} = 2.645751$ ...

$I_2 = \; \sqrt[3]{111} = 4.805895$ ...

$I_3 = \; \sqrt[4]{54} = 2.710806$ ...

$I_4 = \; \sqrt[5]{4} = 1.319507$ ...

6. More Examples

Zeina and Abobala [4] have defined the literal neutrosophic ring:

$R[I] = \{a + bI; a, b \in R\}$, where I = literal indeterminacy, with $I^2 = I$, where a is the determinate part of the neutrosophic number $a + bI$, while bI is its indeterminate part.

A function, $f : R[I] \to R[I]$, by using the AH-Isometry, can be calculated as:

$f(a+bI) = f(a) + [\, f(a+b) - f(a)]I$

For example, $\sin(\frac{\pi}{3}+0.5I)=\sin(\frac{\pi}{3})+[\sin(\frac{\pi}{3}+0.5)-\sin(\frac{\pi}{3})]I$, where the determinate part is $\sin(\frac{\pi}{3})$ and indeterminate part is $[\sin(\frac{\pi}{3}+0.5)-\sin(\frac{\pi}{3})]I$.

Into the original expression $\sin(\frac{\pi}{3}+0.5I)$ it is not clear what is the determinate part and what is the indeterminate part. The AH-Isometry helped us distinguish between these parts.

7. **Theorem of Isomorphism**:

Let $R_n(I)=\{a_0+\sum_{i=1}^{n}a_iI_i;\ \ a_i\in R\}$ be the n-refined neutrosophic real ring.
Define:

$$f: R_n(I)\to R\times R\times..\times R\ \ (n+1\ \ times)$$

$$f(a_0+\sum_{i=1}^{n}a_iI_i)=(a_0,\sum_{i=0}^{n}a_i,\sum_{i\neq 1}^{n}a_i,\sum_{i\neq 1,2}^{n}a_i,\sum_{i\neq 1,2,3}^{n}a_i,\dots,a_0+a_n)$$

Then *f* is a ring isomorphism.

Proof:

It is clear that *f* is well-defined mapping and preserves addition operation. We must prove that it keeps multiplication.

Assume that A = $a_0+\sum_{i=1}^{n}a_iI_i$, B = $b_0+\sum_{i=1}^{n}b_iI_i$, then for n = 2, according to the refined AH-isometry the theorem holds see [3]. Assume that it is true for k, we prove for k+1.

Assume that A = $a_0+\sum_{i=1}^{k+1}a_iI_i$, B = $b_0+\sum_{i=1}^{k+1}b_iI_i$

$A\cdot B=\left(a_0+\sum_{i=1}^{k}a_iI_i+a_{k+1}I_{k+1}\right)\left(b_0+\sum_{i=1}^{k}b_iI_i+b_{k+1}I_{k+1}\right)=\left(a_0+\sum_{i=1}^{k}a_iI_i\right)\left(b_0+\sum_{i=1}^{k}b_iI_i\right)+(a_{k+1}I_{k+1})\left(b_0+\sum_{i=1}^{k}b_iI_i\right)+(a_{k+1}I_{k+1})(b_{k+1}I_{k+1})+\left(a_0+\sum_{i=1}^{k}a_iI_i\right)(b_{k+1}I_{k+1})\ =$

$\left(a_0+\sum_{i=1}^{k}a_iI_i\right)\left(b_0+\sum_{i=1}^{k}b_iI_i\right)+a_{k+1}b_0I_{k+1}+\sum_{i=1}^{k}b_ia_{k+1}I_i+a_{k+1}b_{k+1}I_{k+1}+\left(a_0b_{k+1}I_{k+1}+\sum_{i=1}^{k}b_{k+1}a_iI_i\right)=\left(a_0+\sum_{i=1}^{k}a_iI_i\right)\left(b_0+\sum_{i=1}^{k}b_iI_i\right)+\sum_{i=1}^{k}b_ia_{k+1}I_i+(a_{k+1}b_0+a_{k+1}b_{k+1}+a_0b_{k+1})I_{k+1}+\sum_{i=1}^{k}a_ib_{k+1}I_i,$

put T = $a_0+\sum_{i=1}^{k}a_iI_i$, $S=b_0+\sum_{i=1}^{k}b_iI_i$, then

$A\cdot B=TS+\sum_{i=1}^{k}b_ia_{k+1}I_i+(a_{k+1}b_0+a_{k+1}b_{k+1}+a_0b_{k+1})I_{k+1}+\sum_{i=1}^{k}a_ib_{k+1}I_i.$

On the other hand, we have:

$f(A\cdot B)=f(TS)+f\left(\sum_{i=1}^{k}b_ia_{k+1}I_i\right)+f\left((a_{k+1}b_0+a_{k+1}b_{k+1}+a_0b_{k+1})I_{k+1}\right)+f\left(\sum_{i=1}^{k}a_ib_{k+1}I_i\right)\ =$

$=f(T)f(S)+\left(0,\sum_{i=1}^{k}b_i\,a_{k+1},\sum_{\neq 1}^{n}b_i\,a_{k+1},\dots,b_ka_{k+1}\right)+(0,a_{k+1}b_0+a_{k+1}b_{k+1}+a_0b_{k+1},a_{k+1}b_0+a_{k+1}b_{k+1}+a_0b_{k+1},\dots.,a_{k+1}b_0+a_{k+1}b_{k+1}+a_0b_{k+1})+\left(0,\sum_{i=1}^{k}a_i\,b_{k+1},\sum_{i\neq 1}^{k}a_i\,b_{k+1},\dots,a_kb_{k+1}\right)\ =$

$$\left(a_0,\sum_{i=0}^{k}a_i,\sum_{i\neq 1}^{k}a_i,\sum_{i\neq 1,2}^{k}a_i,\sum_{i\neq 1,2,3}^{k}a_i,\dots,a_0+a_k\right).(b_0,\sum_{i=0}^{k}b_i,\sum_{i\neq 1}^{k}b_i,\sum_{i\neq 1,2}^{k}b_i,\sum_{i\neq 1,2,3}^{k}b_i,\dots,b_0+b_k)+$$

$(0,\ \ a_{k+1},\dots,\ \ a_{k+1})\cdot\left(0,\sum_{i=1}^{k}b_i,\sum_{i\neq 0,1}^{k}b_i,\sum_{i\neq 0,1,2}^{k}b_i,\sum_{i\neq 0,1,2,3}^{k}b_i,\dots,b_k\right)+(0,a_{k+1}b_0+a_{k+1}b_{k+1}+a_0b_{k+1},a_{k+1}b_0+a_{k+1}b_{k+1}+a_0b_{k+1},\dots.,a_{k+1}b_0+a_{k+1}b_{k+1}+a_0b_{k+1})+(0,b_{k+1},\dots,\ \ b_{k+1})\cdot\left(0,\sum_{i=1}^{k}a_i,\sum_{i\neq 0,1}^{k}a_i,\sum_{i\neq 0,1,2}^{k}a_i,\sum_{i\neq 0,1,2,3}^{k}a_i,\dots,a_k\right)=$

$$\left(a_0,\sum_{i=0}^{k}a_i,\sum_{i\neq 1}^{k}a_i,\sum_{i\neq 1,2}^{k}a_i,\sum_{i\neq 1,2,3}^{k}a_i,\dots,a_0+a_k\right)\cdot\left(b_0,\sum_{i=0}^{k}b_i,\sum_{i\neq 1}^{k}b_i,\sum_{i\neq 1,2}^{k}b_i,\sum_{i\neq 1,2,3}^{k}b_i,\dots,b_0+b_k\right)+$$

$(\ \ 0,\ \ a_{k+1},\dots,\ \ a_{k+1}\ \ )\ \ \cdot(\ \ \ 0,\sum_{i=1}^{k}b_i+b_0+b_{k+1},\sum_{i\neq 0,1}^{k}b_i+b_0+b_{k+1},\sum_{i\neq 0,1,2}^{k}b_i+b_0+b_{k+1},\dots b_0+b_k+b_{k+1})+(\ \ 0,\ \ a_0,\dots,\ \ a_0)\cdot(\ \ 0,\ \ b_{k+1},\dots,\ \ b_{k+1})=$

$\left(a_0,\sum_{i=0}^{k+1}a_i,\sum_{i\neq 1}^{k+1}a_i,\sum_{i\neq 1,2}^{k+1}a_i,\sum_{i\neq 1,2,3}^{k+1}a_i,\dots,a_0+a_k\right)\cdot$

$\left(b_0,\sum_{i=0}^{k+1}b_i,\sum_{i\neq 1}^{k+1}b_i,\sum_{i\neq 1,2}^{k+1}b_i,\sum_{i\neq 1,2,3}^{k+1}b_i,\dots,b_0+b_k\right))=f(A)\cdot f(B).$

Thus, f is a ring isomorphism, and it is called the ***n-Refined Neutrosophic AH-Isometry****.*

8. Examples of 3- and 4-Refined Neutrosophic Isomorphisms

For *n = 3*, the *3*-refined neutrosophic case, the formula of *f* is

$$f(a_0+a_1I_1+a_2I_2+a_3I_3)=(a_0,a_0+a_1+a_2+a_3,a_0+a_2+a_3,a_0+a_3).$$

For *n = 4*, the *4*-refined neutrosophic case, the formula of *f* is

$$f(a_0+a_1I_1+a_2I_2+a_3I_3+a_4I_4)$$
$$=(a_0,a_0+a_1+a_2+a_3+a_4,a_0+a_2+a_3+a_4,a_0+a_3+a_4,a_0+a_4).$$

9. Inverse AH-Isomorphism

The formula of inverse isomorphism is:

$f^{-1}: R\times R\times\dots\times R\ \ (n+1\ \ times)\to R_n(I)$ such that*:*

$$f^{-1}(a_0,a_1,a_2,a_3\dots a_n)=a_0+I_1(a_1-a_2)+I_2(a_2-a_3)+I_3(a_3-a_4)+\cdots+\ I_n(a_n-a_0).$$

Application:

Find the inverse of the following *3*-refined neutrosophic real number $x=1+I_1+I_2+I_3$.

Now, we will see how the n-refined AH-isometry is very powerful in simplifying hard computations.

First, we compute: $f(x)=(1,4,3,2)$, $[f(x)]^{-1}=(1,\frac{1}{4},\frac{1}{3},\frac{1}{2})$, hence:

$$x^{-1}=f^{-1}[[f(x)]^{-1}]=1+\left(\frac{1}{4}-\frac{1}{3}\right)I_1+\left(\frac{1}{3}-\frac{1}{2}\right)I_2+\left(\frac{1}{2}-1\right)I_3=1+\left(-\frac{1}{12}\right)I_1+\left(-\frac{1}{6}\right)I_2+\left(-\frac{1}{2}\right)I_3.$$

By an easy computation, we can see:

$$x\cdot x^{-1}=1+\left(-\frac{1}{12}-\frac{1}{12}-\frac{1}{12}-\frac{1}{12}-\frac{1}{6}-\frac{1}{2}+1\right)I_1+\left(-\frac{1}{6}-\frac{1}{6}-\frac{1}{6}-\frac{1}{2}+1\right)I_2+\left(-\frac{1}{2}-\frac{1}{2}+1\right)I_3=1.$$

10. Conclusion

We extended for the first time the Neutrosophic AH-Isometry to n-Refined Neutrosophic AH-Isometry, to find the determinate part of a function, as well as its multiple sub-indeterminate parts.

Many real examples of n-refined neutrosophic numbers, that have a determinate part and many types of sub-indeterminacies, were presented.

The AH-Isometry is designed to find the determinate part and sub-indeterminate parts of a n-refined neutrosophic function of the form $f(a_0 + a_1 I_1 + a_2 I_2 + ... + a_n I_n)$.

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Received: March 1, 2024. Accepted: July 27, 2024

NSS
Neutrosophic Sets and Systems, Vol. 70, 2024
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On The Solutions of Some Different Types of Two-Fold Fuzzy and Neutrosophic Differential Equations

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Abstract:

This paper is dedicated to studying for the first time the concept of two-fold differential equations of different orders and different types, where we present the solutions for two-fold fuzzy differential equations, and for two-fold neutrosophic differential equations. Also, we illustrate many numerical examples to clarify and explain the novelty of this work.

Keywords: two-fold fuzzy algebra, two-fold neutrosophic set, two-fold differential equation, numerical method.

Introduction

Differential equations are one of the most important concepts in mathematics due to their wide applications in all scientific fields, such as computer science, economics, and even in physics. The use of fuzzy logic or neutrosophic logic to study differential equations and their applications in various fields is not new, as we find in [8-9], many results describing the solutions of various rows of neutrosophic differential equations, and also studying their applications in other scientific and vital fields.

The concept of two-fold algebras was introduced for the first time recently in 2023 [1], and these ideas have been used in their application to various algebraic structures such as rings, algebraic modules and spaces, and metric spaces [3,4,6]. In [7], two-fold algebras were used to study some special functions, specifically the gamma function.

Previous studies have motivated us to study the possibility of applying two-fold fuzzy algebras and two-fold neutrosophic algebras for the first time in the field of differential equations, by using two-fold real numbers in defining different classes of differential equations, and also in studying their solutions and the behavior of these solutions.

For details about two-fold neutrosophic/fuzzy numbers and functions, see [5].

Main Discussion

Definition:

Let $F(x,y,y')=0, G(x,\mu,\mu')=0$ be two different differential equations of first order. The two-fold fuzzy differential equation is defined as follows:

$\left(F(x,y,y')\right)_{G(x,\mu,\mu')}=0_0(1)$.

A function $y_\mu=f_\mu$ is called a solution at $x_0\in\mathbb{R}$ if and only if:

$$\begin{cases} F(x_0,f,f')=0 \\ G(x_0,\mu,\mu')=0 \\ \mu(x_0)\in[0,1] \end{cases}$$

The set of all $x_0\in\mathbb{R}$ for which $\mu(x_0)\in[0,1]$ is called the solving set of equation (1).

We denote it by

$SS=\{x_0\in\mathbb{R} \quad ; F(x_0,f,f')=G(x_0,\mu,\mu')=0, \mu(x_0)\in[0,1]\}$.

Example:

Consider the following two-fold fuzzy differential equation:

$(y'-y)_{\mu'(x)-\cos x}=0_0$

It is equivalent to:

$$\begin{cases} y'-y=0 & (1) \\ \mu'(x)-\cos x=0 & (2) \end{cases}$$

The general solution of (1) is:

$f(x)=k\ \ e^x \ \ ; k\in\mathbb{R}$,

The general solution of (2) is:

$\mu(x)=\sin x+l \quad ; l\in\mathbb{R}$.

$SS=\{a\in\mathbb{R}\ ; 0\le \sin a+l\ \le 1\}=\{a\in\mathbb{R}\ ; -l\le \sin a\le 1-l\}$.

The general solution of $(y'-y)_{\mu'-\cos x}=0_0$ is:

$y_\mu=(k\ \ e^x)_{\sin x+l} \quad ; k\in\mathbb{R}\ , l\in\mathbb{R}\ , x\in SS$.

Example:

Consider the following two-fold fuzzy first order differential equation:

$(y'-3)_{\mu'+2}=0_0$, then: $\begin{cases} y'=3 & (1) \\ \mu'=-2 & (2) \end{cases}$

Thus: $\begin{cases} f(x) = y = 3x + c \\ \mu(x) = -2x + a \end{cases}$ $; a, c \in \mathbb{R}$

Also, $0 \leq \mu(x) \leq 1 \quad \Leftrightarrow \frac{a}{2} \geq x \geq \frac{a-1}{2}$.

The general solution of $(y' - 3)_{\mu'+2} = 0_0$ is:

$y_\mu = (3x + c)_{-2x+a} \quad ; \quad \frac{a-1}{2} \leq x \leq \frac{a}{2}$

Definition:

1] The first-second order two-fold fuzzy differential equation is defined as follows:

$\left(F(x, y, y')\right)_{G(x,\mu,\mu',\mu'')} = 0_0$.

2] The second-first order two-fold fuzzy differential equation is defined as follows:

$\left(F(x, y, y', y'')\right)_{G(x,\mu,\mu')} = 0_0$.

3] The second order two-fold fuzzy differential equation is defined as:

$\left(F(x, y, y', y'')\right)_{G(x,\mu,\mu',\mu'')} = 0_0$.

Remark:

The solving set (SS) for each previous kind of equations is defined under the condition

$\mu(x_0) \in [0,1] \ \ \forall \ \ x_0 \in SS$.

Example:

Consider the following second-first order equation:

$(y'' - 2x)_{\mu'-\mu} = 0_0$ it is equivalent to: $\begin{cases} y'' = 2x & (1) \\ \mu' = \mu & (2) \end{cases}$

The general solution of (1) is:

$y = f(x) = \frac{1}{3}x^3 + ax + b \ \ ; a, b \in \mathbb{R}$.

The general solution of (2) is:

$\mu(x) = k \ \ e^x \ \ ; k \in \mathbb{R}$.

$0 \leq \mu(x) \leq 1 \Leftrightarrow 0 \leq k \ \ e^x \ \ \leq 1$:

For $k > 0$: $\quad 0 \leq \ \ e^x \ \ \leq \frac{1}{k} \Leftrightarrow -\infty < x \leq \ln(\frac{1}{k})$

For $k < 0$: $\quad 0 \geq \ \ e^x \ \ \geq \frac{1}{k}$ which is a contradiction.

For $k = 0$: $\quad \mu(x) = 0$ for all $x \in \mathbb{R}$.

The general solution of $(y'' - 2x)_{\mu'-\mu} = 0_0$ is:

$$y_\mu = \begin{cases} (\frac{1}{3}x^3 + ax + b)_0 & ; x \in \mathbb{R} \quad , a, b \in \mathbb{R} \\ (\frac{1}{3}x^3 + ax + b)_k \ e^x & ; k > 0 \quad , -\infty < x \leq \ln\left(\frac{1}{k}\right) \ , a, b \in \mathbb{R} \end{cases}.$$

Example:

Consider the following second order equation:

$(y'' - 6x)_{\mu''-2} = 0_0$, then: $\begin{cases} y'' = 6x & (1) \\ \mu'' = 2 & (2) \end{cases}$

The general solution of (1) is:

$y = f(x) = x^3 + ax + b \ ; a, b \in \mathbb{R}.$

The general solution of (2) is:

$\mu(x) = x^2 + cx + d \ ; c, d \in \mathbb{R}.$

To determine the solving set, we write:

$0 \leq \mu(x) \leq 1 \Leftrightarrow \mu(x) = l \quad ; l \in [0,1]$, there for:

$x^2 + cx + d - l = 0, \ \Delta = c^2 - 4(d - l) \geq 0.$

$SS = \{x \in \mathbb{R} \ ; c^2 - 4(d - l) \geq 0\}$ with $l \in [0,1]$.

The general solution of $(y'' - 6x)_{\mu'-2} = 0_0$ is:

$y_\mu = (x^3 + ax + b)_{x^2+cx+d} \quad ; x \in SS.$

For example: let $d = 0$, then: $SS = \mathbb{R}$.

For $c = d = 1$, $c^2 - 4(d - l) = 1 - 4 + 4l = 4l - 3$, then:

$\begin{cases} 4l - 3 \geq 0 \implies l \geq \frac{3}{4} \\ 4l - 3 \leq 1 \implies l \leq 1 \end{cases} \Leftrightarrow \frac{3}{4} \leq l \leq 1$ and:

$$x = \begin{cases} \dfrac{-c + \sqrt{4l - 3}}{2} \\ \dfrac{-c - \sqrt{4l - 3}}{2} \end{cases}$$

The general solution in this case is:

$y_\mu = (x^3 + ax + b)_{x^2+cx+d} \quad ; x \in \left\{\frac{-c+\sqrt{4l-3}}{2}, \frac{-c-\sqrt{4l-3}}{2}\right\} \ , l \in [\frac{3}{4}, 1].$

Example:

Consider the following first-second order differential equation:

$(y' - \frac{1}{x})_{\mu''-2} = 0_0$, then: $\begin{cases} y' = \frac{1}{x} & (1) \\ \mu'' = 2 & (2) \end{cases}$

The solution of (1) is:

$y = f(x) = \ln x + b \ \ ; x > 0, b \in \mathbb{R}.$

The solution of (2) is:

$\mu(x) = x^2 + cx + d \ \ ; c, d \in \mathbb{R}.$

$SS = \{x \in \mathbb{R} \ \ ; c^2 - 4(d - l) \geq 0 \ \ , 0 \leq l \leq 1\}.$

The general solution is:

$y_\mu = (\ln x + b \ \ )_{x^2+cx+d}$ with: $x \in]0, \infty[\cap SS.$

Definition:

Let $\begin{cases} F_1(x, y, y', \dots, y^n) = 0 \\ F_2(x, t, t', \dots, t^m) = 0 \\ F_3(x, i, i', \dots, i^s) = 0 \\ F_4(x, f, f', \dots, f^j) = 0 \end{cases}$

Be 4 differential equations of orders n, m, s, j respectively.

We define the two-fold neutrosophic differential equation of order (n, m, s, j) as follows:

$\left(F_1(x, y, y', \dots, y^{(n)})\right)_{(F_2,F_3,F_4)} = 0_{(0,0,0)}$ (1)

A two-fold neutrosophic function $(h)_{(t,i,f)}$ is called a solution of (1) if and only if (at the point x_0):

$\begin{cases} F_1(x_0, h, h', \dots, h^{(n)}) = 0 \\ F_2(x_0, t, t', \dots, t^{(m)}) = 0 \\ F_3(x_0, i, i', \dots, i^{(s)}) = 0 \\ F_1(x_0, f, f', \dots, f^{(j)}) = 0 \end{cases}$ with $t(x_0), i(x_0), f(x_0) \in [0,1].$

The solving set is defined as $= S_1 \cap S_2 \cap S_3$;

$\begin{cases} S_1 = \{x \in \mathbb{R} \ \ ; t(x) \in [0,1]\} \\ S_2 = \{x \in \mathbb{R} \ \ ; i(x) \in [0,1]\} \\ S_3 = \{x \in \mathbb{R} \ \ ; f(x) \in [0,1]\} \end{cases}$

Example:

Consider the two-fold neutrosophic differential equation of order (1,1,1,1):

$(y'-2)_{(t'-1,i'-1,f'+3)} = 0_{(0,0,0)}$, thus: $\begin{cases} y' = 2 \\ t' = 1 \\ i' = 1 \\ f' = -3 \end{cases}$

Therefor: $\begin{cases} y = 2x + a_1 \\ t = x + a_2 \\ i = x + a_3 \\ f = -3x + a_4 \end{cases}$

On the other hand, we have:

$$\begin{cases} 0 \le t(x) \le 1 & \Leftrightarrow -a_2 \le x \le 1 - a_2 \\ 0 \le i(x) \le 1 & \Leftrightarrow -a_3 \le x \le 1 - a_3 \\ 0 \le f(x) \le 1 & \Leftrightarrow \dfrac{+a_4}{3} \ge x \ge \dfrac{a_4 - 1}{3} \end{cases}$$

Hence $SS = [-a_2, 1-a_2] \cap [-a_3, 1-a_3] \cap \left[\frac{a_4-1}{3}, \frac{+a_4}{3}\right]$.

The general solution is:

$(2x + a_1)_{(x+a_2, x+a_3, -3x+a_4)}$ $\quad ; x \in SS$.

For example, take: $a_2 = 0$, $a_3 = \frac{1}{2}, a_4 = +1$, then:

$SS = [0,1] \cap \left[\frac{-1}{2}, \frac{1}{2}\right] \cap \left[0, \frac{1}{3}\right] = [0, \frac{1}{3}]$, and the general solution in this case is:

$(2x + a_1)_{(x, x+\frac{1}{2}, -3x+1)}$ $\quad ; x \in \left[0, \frac{1}{3}\right]$.

Example:

Consider the following two-fold neutrosophic differential equation of order (2,1,1,1): $(y'' + \sin x)_{(t'-2, i'+1, f'+2)} = 0_{(0,0,0)}$, hence:

$$\begin{cases} y'' = -\sin x \\ t' = 2 \\ i' = -1 \\ f' = -2 \end{cases} \text{ and } \begin{cases} y = \sin x + ax + b & ; a, b \in \mathbb{R} \\ t = 2x + c_1 \\ i = -x + c_2 \\ f = -2x + c_3 \end{cases} \quad ; c_1, c_2, c_3 \in \mathbb{R}$$

$$\begin{cases} 0 \le t(x) \le 1 & \Leftrightarrow \dfrac{-c_1}{2} \le x \le \dfrac{1 - c_1}{2} \\ 0 \le i(x) \le 1 & \Leftrightarrow c_2 - 1 \le x \le c_2 \\ 0 \le f(x) \le 1 & \Leftrightarrow \dfrac{c_3 - 1}{2} \ge x \ge \dfrac{c_3}{2} \end{cases}$$

$SS = [\frac{-c_1}{2}, \frac{1-c_1}{2}] \cap [c_2 - 1, c_2] \cap \left[\frac{c_3-1}{2}, \frac{c_3}{2}\right]$.

The general solution is:

$y_{(t,i,f)} = (\sin x + ax + b)_{(2x+c_1,-x+c_2,-2x+c_3)}$; $x \in SS$, $a, b, c_1, c_2, c_3 \in \mathbb{R}$.

For example: take $c_1 = -3, c_2 = 2, c_3 = 4$, then:

$SS = \left[\frac{3}{2}, 2\right] \cap [1,2] \cap \left[\frac{3}{2}, 2\right] = \left[\frac{3}{2}, 2\right]$.

The general solution in this case is:

$y_{(t,i,f)} = (\sin x + ax + b)_{(2x-3,-x+2,-2x+4)}$; $a, b \in \mathbb{R}$, $x \in \left[\frac{3}{2}, 2\right]$.

Another case if $c_1 = c_2 = c_3 = 1$, then:

$SS = \left[\frac{-1}{2}, 0\right] \cap [0,1] \cap \left[0, \frac{1}{2}\right] = \{0\}$, so that the corresponding solution is: $y_{(t,i,f)} = (\sin x + ax + b)_{(0,0,0)}$; $a, b \in \mathbb{R}$

Example:

Take the following equation of order (1,2,1,2):

$(y')_{(t''-2,i',f''-f')} = 0_{(0,0,0)}$, then:

$$\begin{cases} y' = 0 \\ t'' = 2 \\ i' = 0 \\ f'' = f' \end{cases} \text{ and } \begin{cases} y = c & ; c \in \mathbb{R} \\ t = x^2 + ax + b & ; a, b \in \mathbb{R} \\ i = m & ; m \in \mathbb{R} \\ f = k\ e^x & ; k \in \mathbb{R} \end{cases}$$

For determining the solving set (SS), we write:

$$\begin{cases} 0 \le i(x) \le 1 \Leftrightarrow m \in [0,1] \\ 0 \le f(x) \le 1 \Leftrightarrow 0 \le e^x \le \frac{1}{k} \Leftrightarrow -\infty < x \le \ln\left(\frac{1}{k}\right) \ ; k > 0 \end{cases}$$

And $0 \le t(x) \le 1 \Leftrightarrow x^2 + ax + b = l \in [0,1] \Leftrightarrow x^2 + ax + b - l = 0 \Leftrightarrow a^2 - 4(b-l) \ge 0$,

with $x = \frac{-a \mp \sqrt{a^2 - 4(b-l)}}{2}$

$SS = \left]-\infty, \ln\left(\frac{1}{k}\right)\right] \cap I$; $I = \{x \in \mathbb{R}$; $x = \frac{-a \mp \sqrt{a^2-4(b-l)}}{2}$, $l \in [0,1], a^2 - 4(b-l) \ge 0\}$.

For example if $k = \frac{1}{2}$, $a = b = 0$, then:

$x = \sqrt{l}$; $0 \le l \le 1$, and $SS =]-\infty, \ln 2] \cap \{\sqrt{l}$; $0 \le l \le 1\} = [0, \ln 2]$; $l \in [0, (\ln 2)^2]$.

The solution in this case is:

$y_{(t,i,f)} = (c)_{(x^2, m, \frac{1}{2}e^x)}$; $x \in [0, \ln 2], m \in [0,1]$.

Example:

Consider the following equation of order (2,1,1,1):

$(y'' - 1)_{(t'-t,i'-i,f'-f)} = 0_{(0,0,0)}$, hence:

$$\begin{cases} y'' = 1 \\ t' = t \\ i' = i \\ f' = f \end{cases} \text{ and } \begin{cases} y = \frac{x^2}{2} + ax + b \\ t = k_1 e^x \\ i = k_2 e^x \\ f = k_3 e^x \end{cases} ; a, b, k_i \in \mathbb{R}.$$

$$\begin{cases} \\ 0 \le t(x) \le 1 \\ 0 \le i(x) \le 1 \\ 0 \le f(x) \le 1 \\ \\ \end{cases} \Leftrightarrow \begin{cases} -\infty < x \le \ln\left(\frac{1}{k_1}\right) \\ -\infty < x \le \ln\left(\frac{1}{k_2}\right) \\ -\infty < x \le \ln\left(\frac{1}{k_3}\right) \end{cases} ; \quad k_1, k_2, k_3 > 0$$

$$SS = \left]-\infty, \ln\left(\frac{1}{k_1}\right)\right] \cap \left]-\infty, \ln\left(\frac{1}{k_2}\right)\right] \cap \left]-\infty, \ln\left(\frac{1}{k_3}\right)\right] = \left]-\infty, \ln\left(\frac{1}{\max(k_1,k_2,k_3)}\right)\right].$$

The general solution is:

$$y_{(t,i,f)} = \left(\frac{x^2}{2} + ax + b\right)_{(k_1 e^x, k_2 e^x, k_3 e^x)} \quad ; x \in \left]-\infty, \ln\left(\frac{1}{\max(k_1,k_2,k_3)}\right)\right].$$

Example:

Consider the following equation of order (3.1,1,1):

$(y''' - G)_{(t',i'+2,f'-2)} = (0)_{(0,0,0)}$, then:

$$\begin{cases} y''' = 6 \\ t' = 0 \\ i' = -2 \\ f' = 2 \end{cases} \text{ and } \begin{cases} y = x^3 + ax^2 + bx + c \\ t' = m_1 \\ i' = -2x + m_2 \\ f = 2x + m_3 \end{cases} \quad \text{with } a, b, c, m_1, m_2, m_3 \in \mathbb{R}.$$

$$\begin{cases} \\ 0 \le t(x) \le 1 \\ 0 \le i(x) \le 1 \\ 0 \le f(x) \le 1 \\ \\ \end{cases} \Leftrightarrow \begin{cases} m_1 \in [0,1] \ , x \in \mathbb{R} \\ x_2 \in \left[\frac{m_2 - 1}{2}, \frac{m_2}{2}\right] \\ x_3 \in \left[\frac{-m_3}{2}, \frac{-m_3 + 1}{2}\right] \end{cases}$$

$$SS = \left[\tfrac{m_2-1}{2}, \tfrac{m_2}{2}\right] \cap \left[\tfrac{-m_3}{2}, \tfrac{-m_3+1}{2}\right].$$

For $m_2 = 1 \ , m_3 = \frac{-1}{2}$, then $SS = \left[0, \frac{1}{2}\right] \cap \left[\frac{1}{4}, \frac{3}{4}\right] = \left[\frac{1}{4}, \frac{1}{2}\right]$.

The general solution in this case is:

$$y_{(t,i,f)} = \left(x^3 + ax^2 + bx + c \quad\right)_{(m_1, -2x + \frac{1}{2}, 2x - \frac{1}{2})} \quad ; x \in \left[\frac{1}{4}, \frac{1}{2}\right], m_1 \in [0,1].$$

Conclusion

In this paper, we have studied for the first time the concept of two-fold differential equations of different orders and different types, where we have presented the solutions for two-fold

fuzzy differential equations, and for two-fold neutrosophic differential equations. Also, we illustrated many numerical examples to clarify and explain the novelty of this work.

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Received: March 22, 2024. Accepted: July 29, 2024

NSS
Neutrosophic Sets and Systems, Vol. 70, 2024
University of New Mexico

Analyzing Interdisciplinary Education in General Medicine Using Smarandache's Multivalued Logic Hypothesis Theory and Plithogenic Probability

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Abstract. Smarandache extended the concept of falsifiability to include hypotheses that involve multiple values. Smarandache's extension of classical logic introduces Partial Falsifiability within multi-valued logic, which allows for the inclusion of partial truth values, uncertainties, and false values, unlike traditional falsifiability which only classifies hypotheses as completely true or false. Using this approach this study investigates specifically the concept of interdisciplinarity in the field of integral general medicine, with a specific emphasis on the integration of nursing procedures. The method employs sentiment analysis of scientific questions using the Consensus tool and plithogenic statistics to assess complex hypotheses. The findings indicate that the combination of nursing procedures and interprofessional approaches greatly improves teamwork, communication, and clinical competence. The results suggest a high probability of truth (0.67), some indeterminacy (0.33), and a low probability of falsity (0.17). Future research will focus on utilizing alternative sentiment analysis tools and incorporating the concept of fallibility into hypotheses using Smarandache's multivalued logic in different contexts.

Keywords: interdisciplinarity, general integral medicine, nursing procedures falsifiability of a hypothesis, multi-valued logics, plithogenic statistics

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1 Introduction

The assessment of sentiment in scientific literature has received considerable attention in recent years, primarily because of the increasing accessibility of scientific publications. Sentiment analysis in this context specifically aims to interpret the emotions and attitudes of the authors as conveyed through the citations/themes found in scholarly databases [1].

In a recent publication, Smarandache [2] expanded the notion of falsifiability to encompass hypotheses that involve multiple values in logic. In classical logic, falsifiability traditionally deals with hypotheses that can be definitively classified as either completely true or completely false. Florentin's extension incorporates the concept of Partial Falsifiability into the realm of multi-valued logic, allowing hypotheses to possess partial truth values, uncertainties, and false values [3]. In the context of neutrosophic logic, a hypothesis $NLH(T,I,F)$ is considered falsifiable if certain conditions exist that allow the hypothesis to be negated, resulting in $\neg NLH(F,1-I,T)$, where t, i, and f are degrees of truth, indeterminacy, and falsity, respectively, and they all fall within the range of [0,1]. In the probabilistic interpretation, a neutrosophic probabilistic hypothesis [4], $NPH(T,I,T)$ is defined with t, i, and f values ranging from 0 to 1, representing the probability, indeterminacy, and falsity, respectively. This approach offers a more comprehensive and intricate assessment of the falsifiability and probability of hypotheses in intricate scientific like medical education and social science contexts.

The Consensus Meter [5] is a novel tool in Consensus AI that functions as a gauge to quantify agreement and disagreement among a group of papers(Figure 1). The system functions by posing straightforward binary inquir ies, enabling a swift evaluation of the degree of agreement or disagreement among participants. The responses are categorized as "Yes," "No," or "Possibly," which offers a distinct perspective on the group's position regarding a particular subject [6]. The Consensus Meter is a tool that utilizes neutrosophic logic/probability to assess the falsifiability and probability of a hypothesis. It offers a nuanced evaluation of the hypothesis's validity in various conditions.

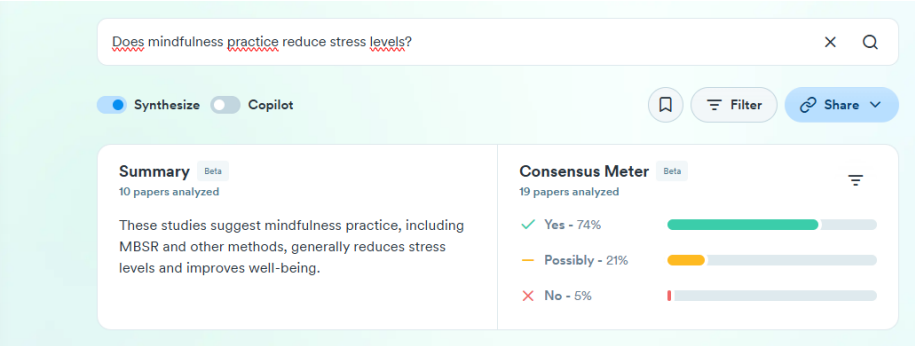

Figure 1: Consensus Meter Interface for Assessing the Impact of Mindfulness on Stress Reduction

A scientific question as "Does X result in Y?", explores the cause-and-effect connection between two variables. This question type seeks to ascertain whether alterations in variable X have a direct impact on variable Y. An illustration of this is the inquiry, "Does smoking induce lung cancer?", which investigates the correlation between smoking and the development of lung cancer. These inquiries are pivotal for establishing causal relationships [7]. Plithogenic probability, as defined in [8], assigns multiple probabilities to each event E in a probability space U, in contrast to classical probability, imprecise probability, and neutrosophic probability, which assign only one probability. The plithogenic probability distribution function, P(x), for a random variable x, is defined by multiple plithogenic probability distribution sub-functions. Each sub-function quantifies the probability (relative to a specific attribute value) of the occurrence of the value x. The probabilities of occurrence can be represented using classical, imprecise, or neutrosophic probabilities, depending on the level of chance involved. This approach enables the separation of the probabilities of occurrence for each question that constitutes the hypothesis being analyzed.

The incorporation of nursing procedures and interprofessional approaches into medical education significantly improves teamwork, communication, clinical proficiency, and patient care outcomes [9]. These educational methodologies not only enhance the professional competencies of students but also foster a more cooperative and patient-centered healthcare milieu [10]. However, effectively executing these strategies requires overcoming practical obstacles and ensuring their continuous integration into educational curricula [11]. One of the difficulties in implementing this approach is the need to coordinate curricula across various professional programs and ensure their long-term integration into existing educational frameworks [12]. This paper studies this approach to medical education using Smarandache's partial falsifiability hypothesis theory, sentiment analysis with the Consensus tool, and plithogenic statistics, extending the research tool based on set theory [13].

2 Prelininaries

2.1 Plithogenic Probability

Neutrosophic (or Indeterminate) Data is characterized by its inherent vagueness, lack of clarity, incompleteness, partial unknowns, and conflicting information [14, 15]. The data can be classified as either quantitative (metrical), qualitative (categorical), or a combination of both. Plithogenic Variate Data[16] describes the connections or correlations between Neutrosophic variables. A Neutrosophic Variable [17, 18], which can be a function or operator, deals with neutrosophic data in either its arguments, its values, or both. Complex problems often require multiple measurements and observations due to their multidimensional nature like measures needed on scientific inquiries. Neutrosophic variables can exhibit dependence, independence, partial dependence, partial independence, or partial indeterminacy como in science [19].

A Plithogenic Set [20, 21] is a non-empty set P whose elements within the domain of discourse U ($P \subseteq U$) are characterized by one or more attributes $A_1, A_2, \cdots, A_m$, where m is at least 1. where each attribute can have a set of possible values within the spectrum S of values (states), such that S can be a finite, infinite, discrete, continuous, open, or closed set.

Each element $x \in P$ is characterized by all the possible values of the attributes that are inside the set $V = \{v_1, v_2, \cdots, v_n\}$. The value of an attribute has a degree of appurtenance $d(x, v)$ of an element x in the set P, based on a specific criterion. The degree of appurtenance can be either fuzzy, intuitionistic fuzzy, or neutrosophic, among others [22].

That means,

$\forall x \in P, d: P \times V \rightarrow \mathcal{P}([0,1]^z)$ (1)

Where $d(x, v) \subseteq [0,1]^z$ and $\mathcal{P}([0,1]^z)$ is the power set of $[0,1]^z$. $z = 1$ (the fuzzy degree of appurtenance), $z = 2$ (the intuitionistic fuzzy degree of appurtenance), or $z = 3$ (the neutrosophic degree of appurtenance).

Plithogenic Probability [23], derived from Plithogenic Variate Analysis, represents a multi-dimensional probability ("plitho" meaning "many" and synonymous with "multi"). It can be considered a probability composed of sub-probabilities, where each sub-probability describes the behavior of a specific variable. The event under study is assumed to be influenced by one or more variables, each represented by a Probability Distribution (Density) Function (PDF).

Consider an event E in a given probability space, whether classical or neutrosophic, determined by $n \geq 2$ variables. $v_1, v_2, \ldots, v_n$, denoted as $E(v_1, v_2, \ldots, v_n)$. The multi-variate probability of event E occurring denoted as MVP(E), relies on multiple probabilities. Specifically, it depends on the probability of event E occurring with respect to each variable: $P1(E(v_1))$ for variable v_1, $P2(E(v_2))$ for variable v_2, and so forth. Therefore, $MVP(E(v_1, v_2, \ldots, v_n))$ is represented as $(P1(E(v_1)), P2(E(v_2)), \ldots, Pn(E(v_n)))$. The variables $v_1, v_2, \ldots, v_n$, and the probabilities $P_1, P_2, \ldots, P_n$, may be classical or have some degree of indeterminacy [24].

To transition from Plithogenic Neutrosophic Probability (PNP) to Univariate Neutrosophic Probability UNP, we employ the conjunction operator [25]:

$UNP(v_1, v_2, \ldots, v_n) = v_1 \wedge_{i=1}^{n} v_n$ (2)

$\wedge$ In this context, it is a neutrosophic conjunction (t-norm). If we take $\wedge_p$ as the plithogenic conjunction between probabilities of the PNP type, where $(T_A, I_A, F_A) \wedge_p (T_B, I_B, F_B) = (T_A \wedge T_B, I_A \vee I_B, F_A \vee F_B)$, such that $\wedge$ is the t-norm minimum of fuzzy logic and $\vee$ the t-conorm maximum[26, 27].

2 Material and Methods

Figure 2 contains a diagram of the steps to follow in this method based on ."Partial Falsifiability of Fuzzy and Fuzzy-Extension Hypotheses" as proposed by Smarandache [2]. The research process entails a sequence of systematic processes aimed at examining a hypothesis, assessing its soundness, and drawing conclusions based on detailed data analysis. Below is a detailed description of each phase of the procedure

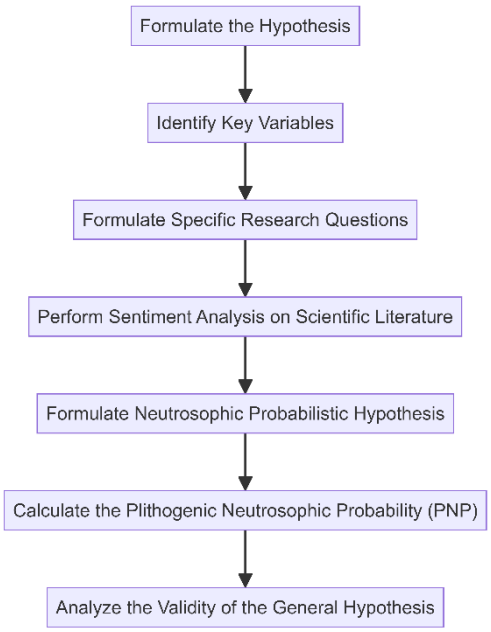
Formulate the Hypothesis
Identify Key Variables
Formulate Specific Research Questions
Perform Sentiment Analysis on Scientific Literature
Formulate Neutrosophic Probabilistic Hypothesis
Calculate the Plithogenic Neutrosophic Probability (PNP)
Analyze the Validity of the General Hypothesis

Figure 2: Scheme of the proposed method

a. Formulate the Hypothesis

Begin by explicitly stating the hypothesis that you intend to examine. Make sure that indicates a cause-and-effect relationship between the variables. For example, "Increased study time leads to higher exam scores."

b. Identify Key Variables

Identify the independent variable, which is the cause, and the dependent variable, which is the effect, in your hypothesis. This aids in directing your research inquiries towards the exact relationship to investigate.

c. Formulate Specific Research Questions

Split the hypothesis into precise research questions formulated as "Does X cause Y?" This enables a comprehensive and focused examination of the postulated correlation.

d. Perform sentiment analysis on scientific literature.

To conduct sentiment analysis on a research paper and quantify the occurrences of "Yes," "Possibility/Indeterminacy," and "No," a sentiment analysis tool for scientific statements is needed . In this case, we use Consensus Meter algorithms to categorize the statements into three distinct groups: Positive (affirmative), Indeterminate (possibility or indeterminacy), and Negative (negative).

e. Formulate Neutrosophic Probabilistic Hypothesis

Determine the ratios of each category to construct the neutrosophic probability hypothesis (T, I, F), where T denotes the truth value, I represents the indeterminacy, and F indicates the falsity.

f. Calculate the Plithogenic Neutrosophic Probability (PNP)

By utilizing the neutrosophic probabilities assigned to each question the Univariate Neutrosophic Probability (UNP) is calculated to assess the soundness of the overall hypothesis. This process entails combining the separate probabilities to offer a thorough assessment of the overall hypothesis.

$$UNP(v_1, v_2, \dots, v_n) = (Min(t_1, t_n, \dots, t_n), Max(i_1, i_n, \dots, i_n), Max(f_1, f_n, \dots, f_n)) \quad (3)$$

Where:

$T_1, T_2, \dots, T_n$: are the truth probabilities values of each question.

$I_1, I_2, \dots, I_n$: are the indeterminacy probabilities values of each question.

$F_1, F_2, \dots, F_n$: are the falsity probabilities values of each question

g. Analyze the validity of the general hypothesis

In this case, the negation of NPH is represented as [28]:

$$(T, I, F) = (F, I, T) \quad (4)$$

This step entails analyzing the negated neutrosophic probabilities to evaluate the overall soundness and dependability of the general hypothesis. By evaluating the levels of falsehood, uncertainty, and truthfulness, one can ascertain the degree to which the hypothesis is valid, ambiguous, or incorrect depending on the scientific literature.

3 Case Study

Incorporating nursing procedures and interprofessional approaches into medical education greatly improves collaboration, communication, clinical proficiency, and patient outcomes in actual clinical environments. Incorporating these components into medical curricula fosters a cooperative atmosphere in which healthcare practitioners acquire the skills to collaborate efficiently, thereby enhancing the quality of patient care. Through prioritizing interprofessional education, medical students acquire a more profound comprehension of the responsibilities and contributions of diverse healthcare providers, resulting in enhanced coordination and increased efficiency in healthcare delivery.

Furthermore, incorporating nursing procedures into medical education aids in refining the clinical aptitude of students. This approach guarantees that upcoming healthcare professionals possess practical knowledge and essential competencies necessary for patient care. Additionally, it promotes a setting that values ongoing education and adjustment, which is crucial in the constantly changing healthcare industry. The outcome is an enhanced and self-assured healthcare workforce capable of providing exceptional care and adjusting to the ever-changing demands of patients.

Although there are potential advantages, there are also various constraints and difficulties that could lead to uncertainty and potentially undermine the hypothesis. Differences in the way educational institutions implement programs, including variations in resources, expertise, and commitment, can result in inconsistent outcomes and substantial disparities in program effectiveness. Variations in educational environments, encompassing cultural, social, and organizational elements, can impact the assimilation and achievement of nursing protocols and interprofessional methodologies, leading to diverse levels of effectiveness in different settings. Practical barriers, such

as the coordination of curricula across multiple professional programs, conflicts in scheduling, and divergent academic priorities, can hinder the smooth integration. Ensuring sustainability and long-term integration necessitates ongoing backing, resources, and dedication from stakeholders. Without sustained endeavor, the initial advantages may dwindle. Another challenge arises from the resistance to change exhibited by faculty, students, and institutions, which can be attributed to deeply ingrained practices, limited awareness, or perceived risks to professional autonomy. Furthermore, the process of determining and analyzing the effects of interprofessional education on clinical outcomes is intricate and necessitates reliable tools and methodologies. Insufficient evaluation mechanisms can result in inconclusive or deceptive findings, thereby exacerbating the uncertainty surrounding the matter.
In this case study the hypothesis is stated as follows.
The inclusion of nursing procedures and interprofessional approaches in medical education improves teamwork, communication, clinical competence, and patient care outcomes in real clinical settings.
To investigate this hypothesis, we followed a systematic research process based on the notion of "Partial Falsability of Fuzzy and Fuzzy-Extension Hypotheses" proposed by Smarandache [2].

Below is a description of each question along with the variable intended to be measured:

Teamwork and Communication

Q1: Does the inclusion of nursing procedures in medical education lead to better teamwork and communication in clinical settings?

Variable: Quality of teamwork and communication in clinical settings.

Patient Care Outcomes

Q2: Does integrating nursing education into medical training improve patient care outcomes?

Variable: Patient care outcomes (e.g., recovery rates, patient satisfaction).

Practical Challenges in Clinical Settings

Q3: Can the interdisciplinary approach in medical education realistically address the challenges faced in practical, real-world medical settings?

Variable: Ability to address practical challenges in clinical settings through an interdisciplinary approach.

Clinical Competence

Q4: Does integrating interdisciplinary procedures into medical education improve clinical competence in graduates?

Variable: Level of clinical competence of graduates.

These questions allow for the measurement of different aspects and outcomes of the interdisciplinary approach in medical education and how it affects the training and performance of healthcare professionals.

Sentiment about each question is investigated using the Consensus Tool (Table 1) that classifies paper with Positive, Indeterminate, and Negative stances about a specific question

Questions\Example of Stances in Papers	Positive	Indeterminacy	Negative	Neutrosophic Probability
Q1	[29-34]	[35]		(0.86, 0.14, 0)
Q2	[36-39]	[40]	[41]	(0.67, 0.17, 0.17)
Q3	[41-44]	[45]		(0.67, 0.33, 0)
Q4	[46-51]	[52, 53]		(0.75, 0.25, 0)

Table 1: Sentiment Evaluation on Research Questions in Interdisciplinary Medical Education using the Consensus Tool

Figure 3 shows the probabilities of positive, indeterminate, and negative sentiments for questions Q1, Q2, Q3, and

Q4.

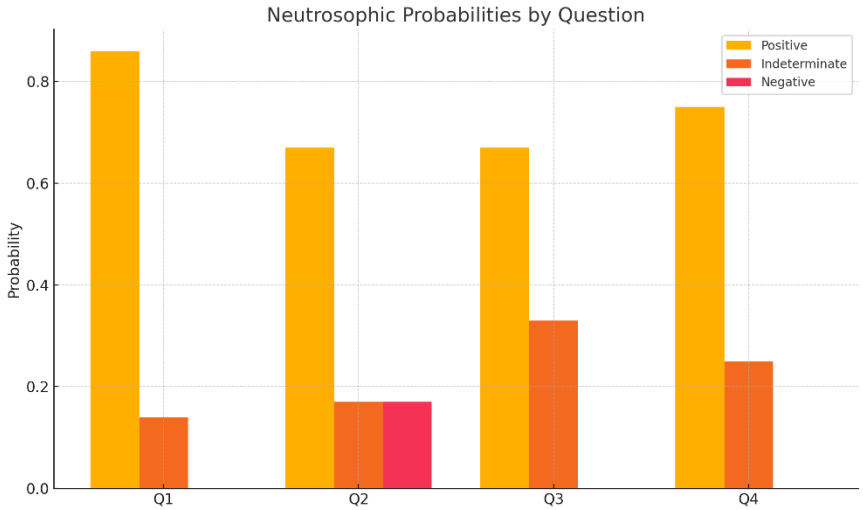

Figure 2: Neutrosophic Probabilities for Sentiment Analysis of Research Questions

Then, the univariate neutrosophic probability is calculated as follows:

$$UNP(H) = \wedge_{p_{i=1}}^{4} \text{PNP}(Q_i) \tag{5}$$

$$UNP\ (0.86, 0.14, 0), (0.67, 0.17, 0.17), (0.67, 0.33, 0)\ (0.75, 0.25, 0)$$

$$= (min(0.86, 0.67, 0.67, 0.65), max(0.14, 0.17, 0.33, 0.25), max(0, 0.17, 0, 0))$$

$$= (0.67, 0.33, 0.17)$$

The value (0.67, 0.33, 0.17) for a neutrosophic probabilistic hypothesis indicates the following:

- **0.67:** There is a 67% probability that the hypothesis is true. This means there is a high likelihood that the inclusion of nursing procedures and interprofessional approaches in medical education improves teamwork, communication, clinical competence, and patient care outcomes in real clinical settings.
- **0.33:** There is a 33% indeterminacy, meaning there is some uncertainty about the validity of the hypothesis.
- **0.17:** There is a 17% probability that the hypothesis is false.

Negation of the Hypothesis
The negation of the hypothesis yields the values (0.17, 0.33, 0.67), which can be interpreted as follows:

- **0.17:** There is a 17% probability that the negation of the hypothesis is true. This means there is a relatively low likelihood that excluding nursing procedures and interprofessional approaches from medical education would improve teamwork, communication, clinical competence, and patient care outcomes.
- **0.33:** There is a 33% indeterminacy, indicating some uncertainty about the validity of the negated hypothesis. This mirrors the uncertainty present in the original hypothesis.
- **0.67:** There is a 67% probability that the negation of the hypothesis is false. This means there is a high likelihood that excluding nursing procedures and interprofessional approaches from medical education would not improve, or might even harm, teamwork, communication, clinical competence, and patient care outcomes.

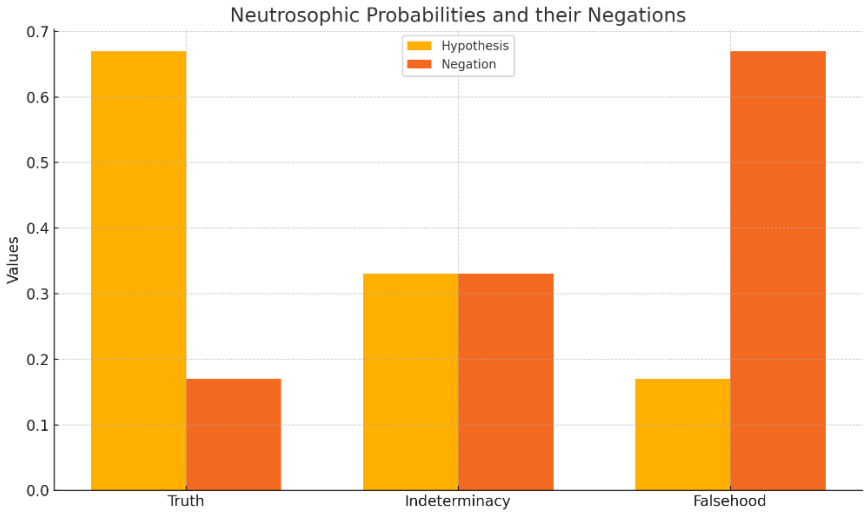

Figure 3: Neutrosophic Probabilities and Their Negations [28].

The results of the neutrosophic probabilistic hypothesis analysis suggest a strong likelihood that incorporating nursing procedures and interprofessional approaches into medical education has positive effects on important clinical and educational outcomes[54]. However, the presence of indeterminacy highlights the need for further research and consideration of contextual factors to fully validate these findings. The high falsity value for the negated hypothesis reinforces the positive impact of these educational approaches.

The Consensus Meter can be considered a sentiment analysis tool for scientific statements. This type of tool uses language models to classify and summarize responses to specific questions, providing an aggregated view of scientific opinion on a given topic. However, it has limitations, including the potential for incorrect classifications about 10% of the time and missing specific nuances of the original question. Additionally, it does not assess research quality and only includes a snapshot of available studies, not all existing research. Future research will explore the utilization of alternative sentiment analysis methods [55].

4. Conclusion

The assessment of the interdisciplinary influence in the field of general medicine, through the utilization of sentiment analysis and plithogenic statistics, uncovers noteworthy observations. An analysis of the sentiments expressed by authors towards cited works can be achieved through sentiment analysis of scientific citations. On the other hand, plithogenic probability takes into account multiple variables that have different levels of truth, indeterminacy, and falsity. The findings of this study suggest that incorporating nursing procedures and interprofessional approaches into medical education is likely to enhance teamwork, communication, clinical competence, and patient care outcomes. These conclusions are supported by a neutrosophic probabilistic hypothesis, which indicates a high probability of truth (0.67), some uncertainty (0.33), and a low probability of falsehood (0.17).

Future research will prioritize the utilization of alternative sentiment analysis tools for scientific inquiries or hypotheses. Furthermore, it would entail utilizing the concept of fallibility in hypotheses with multivalued logics of Smarandache in various situations.

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Received: March 1, 2024. Accepted: July 29, 2024

NSS
Neutrosophic Sets and Systems, Vol. 70, 2024
University of New Mexico

Assessment of three strategies for teaching an AI literacy program, based on a neutrosophic 2-tuple linguistic model hybridized with the ARAS method

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Abstract. This study analyzes some general Artificial Intelligence competency frameworks, aimed at teachers, and it proposes a comprehensive Artificial Intelligence literacy training program, designed to develop in university teachers the necessary skills to integrate Artificial Intelligence into their curricula and research. Through a mixed theoretical-practical approach, the program addresses the gaps identified in the existing literature. The program covers the foundations of Artificial Intelligence to its specific application in teaching, highlighting ethical aspects, critical thinking, and its integration into pedagogy. Based on frameworks such as DigiComEdu, the program is presented as a strategic response to empower teachers, ensuring their preparation to lead the responsible implementation of AI in the classroom and research. One of the issues to consider is how to apply the program. To this end, there were three options: (1) Teaching classes that are 80% theoretical and 20% practical (2) Teaching classes that are 50% theoretical and 50% practical (3) Teaching classes that are 20% theoretical and 80% practical. For the general evaluation and ranking of each of the program's teaching alternatives, a hybrid method between the neutrosophic 2-tuple linguistic model and the Additive Ratio Assessment System (ARAS) was used. Among the advantages of using this method are the possibility for experts to use natural language in their evaluations, the accuracy of the evaluations as more possible states of knowledge are incorporated, and finally the simplicity in the application of the method.

Keywords: Neutrosophic 2-tuple linguistic model, ARAS method, AI literacy, AI competencies, Artificial Intelligence, AI literacy program, Teacher training.

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1 Introduction

Artificial intelligence refers to a type of algorithm or computerized systems that imitate human intellectual processes, such as the ability to generalize, reason, discover meaning, and learn from past experiences [1]. The term “artificial” bears a close similarity to the actual mental decision-making processes in human brains. For example, mathematical reasoning and causal reasoning tasks activate an important brain network called the task-positive network, which includes several brain regions associated with attention and cognition.

The significance of Artificial Intelligence (AI) in the university environment is not only limited to the implementation of advanced technologies but also redefines the very nature of research and education. In research, AI has proven to be an invaluable tool, enabling massive data analysis, identifying complex patterns, and accelerating scientific discoveries. In the educational field, AI introduces innovative personalized teaching and learning methods, adapting to the individual needs of students. However, this shift towards integrating AI into education and research poses particular challenges for university educators, who must acquire specialized skills to fully realize the potential of this constantly evolving technology.

Given this panorama, the need arises to comprehensively address AI literacy for teachers, particularly university teachers. Understanding AI competence frameworks is an essential component of empowering educators, allowing them to not only use advanced AI tools but also foster literacy in these new resources among their students. Furthermore, the implementation of a specific training system focused on AI literacy stands as a key strategy to close the gap between the rapid evolution of technology and the effective preparation of university teachers. The present study proposes a training system that empowers teachers to lead effectively in educational environments impacted by AI.

Within the framework of this study, a mixed and non-systematic exploratory methodology has been adopted, guided by the flexibility necessary to address the dynamic, and constantly evolving, nature of the field of artificial intelligence literacy for university teachers. Following the guidelines of the European Framework for the Digital Competence of Educators (DigCompEdu), our approach has focused on the identification and analysis of international initiatives related to AI literacy, specifically focusing on the university context.

Data collection encompassed an exhaustive search of academic websites, social media, educational and scientific databases, as well as digital publishing platforms. This diversified strategy allowed us for a variety of perspectives and approaches to AI literacy. The selection of initiatives was based on perceived relevance and pertinence to the topic of study, prioritizing those with significant scope, notable innovations, or unique approaches to AI literacy.

The data analysis was carried out using a qualitative approach, seeking to identify the artificial intelligence competencies necessary for university teachers. This qualitative exploration sought to outline common trends and strategies, as well as areas of focus within the selected initiatives that relate directly to the development of skills essential for the effective and ethical use of AI. Special attention was given to the competencies that are considered fundamental to train educators in the integration of AI in their pedagogical and research practices. Thus, all of this knowledge collection and analysis yielded to an IA literacy program for teachers.

Specifically, evaluation work is carried out on three possible ways of applying the proposed program to teachers, which in a simplified manner are differentiated into (1) Teaching classes that are 80% theoretical and 20% practical (2) Teaching classes that are 50% theoretical and 50% practical (3) Teaching classes 20% theoretical 80% practical. A ranking was obtained according to 5 experts of the possible ways of teaching the program based on a new model that appeared in the literature where the neutrosophic 2-tuple linguistic model is hybridized with the Additive Ratio Assessment System (ARAS) decision-making method [2]. The neutrosophic 2-tuple linguistic model is a generalization of the well-known 2-tuple linguistic model, which is a Computing with Words (CWW) method to perform symbolic linguistic operations where no information is lost [3]. The generalization to the neutrosophic framework allows us to obtain greater accuracy in capturing the opinion of the experts, since apart from the opinion on each attribute, there is also the opinion against and the indeterminacy.

ARAS belongs to the series of decision-making methods, which is characterized by its simplicity. It consists of comparing the values of the attributes with a pre-established ideal value [4].

The study is carried out based on an Artificial Intelligence literacy training program that is an adaptation of the DigCompEdu in universities or higher education centers in Latin America. Specifically, it is studied within the campus of the Bolivarian University of Ecuador. Some readings about Neutrosophy applied to Artificial Intelligence and Education can be found in [5-7], for Artificial Intelligence in [8-11], and education in [12, 13].

The paper is divided into a Materials and Methods section with the fundamental notions about the Neutrosophic 2-tuple Linguistic Model, the ARAS method, and the hybrid between both of them introduced in [4]. Section 3 contains the results of evaluating three didactic strategies for applying the proposed AI literacy program using the model explained above. The last section is dedicated to presenting the conclusions.

2 Materials and Methods

2.1 Basic Concepts of Neutrosophic 2-tuple Linguistic Model

Definition 1 ([14-16]). Let $S = \{s_0, s_1, \dots, s_g\}$ be a set of linguistic terms and $\beta \in [0, g]$ is a value that represents the result of a symbolic operation, then the linguistic 2-tuple that expresses the information equivalent to β is obtained using the following function:

$$\Delta: [0, g] \to S \times [-0.5, 0.5)$$
$$\Delta(\beta) = (s_i, \alpha) \quad (1)$$

Where s_i is such that $i = \text{round}(\beta)$ and $\alpha = \beta - i, \alpha \in [-0.5, 0.5)$ and "round" is the usual rounding operator, s_i is the index label closest to β and α is the value of the symbolic translation.

It should be noted that $\Delta^{-1}: \langle S \rangle \to [0, g]$ is defined as $\Delta^{-1}(s_i, \alpha) = i + \alpha$. Thus, a linguistic 2-tuple $\langle S \rangle$ is identified with its numerical value in $[0, g]$.

Suppose that $S = \{s_0, \dots, s_g\}$ is a *2-Tuple Linguistic Set* (2TLS) with odd cardinality g+1. It is defined for

$(s_T, a), (s_I, b), (s_F, c) \in$ L and a, b, c $\in$[0, g], where $(s_T, a), (s_I, b), (s_F, c) \in$ L independently express the degree of truthfulness, indeterminacy, and falsehood by 2TLS. *2-Tuple Linguistic Neutrosophic Number* (2TLNN) is defined as follows:

$l_j = \{(s_{T_j}, a), (s_{I_j}, b), (s_{F_j}, c)\}$ (2)

Where $0 \leq \Delta^{-1}(s_{T_j}, a) \leq g$, $0 \leq \Delta^{-1}(s_{I_j}, b) \leq g$, $0 \leq \Delta^{-1}(s_{F_j}, c) \leq g$, and $0 \leq \Delta^{-1}\left(s_{T_j}, a\right) + \Delta^{-1}\left(s_{I_j}, b\right) + \Delta^{-1}(s_{F_j}, c) \leq 3g$.

The scoring and accuracy functions allow us to rank 2TLNN.

Let $l_1 = \{(s_{T_1}, a), (s_{I_1}, b), (s_{F_1}, c)\}$ be a 2TLNN in L, the scoring and accuracy functions in l_1 are defined as follows, respectively:

$$S(l_1) = \Delta\left\{\frac{2g + \Delta^{-1}(s_{T_1}, a) - \Delta^{-1}(s_{I_1}, b) - \Delta^{-1}(s_{F_1}, c)}{3}\right\}, \Delta^{-1}(S(l_1)) \in [0, g] \quad (3)$$

$$H(l_1) = \Delta\left\{\frac{g + \Delta^{-1}(s_{T_1}, a) - \Delta^{-1}(s_{F_1}, c)}{2}\right\}, \Delta^{-1}(H(l_1)) \in [0, g] \quad (4)$$

2.2 Additive Ratio Assessment System (ARAS)

The first step in solving the multi-criteria decision-making problem with the support of the ARAS method is to create the $m \times n$ following matrix with m feasible alternatives per row evaluated according to n criteria given by columns [4].

$$X = \begin{bmatrix} x_{01} & \cdots & x_{0j} & \cdots & x_{0n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ x_{i1} & \cdots & x_{ij} & \cdots & x_{in} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ x_{m1} & \cdots & x_{mj} & \cdots & x_{mn} \end{bmatrix} i = 1,2,\dots,m; j = 1,2,\dots,n. \quad (5)$$

Where m is the number of alternatives, n is the number of criteria that describe each alternative, x_{ij} represents the evaluation of the ith alternative according to the jth criterion, while x_{0j} is the optimal value of the jth criterion.

When the jth criterion is unknown then it is taken:

$x_{0j} = max_i\, x_{ij}$, if $max_i\, x_{ij}$ $is\ preferable$

$x_{0j} = min_i\, x_{ij}^*$, if $min_i\, x_{ij}^*$ $is\ preferable$ (6)

The criteria whose values are maximum are normalized with the following Equation:

$\bar{x}_{ij} = \frac{x_{ij}}{\sum_{i=0}^{m} x_{ij}}$ (7)

The criteria whose values are minimum are normalized with the following Equations:

$x_{ij} = \frac{1}{x_{ij}^*}, \bar{x}_{ij} = \frac{x_{ij}}{\sum_{i=0}^{m} x_{ij}}$ (8)

In the other stage, the weights w_j are taken for the criteria, where $w_j \in (0,1)$ satisfy:

$\sum_{j=1}^{n} w_j = 1$ (9)

Thus, the matrix is obtained:

$$\hat{X} = \begin{bmatrix} \hat{x}_{01} & \cdots & \hat{x}_{0j} & \cdots & \hat{x}_{0n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \hat{x}_{i1} & \cdots & \hat{x}_{ij} & \cdots & \hat{x}_{in} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \hat{x}_{m1} & \cdots & \hat{x}_{mj} & \cdots & \hat{x}_{mn} \end{bmatrix} i = 1,2,\dots,m; j = 1,2,\dots,n. \quad (10)$$

This is the matrix with elements normalized by the Equation:

$\hat{x}_{ij} = \bar{x}_{ij} w_j; j = 1,2,\dots,n$ (11)

The values are calculated below:

$O_i = \sum_{j=1}^{n} \hat{x}_{ij}; i = 1,2,\dots,m.$ (12)

Where O_i is the value of the optimality function of the alternative i.

The highest value for O_i is the best and the lowest value is the worst.

The degree of usefulness of the ith alternative is obtained by comparing it with the ideal degree O_0, which is obtained according to Equation 13:

$$K_i = \frac{o_i}{o_0}; i = 1,2,\dots,m. \quad (13)$$

Figure 1 contains a diagram of the algorithm to follow in the ARAS method.

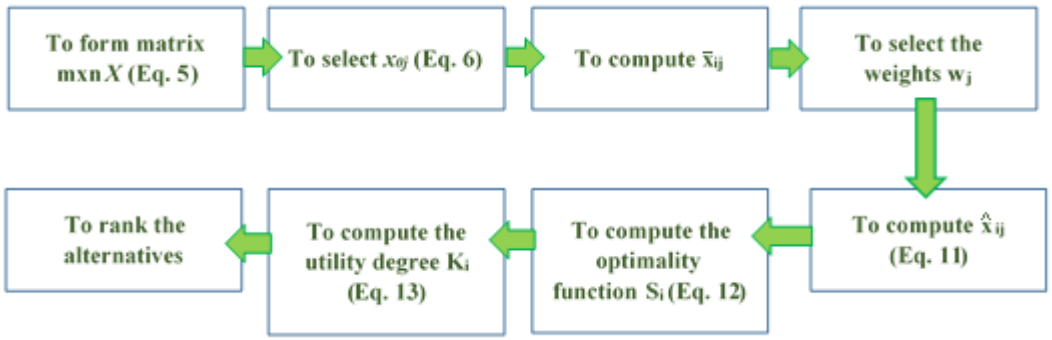
To form matrix mxn X (Eq. 5)
To select x_{0j} (Eq. 6)
To compute $\bar{x}_{ij}$
To select the weights w_j
To compute $\hat{x}_{ij}$ (Eq. 11)
To compute the optimality function S_i (Eq. 12)
To compute the utility degree K_i (Eq. 13)
To rank the alternatives

Figure 1: Scheme of the ARAS algorithm for multi-criteria decision-making. Source: [4].

2. 3 The algorithm of hybridized neutrosophic 2-tuple Linguistic model and ARAS Model

In this subsection, we recall the characteristics of the model that combines the neutrosophic linguistic 2-tuple model with the ARAS multi-criteria decision-making model [4]. See other approaches to Neutrosophic ARAS in [17, 18]. We explain this in the following steps:

1. Let us start from a set $E = \{e_1, e_2, \dots, e_k\}$ of $k \geq 1$experts, and $C = \{c_1, c_2, \dots, c_n\}$ are n criteria to measure m feasible alternatives $A = \{a_1, a_2, \dots, a_m\}$, as in the original ARAS method.

For each criterion, there is a linguistic measurement scale consisting of a set $S_i = \{s_1^i, s_2^i, \dots, s_{l_i}^i\}$ where l_i is an odd number.

2. Each expert e_p evaluates each alternative a_j according to the criterion c_i, giving a triad $(s_{jpq_T}^i, s_{jpq_I}^i, s_{jpq_F}^i)$ where $s_{jpq_T}^i \in S_i$ means the linguistic evaluation given by the expert p that the alternative meets the given criterion. See that q_T is the index of the linguistic term within the set S_i. Similarly, $s_{jpq_I}^i$ indicates the linguistic label for indeterminacy and $s_{jpq_F}^i$ for falsehood.

   This includes the evaluation of an ideal alternative a_0 for each of the criteria. The evaluations of this ideal alternative are obtained from experts either as the maximum of the values obtained when the maximum is preferable, or the minimum when this is preferable.

3. The triads $(s_{jpq_T}^i, s_{jpq_I}^i, s_{jpq_F}^i)$ are aggregated for each criterion and each alternative for all experts. To do this, the arithmetic mean is used concerning all the experts of the given evaluations. In this way, the following values are obtained:

   $(\bar{s}_{jq_T}^i, \bar{s}_{jq_I}^i, \bar{s}_{jq_F}^i)$ with $\beta_T = \frac{\sum_{p=1}^k q_T}{k}$, $\beta_I = \frac{\sum_{p=1}^k q_I}{k}$, and $\beta_F = \frac{\sum_{p=1}^k q_F}{k}$ (14)

   To simplify the notations, there are $r_j^i = \left(\beta_T{}_j^i, \beta_I{}_j^i, \beta_F{}_j^i\right) \in [0, l_i]^3$ where beta values are obtained for each of the evaluations for truthfulness, falsity, and indeterminacy, respectively.
   This is how the matrix that appears in Equation 15 is formed.

$$X_\beta = \begin{bmatrix} r_0^1 & \cdots & r_o^j & \cdots & r_0^n \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ r_i^1 & \cdots & r_i^j & \cdots & r_i^n \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ r_m^1 & \cdots & r_m^j & \cdots & r_m^n \end{bmatrix} i = 1,2,\dots,m; j = 1,2,\dots,n \quad (15)$$

4. Now the values of the matrix X_β are normalized as follows:

   The criteria whose ideal values are maximum are normalized with Equation 16.

$$\bar{r}_i^j = \frac{r_i^j}{\sum_{i=0}^m r_i^j} \quad (16)$$

Thus, $\bar{r}_j^i = \left(\frac{{\beta_T}_j^i}{\sum_{t=0}^m {\beta_T}_j^t}, \frac{{\beta_I}_j^i}{\sum_{t=0}^m {\beta_I}_j^t}, \frac{{\beta_F}_j^i}{\sum_{t=0}^m {\beta_F}_j^t} \right)$.

The criteria whose values are minimum are normalized with the following Equations:

$$r_i^j = \frac{1}{r_i^{j*}}, \bar{r}_i^j = \frac{r_i^j}{\sum_{i=0}^m r_i^j} \quad (17)$$

5. Then, we have the values $\bar{r}_i^j = (\gamma_T, \gamma_I, \gamma_F) \in [0,1]^3$. To convert them to a scalar value in $[0,1]$, formula 18 is used.
   $\lambda(\bar{r}_i^j) = \frac{2+\gamma_T-\gamma_I-\gamma_F}{3}$ (18)
   Let us call it $\bar{x}_{ij} = \lambda(\bar{r}_i^j)$.
6. The original ARAS method is applied to the above results, applying Equations 7-13 to the values $\bar{x}_{ij}$.

Figure 2 contains a schematic of the proposed algorithm.

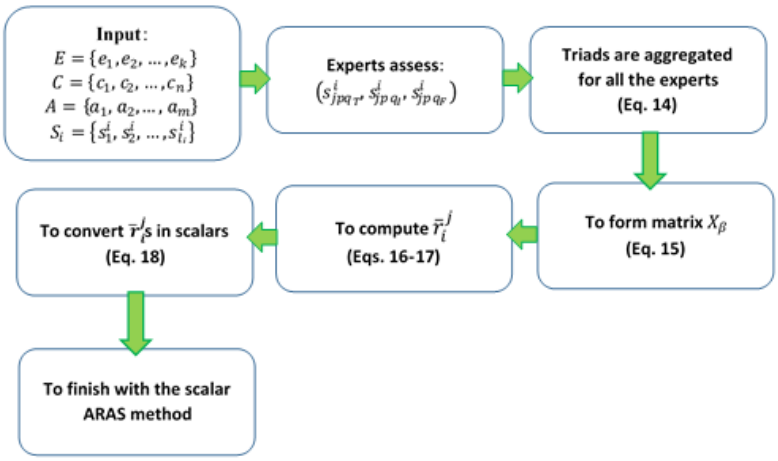
Input:
$E = \{e_1, e_2, \ldots, e_k\}$
$C = \{c_1, c_2, \ldots, c_n\}$
$A = \{a_1, a_2, \ldots, a_m\}$
$S_i = \{s_1^i, s_2^i, \ldots, s_{l_i}^i\}$
Experts assess:
$(s_{jpq_T}^i, s_{jp\,q_I}^i, s_{jp\,q_F}^i)$
Triads are aggregated for all the experts (Eq. 14)
To form matrix X_β (Eq. 15)
To compute $\bar{r}_i^j$ (Eqs. 16-17)
To convert $\bar{r}_i^j$s in scalars (Eq. 18)
To finish with the scalar ARAS method

Figure 2: Scheme of the ARAS algorithm hybridized with the neutrosophic 2-tuple linguistic model for multi-criteria decision-making. Source [4].

3 Analyzing the AI Literacy Program

3.1 Keynotes on DigCompEdu

The European Framework for the Digital Competence of Educators (DigCompEdu) is a scientifically solid framework, which describes the meaning of digitally competent educators. DigCompEdu identifies the digital competencies necessary to perform the teaching role effectively. The framework is designed to be used by professionals at all educational levels, from early childhood education to higher education, as well as by educators working with students with special needs or in non-formal learning contexts.

DigCompEdu is composed of six areas of competence:

- Competency 1: To understand the basic concepts of digital technology;
- Competency 2: To use digital technologies for learning and teaching;
- Competency 3: To create digital content;
- Competency 4: To evaluate the use of digital technologies;
- Competency 5: To participate in the digital society;
- Competency 6: To guide students in the use of digital technologies.

Additionally, the framework also provides a progression of competencies, allowing educators to evaluate their progress over time (Figure 3).

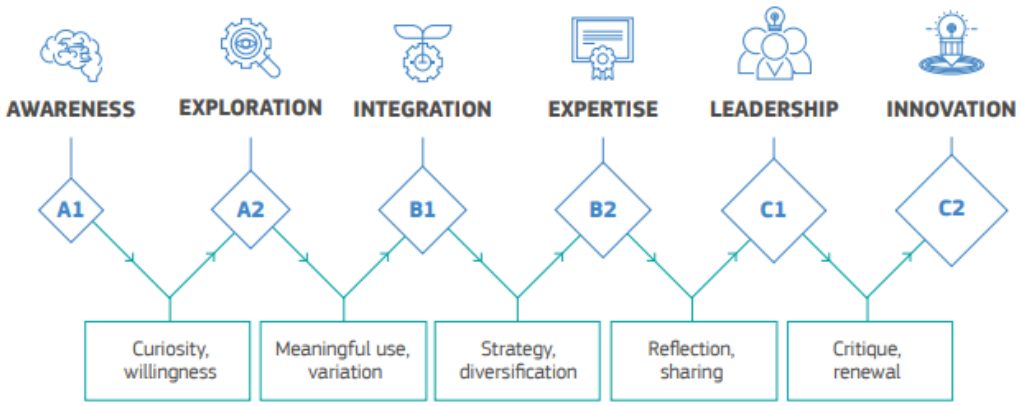

Figure 3 . DigiCompEdu Progression Model.

Although DigCompEdu is a robust framework, it could be updated by adding a new competence area on AI literacy. This competence area would cover topics such as the basic concepts of AI, its applications in education, and its ethical implications; and the review of existing competence areas to reflect the role of AI in education such as the "Use digital technologies for learning and teaching". Given the interest, development, and widespread use of DigComp in general and DigCompEdu in particular, we understand that it is a proposal that can be valid as a guide to propose a teacher training program in the short and medium term.

Concerning the integration of Generative AI in the Latin American educational context, we have to highlight the importance of developing digital teaching competencies within a framework adapted to the cultural, disciplinary, and technological access particularities of the region. Questioning the direct applicability of the European DigCompEdu framework, an adaptation is advocated, which respects diversity and guarantees equitable and culturally relevant digital training for teachers, see Figure 4. This perspective emphasizes the need to consider the different realities of educational communities and, if necessary, promote the creation and customization of frameworks that maximize local strengths and opportunities.

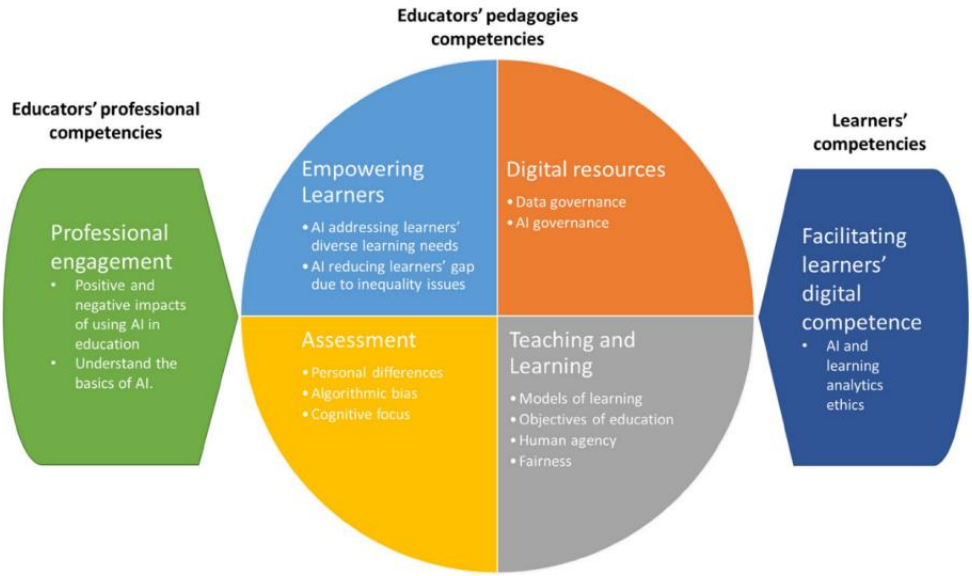

Figure 4. DigCompEdu framework for teachers' AI competence.

On the other hand, the importance of comprehensive professional development for teachers is highlighted,

which not only covers knowledge of AI but also a deep understanding of its ethical implications and limitations. Likewise, the importance of staying up to date on emerging technologies such as the Metaverse, blockchain, and cloud computing is highlighted, as disruptive technologies that students will face in their professional future, for which we must prepare them, but first, they must be trained in this sense.

In this context, a training program is proposed that is distinguished by its flexibility and adaptability, which allows its effective implementation in various educational contexts, whether in institutions of the same level or at different educational levels. Its modular design must cover everything from the fundamentals to the specific application of artificial intelligence in education, allowing it to be adjusted according to the particular needs of each institution. This feature not only ensures that the program is relevant to different academic realities, but also facilitates the incorporation of context-specific pedagogical approaches.

The scalability of the program is supported by its well-defined structure and the inclusion of periodic evaluations, which allow measuring participants' progress. The results obtained from these evaluations will serve to provide feedback to adapt the program in real-time and provide valuable data for future implementations in other educational institutions.

The institution Bolivarian University of Ecuador is distinguished by its youth and determined commitment to entering into scientific research. In line with an innovative vision, the Bolivarian University of Ecuador recognizes the urgent need to prepare its teachers and students to address emerging challenges in technological development, especially AI. As an incipient and dynamic educational entity, this academic institution is in a crucial stage of its evolution, seeking not only to consolidate its presence in the academic field but also to place its professionals at the forefront of educational and technological transformations.

Based on the analyses carried out on the AI competence frameworks, aimed at teachers, the following training program is proposed:

General goal: To develop the necessary skills for teachers to use Artificial Intelligence effectively and ethically in the teaching and learning process.

Program Duration: The training program can be carried out over 3 months, with regular training sessions and practical activities.

Module 1: AI Fundamentals:

- History and evolution of Artificial Intelligence.
- Basic concepts. Types of AI and their applications in various sectors
- Fundamental principles of operation of AI.
- Ethics and responsibility in the development of AI.

Module 2: Application of AI in Education:

- Personalization of learning: Strategies and tools.
- Intelligent feedback and adaptive assessment.
- Integration of AI in curricular planning.
- Educational gamification that is supported by AI.
- Impact of AI on educational equity and diversity.

Module 3: Ethics and Critical Thinking about AI:

- Ethical and social implications of AI in the educational field.
- Reflection on equity and diversity in the use of AI.
- Ethical decision-making in educational situations with AI.
- Individual and collective responsibility in the use of AI.

Module 4: Integration of AI in Pedagogy:

- Design of innovative educational activities with AI.
- Effective AI-assisted teaching strategies.
- Evaluation of the impact of AI on the learning process.
- Development of educational content with a focus on AI.

Module 5: Application of AI in Research (Optional: Aimed at University Professors):

- Use of AI in research data collection and analysis.
- Advanced AI tools and techniques for research.
- Innovations and advances in research facilitated by AI.
- Integration of AI in the publication and dissemination of results.
- Ethics in research supported by AI.

Training methodology:

The training program will adopt a holistic approach, combining theoretical sessions, workshops, practical activities, group discussions, and case studies. The active participation of teachers will be encouraged by promoting collaboration and the exchange of experiences. Additional resources, such as supplemental readings and access to educational AI platforms, will be provided to enrich learning.

Evaluation and monitoring:

Periodic evaluations will be carried out to measure teachers' progress in developing AI competencies. Additionally, individualized feedback and follow-up sessions will be conducted to address the specific needs of participants. To this end, as the next phase of research, a test is being developed to measure AI literacy based on the analyzed frameworks, which can help diagnosing the current and desired state considering the development of teachers' AI competencies.

The AI literacy framework is presented as a comprehensive and well-structured tool for the development of artificial intelligence competencies among educators. It consists of seven key areas ranging from the fundamentals of AI to preparing educators for the future of work. The clear progression and three levels of mastery provide a gradual and adaptive approach for educators who want to improve their AI skills. Its holistic approach and clear structure position it as a guide to consider for continuous professional development in an educational context transformed by AI. The framework consists of seven key areas:

1. AI Fundamentals: It covers the basics of AI, such as the history of AI, different types of AI, and the working principles of AI.
2. Data flow: It focuses on the use of data to train and improve AI systems. Educators at this level must demonstrate a basic understanding of data collection, preparation, and analysis techniques. They must also be able to identify and interpret data patterns relevant to AI applications in education.
3. Critical Thinking and Fact Checking: It explores how educators can critically evaluate AI technologies, communicate and collaborate effectively with AI, and use AI as a tool online, at home, and in the workplace. They must be able to assess the credibility and reliability of information generated by AI, identify potential biases and limitations, and use AI ethically and responsibly.
4. Diverse AI use cases: It examines how AI is used in different fields, such as education, medicine, and business. Educators must be able to understand the potential benefits and challenges of AI in various contexts and identify suitable applications for AI in their teaching practice.
5. Ethics of AI: It recognizes the ethical and social implications of using AI. Educators should be aware of potential ethical issues surrounding AI, such as bias, discrimination, and privacy, and be able to make informed decisions about integrating AI into their classrooms.
6. AI Pedagogy: It focuses on educators' ability to use AI to improve teaching and learning. They must be able to identify and select appropriate AI tools and resources that suit their teaching objectives and the needs of their students. They must also be able to seamlessly integrate AI into their lessons and adapt their pedagogy to accommodate AI-enhanced learning experiences.
7. Future of Work: It explores how AI is transforming the world of work and how educators can prepare students for future success. They must understand the impact of AI on various professions and be able to guide students in developing the skills and competencies necessary to thrive in an AI-driven workforce.

The framework also defines three levels of mastery:

1. Introductory: Students at this level develop a basic understanding of the topic. They can identify and describe key concepts but may lack in-depth knowledge or practical application of the subject.
2. Intermediate: At this level, learners demonstrate a deeper understanding of the topic. They can analyze, evaluate, and synthesize information, and can apply their knowledge to solve problems or make informed decisions.
3. Advanced: Lastly, they do not only deeply understand the topic, but they also contribute to it. They can create and curate content, provide intellectual leadership, and engage in strategic activities within the field. They can also actively participate in the community, consult with others, and share their experience.

So, these preceding aspects serve as the attributes and scale to assess the proposed program. See the next subsection.

3.2 Proposed assessment at Bolivarian University of Ecuador (BUE)

To evaluate the relevance of the program, five academics from the BUE were involved, who are acknowledged pedagogical authorities and knowledgeable about AI applied to teaching. As previously specified, the five experts must evaluate three alternatives (A1) Teaching classes 80% theoretical 20% practical (A2) Teaching classes 50% theoretical and 50% practical (A3) Teaching classes 20% theoretical 80% practical.

The evaluation criteria were specified previously, they are based on evaluating how the alternative allows achievements in the following aspects:

C1: AI Fundamentals.

C2: Data flow.

C3: Critical Thinking and Fact-Checking.

C4: Diverse AI use cases.

C5: Ethics of AI.
C6: AI Pedagogy.
C7: Future of Work.
The evaluation scale contains three elements:
$S = \{s_1, s_2, s_3\}$ where they mean the following:
s_1: By applying this alternative in the measured aspect, the teacher will reach an "Introductory" level in the short or medium term.
s_2: By applying this alternative in the measured aspect, the teacher will reach an "Intermediate" level in the short or medium term.
s_3: By applying this alternative in the measured aspect, the teacher will reach an "Advanced" level in the short or medium term.
a_0 was defined with the following values (s_3, s_3, s_1) for all the criteria, where the first two terms are equal because it means that there is no indeterminacy.
The five specialists issued their opinions after carefully studying the program, consulting the designers if necessary, and analyzing possible variants of class delivery.
The results of the experts' evaluation of each of the attributes and alternatives are summarized below.

Criterion/Alternative	a_0 = Optimal	A1	A2	A3
C_1	(3,3,1)	(3,3,1)	(3,2,1)	(2,2,1)
C_2	(3,3,1)	(2,2,1)	(3,2,1)	(3,3,1)
C_3	(3,3,1)	(3,3,1)	(3,2,1)	(2,2,1)
C_4	(3,3,1)	(3,3,1)	(3,2,1)	(2,2,1)
C_5	(3,3,1)	(3,3,1)	(3,2,1)	(2,2,1)
C_6	(3,3,1)	(2,2,1)	(3,3,1)	(2,2,1)
C_7	(3,3,1)	(2,2,1)	(3,3,1)	(3,2,1)

Table 1: Matrix of evaluations with the triple betas added for all experts for each alternative.

In Table 1 the triads appear in the form of evaluation indices, where each index is the result of calculating the median and not the arithmetic means of the evaluations given by the experts, which are the beta values. Note that the tables are represented with the transposed matrices.

Table 2 summarizes the normalized triads.

Criterion/Alternative	a_0 = Optimal	A1	A2	A3
C_1	(0.27,0,0.25)	(0.27,0,0.25)	(0.27,0.25,0.25)	(0.18,0,0.25)
C_2	(0.27,0,0.25)	(0.18,0,0.25)	(0.27,0.25,0.25)	(0.27,0,0.25)
C_3	(0.27,0,0.25)	(0.27,0,0.25)	(0.27,0.25,0.25)	(0.18,0,0.25)
C_4	(0.27,0,0.25)	(0.27,0,0.25)	(0.27,0.25,0.25)	(0.18,0,0.25)
C_5	(0.27,0,0.25)	(0.27,0,0.25)	(0.27,0.25,0.25)	(0.18,0,0.25)
C_6	(0.3,0,0.25)	(0.2,0,0.25)	(0.3,0,0.25)	(0.2,0,0.25)
C_7	(0.27,0,0.25)	(0.18,0,0.25)	(0.27,0,0.25)	(0.27,0.25,0.25)

Table 2: Matrix of evaluations with the triple betas from Table 1 normalized.

Note that in Table 2 in the indeterminacy component, the formula $abs(t - i)$ was used and then normalized. Next, Equation 18 is applied in Table 3.

Criterion/Alternative	a_0 = Optimal	A1	A2	A3
C_1	0.67333	0.67333	0.59	0.64333
C_2	0.67333	0.64333	0.59	0.67333
C_3	0.67333	0.67333	0.59	0.64333
C_4	0.67333	0.67333	0.59	0.64333
C_5	0.67333	0.67333	0.59	0.64333
C_6	0.68333	0.65	0.68333	0.65

C_7	0.67333	0.64333	0.67333	0.59

Table 3: Matrix of evaluations with the results of Table 2 converted into numerical values.

The data obtained from Table 3 are normalized according to Equation 7.

Criterion/Alternative	a_0 = Optimal	A1	A2	A3
C_1	0.260981632	0.26098163	0.22868306	0.24935368
C_2	0.260981632	0.24935368	0.22868306	0.26098163
C_3	0.260981632	0.26098163	0.22868306	0.24935368
C_4	0.260981632	0.26098163	0.22868306	0.24935368
C_5	0.260981632	0.26098163	0.22868306	0.24935368
C_6	0.256249391	0.24375061	0.25624939	0.24375061
C_7	0.260981632	0.24935368	0.26098163	0.22868306

Table 4: Matrix of the normalized elements of Table 3.

The summary of the results of the classic ARAS applied to the data in Table 4 can be observed in Table 5 with the weights $w_j = \frac{1}{7}$.

Alternative	a_0 = Optimal	A1	A2	A3
Optimality	0.260305597	0.25519779	0.23723519	0.24726143
Utility	1	0.98037763	0.91137182	0.94988903
Ranking	-	1	3	2

Table 5: Optimality and utility values of the ARAS crisp method for the results of Table 4.

According to the results of Table 5 and the experts' criteria, it is preferred to apply a pedagogical strategy with a proportion of 80% theoretical and 20% practical.

Conclusion

Artificial Intelligence is one of the scientific disciplines that is currently causing the greatest social impact in today's world. Its correct application depends on access to technological means, but also on the preparation of citizens for their correct use, including ethics. Its teaching is essential at all school levels so that students are then able to incorporate it into their daily lives and also professional. The purpose of this paper was to determine the pedagogical strategy to follow for a program where DigiComEdu is adapted from the European level to the conditions of a Latin American country, specifically at the Bolivarian University of Ecuador. A group of experts was consulted to evaluate three strategies: (1) Teaching classes 80% theoretical 20% practical (2) Teaching classes 50% theoretical and 50% practical (3) Teaching classes 20% theoretical 80% practical. The criteria to be evaluated were seven. A linguistic scale was used to carry out the evaluations and a hybrid method of neutrosophic linguistic 2-tuple model and the ARAS method was applied for decision making. It was concluded that the best strategy is teaching 80% theoretical classes and 20% practical classes.

GRATITUDE:

To the Universidad Bolivariana del Ecuador for the support and financing for the development of the Research Project "Right to education in new technologies" making this possible to achieve the established objective.

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Received: March 5, 2024. Accepted: July 30, 2024

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ISSN (print): 2331-6055, ISSN (online): 2331-608X
Impact Factor: 1.739

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